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Differential Operators on Hermite Sobolev Spaces

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Abstract

In this paper, we compute the Hilbert space adjoint ∂^* of the derivative operator ∂ on the Hermite Sobolev spaces $\mathcal{S}_q$. We use this calculation to give a different proof of the 'Monotonicity Inequality' for the class of differential operators (L, A) for which the inequality was proved in [2]. We also prove the Monotonicity Inequality for (L, A), when these correspond to the Ornstein-Uhlenbeck diffusion.

Keywords: Monotonicity Inequality, Stochastic Partial Differential Equations, Hermite Sobolev Spaces, Tempered Distributions, Adjoint Operators.
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1 Introduction

Let $\mathcal{S}$ be the space of rapidly decreasing smooth functions and $\mathcal{S}'$ be its dual, the space of tempered distributions on $\mathbb{R}^d$. Let $A_i, L : \mathcal{S}' \to \mathcal{S}', i = 1, \cdots, r$ be partial differential operators (to be specified below) and $A = (A_1, \cdots, A_r)$. Consider the existence and uniqueness problem for stochastic partial differential equations of the form

$$dY_t = L(Y_t)\ dt + A(Y_t).dB_t$$

where (B_t) is a r-dimensional Brownian motion and (Y_t) an $\mathcal{S}'$ valued process, with Y_0 a given $\mathcal{S}'$-valued random variable. In many cases (see [1], [2], [4], [5], [7], [8]) a sufficient condition for existence and uniqueness is the so called 'Monotonicity Inequality' for the pair of operators (L, A) (a related inequality, called the coercivity inequality is also considered in the context of stochastic partial differential equations, but in the setting of a Gelfand triple of Hilbert spaces (see [5], [6])). In [2] the Monotonicity inequality was proved for constant coefficient differential operators L and A of the form

$$L = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^t)_{ij}\partial^2_{ij} - \sum_{i=1}^{d} b_i\partial_i$$

and

$$A_i = -\sum_{j=1}^{d} \sigma_{ji}\partial_j.$$

The object of this note is to provide a new proof that is more conceptual than the essentially computational proof given in [2].

The Monotonicity inequality can be stated as follows : Let $\|.\|$ be a Hilbertian semi-norm on $\mathcal{S}'$ with corresponding inner product $\langle\cdot,\cdot\rangle$. Then the pair of operators (L, A) satisfies the 'Monotonicity Inequality' for the semi-norm $\|.\|$ if

$$\langle\phi, L\phi\rangle + \sum_{i=1}^{r} \|A_i\phi\|^2 \leq \ C\|\phi\|^2, \ \forall\phi \in \mathcal{S}. \tag{1.1}$$

In practice, the norm $\|.\|$ is taken as one of the Hermite Sobolev norms $\|.\|_q, q \in \mathbb{R}$ defined in Section 2, with the corresponding Hilbert spaces denoted by $(\mathcal{S}_q, \|\cdot\|_q)$ and $\mathcal{S} \subset \mathcal{S}_q \subset \mathcal{S}'$. The proof in [2] involved expanding ϕ along an ONB $\{h_{n,q}\}$ in $\mathcal{S}_q$, where $h_{n,q}$ are multiples of the usual Hermite functions h_n, that form an ONB in $\mathcal{S}_0 := \mathcal{L}^2(\mathbb{R}^d, dx) \equiv \mathcal{L}^2(\mathbb{R}^d)$ [where dx denotes the Lebesgue measure]. The left hand side in the inequality above can then be computed using linearity, in terms of the action of L and A_i on the $h_{n,q}$, which in turn can be computed, using the recurrence relation for the action of the derivatives ∂_i on the Hermite functions, viz. $\partial_i h_n$. It was shown in [2] that the resulting series was essentially the same as that for $\|\phi\|^2$, by showing that certain sequences appearing in successive terms of the series was bounded (see Lemma (2.2), [2]).

In this paper we show that the set of computations mentioned in the previous paragraph, involve essentially two main steps : One, calculation of the adjoint ∂_i^* of the derivative operator ∂_i in the Hilbert Space corresponding to the norm $\|.\|_p$. We calculate it as $\partial_i^* = -\partial_i + T_{\partial_i}$, where T_{∂_i} is a bounded operator on $(\mathcal{S}_p, \|\|_p)$. The proof that T_{∂_i} is a bounded operator involves a 'first-order' version of the inequalities proved in Lemma (2.2) of [2] (see Lemma (2.2) below). Two, an 'integration by parts', using the adjoint computed in step one, that results in cancelation of the unbounded terms in the series for the LHS of inequality in (1.1), and that leaves only the action of the bounded operator T_{∂_i} on ϕ. Our proof thus is a generalization of the proof in the case $p = 0$ (i.e. $\mathcal{L}^2(\mathbb{R}^d)$), for which it follows trivially by 'integration by parts'.

In Section 2, we calculate the adjoint ∂_i^* and prove the representation mentioned above. In Section 3, we prove the Monotonicity Inequality when the operators L and A are constant coefficient operators as above . Section 4 is devoted to a proof of the Monotonicity Inequality for variable coefficients when L and A correspond to the Ornstein-Uhlenbeck diffusion (see [8]). Some applications of this will be developed in a separate article. We prove our results only when the underlying state space has dimension $d = 1$. We only state the results for higher dimensions. The proofs for higher dimensions are similar to the one dimensional case.

2 The Adjoint of the Derivative on $\mathcal{S}_q$

Let $\mathbb{Z}_+^d := \{n = (n_1, \cdots, n_d) : \ n_i$ non-negative integers$\}$. If $n = (n_1, \cdots, n_d)$, we define $|n| := n_1 + \cdots + n_d$. Let $\{h_n : n \in \mathbb{Z}_+^d\}$ be the orthonormal basis

(ONB) in $\mathcal{L}^2(\mathbb{R}^d)$ given by the Hermite functions (see [3], [9]). Let $\langle\cdot,\cdot\rangle$ represent the $\mathcal{L}^2(\mathbb{R}^d)$ inner product. For any fixed $q \in \mathbb{R}$, consider the following formal sums

$$\begin{cases} \langle f,g\rangle_q := \sum_{k=0}^{\infty}\sum_{|n|=k}(2k+d)^{2q}\langle f,h_n\rangle\langle g,h_n\rangle, \\ \|f\|_q^2 := \sum_{k=0}^{\infty}\sum_{|n|=k}(2k+d)^{2q}\langle f,h_n\rangle^2 \end{cases} \tag{2.1}$$

Let $\mathcal{S} \subset \mathcal{L}^2(\mathbb{R}^d)$ be the space of smooth rapidly decreasing $\mathbb{R}$-valued functions on $\mathbb{R}^d$ with the topology given by L. Schwartz (see [10]). Then $(\mathcal{S}, \|.\|_q)$ are pre-Hilbert spaces and completing them one obtains the Hilbert spaces $(\mathcal{S}_q, \|.\|_q)$ (see [3]).

For any $\phi \in \mathcal{S}$, we have $\phi = \displaystyle\sum_{k=0}^{\infty}\sum_{|n|=k}\phi_n h_n$. We are going to use the following convention: $n_i < 0$, for some i will mean $\phi_n = 0, h_n = 0$.

Let $\mathcal{S}'$ denote the space of tempered distributions. In [3], it was shown that $(\mathcal{S}_{-q}, \|.\|_{-q})$ are dual to $(\mathcal{S}_q, \|.\|_q)$. Furthermore, the following are also known:

$$\begin{cases} \mathcal{L}^2(\mathbb{R}^d) = (\mathcal{S}_0, \|.\|_0), \\ \text{for } p<q, (\mathcal{S}_q, \|.\|_q) \subset (\mathcal{S}_p, \|.\|_p), \\ \mathcal{S} = \bigcap_{q\in\mathbb{R}}(\mathcal{S}_q, \|.\|_q), \\ \mathcal{S}' = \bigcup_{q\in\mathbb{R}}(\mathcal{S}_q, \|.\|_q) \end{cases}$$

From now onwards we shall work with dimension $d = 1$. Any results mentioned in the following can be generalized to multi-dimensions.

Consider the derivative map denoted by $\partial : \mathcal{S} \to \mathcal{S}$. We can extend this map by duality to $\partial : \mathcal{S}' \to \mathcal{S}'$ as follows: for $\psi \in \mathcal{S}'$,

$$\langle \partial\psi, \phi\rangle := -\langle \psi, \partial\phi\rangle, \ \forall \phi \in \mathcal{S}.$$

The following relation is wellknown (see [9])

$$\partial h_n = \sqrt{\frac{n}{2}}h_{n-1} - \sqrt{\frac{n+1}{2}}h_{n+1}, \ \forall n \geq 0, \tag{2.2}$$

and hence it is easy to see that $\partial : \mathcal{S}_{q+\frac{1}{2}} \to \mathcal{S}_q$ is a bounded linear operator. Since $\mathcal{S}$ is dense in $\mathcal{S}_q$, ∂ is a densely defined closed operator on $\mathcal{S}_q$ with $Dom(\partial) \supset \mathcal{S}_{q+\frac{1}{2}}$.

Let ∂^* denote the Hilbert space adjoint of ∂ on $\mathcal{S}_q$. For convenience of notation, we shall not introduce q in ∂^*, though it should be understood that we are working for a fixed $q \in \mathbb{R}$. Now $\partial^* : Dom(\partial^*) \subset \mathcal{S}_q \to \mathcal{S}_q$ with $Dom(\partial^*) = \{\phi \in \mathcal{S}_q : Dom(\partial) \ni \psi \to \langle \partial\psi, \phi\rangle_q$, is a bounded linear functional$\}$. Note that $\mathcal{S} \subset Dom(\partial^*)$.

For $\phi \in Dom(\partial^*)$, ∂^* satisfies

$$\langle \partial\psi, \phi\rangle_q = \langle \psi, \partial^*\phi\rangle_q, \ \psi \in Dom(\partial).$$

Observe that by integration by parts, for $q = 0$, we have $\partial^* = -\partial$. In the next Theorem, we compute ∂^* in $\mathcal{S}_q$ explicitly and the resulting formula generalizes the above relation to the case $q \neq 0$.

Consider the following two sequences:

$$a_n := \sqrt{\frac{n}{2}} \left[\left(\frac{2n-1}{2n+1} \right)^{2q} - 1 \right], \; b_n := \sqrt{\frac{n+1}{2}} \left[1 - \left(\frac{2n+3}{2n+1} \right)^{2q} \right] \tag{2.3}$$

We now define linear operators $\tilde{A}, \tilde{B}, U_{+1}, U_{-1}$ on $\mathcal{S}$ via the formal expressions: for $\phi = \sum_{n=0}^{\infty} \phi_n h_n \in \mathcal{S}$,

$$\tilde{A}\phi := \sum_{n=0}^{\infty} a_n \psi_n h_n, \; \tilde{B}\phi := \sum_{n=0}^{\infty} b_n \psi_n h_n, \tag{2.4}$$

$$U_{+1}\phi := \sum_{n=0}^{\infty} \psi_{n+1} h_n, \; U_{-1}\phi := \sum_{n=0}^{\infty} \psi_{n-1} h_n. \tag{2.5}$$

We then have,

Theorem 2.1. *$\tilde{A}, \tilde{B}, U_{+1}, U_{-1}$ are bounded linear operators on $(\mathcal{S}, \|\cdot\|_q)$ and hence can be extended to bounded linear operators in $(\mathcal{S}_q, \|\cdot\|_q)$. Further we have, for any $\phi, \psi \in \mathcal{S}$,*

$$\langle \partial\phi, \psi \rangle_q + \langle \phi, \partial\psi \rangle_q = \langle \phi, (\tilde{A}U_{-1} + \tilde{B}U_{+1})\psi \rangle_q \tag{2.6}$$

and hence we obtain

$$\partial^* = -\partial + T_\partial \text{ on } \mathcal{S},$$

where $T_\partial = \tilde{A}U_{-1} + \tilde{B}U_{+1}$.

For the proof of Theorem (2.1), we need the following two Lemmas (2.2), (2.3).

Lemma 2.2. *The sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ satisfy the following inequality:*

$$|a_n| \leq \frac{M}{\sqrt{n}}, \; |b_n| \leq \frac{M}{\sqrt{n}}, \; \forall n \in \mathbb{N}$$

for some $M > 0$. Consequently, both the sequences are bounded. We take

$$M_{1,q} := \sup_{n \in \mathbb{N}\cup\{0\}} |a_n| \text{ and } M_{2,q} := \sup_{n \in \mathbb{N}\cup\{0\}} |b_n|. \tag{2.7}$$

Lemma 2.3. *For any $\phi, \psi \in \mathcal{S}$, we have*

$$\langle \partial\phi, \psi \rangle_q + \langle \phi, \partial\psi \rangle_q = \sum_{n=0}^{\infty} (2n+1)^{2q} \phi_n a_n \psi_{n-1} + \sum_{n=0}^{\infty} (2n+1)^{2q} \phi_n b_n \psi_{n+1} \tag{2.8}$$

Proof of Theorem (2.1).
Given $\phi \in \mathcal{S}$,

$$\begin{aligned}
\|\tilde{A}\phi\|_q^2 &= \sum_{n=0}^{\infty}(2n+1)^{2q}|a_n|^2\phi_n^2 \\
&\leq \left(\sup_{n\in\mathbb{N}\cup\{0\}\}}|a_n|^2\right)\sum_{n=0}^{\infty}(2n+1)^{2q}\phi_n^2 \\
&\leq M_{1,q}^2\|\phi\|_q^2 \text{ (by Lemma (2.2))}
\end{aligned}$$

Therefore, $\|\tilde{A}\phi\|_q \leq M_{1,q}\|\phi\|_q$, i.e. $\|\tilde{A}\|_q \leq M_{1,q}$.
Similarly, $\|\tilde{B}\|_q \leq M_{2,q}$.
Again,

$$\begin{aligned}
\|U_{+1}\phi\|_q^2 &= \sum_{n=0}^{\infty}(2n+1)^{2q}|\phi_{n+1}|^2 \\
&= \sum_{n=0}^{\infty}\frac{(2n+1)^{2q}}{(2n+3)^{2q}}(2n+3)^{2q}|\phi_{n+1}|^2 \\
&\leq \left(\sup_{n\in\mathbb{N}\cup\{0\}}\frac{(2n+1)^{2q}}{(2n+3)^{2q}}\right)\sum_{n=1}^{\infty}(2n+1)^{2q}|\phi_n|^2 \\
&\leq \left(\sup_{n\in\mathbb{N}\cup\{0\}}\left(\frac{2n+1}{2n+3}\right)^{2q}\right)\|\phi\|_q^2
\end{aligned}$$

from which we obtain $\|U_{+1}\|_q \leq \sup\limits_{n\in\mathbb{N}\cup\{0\}}(\frac{2n+1}{2n+3})^q$.
Similarly, $\|U_{-1}\|_q \leq \sup\limits_{n\in\mathbb{N}}(\frac{2n+1}{2n-1})^q$.
Linearity of $\tilde{A}, \tilde{B}, U_{+1}, U_{-1}$ is clear from definition. Using Lemma (2.3), we now have

$$\langle\partial\phi,\psi\rangle_q + \langle\phi,\partial\psi\rangle_q = \langle\phi,(\tilde{A}U_{-1}+\tilde{B}U_{+1})\psi\rangle_q.$$

Since each of $\tilde{A}, \tilde{B}, U_{+1}, U_{-1}$ is a bounded linear operator on $(\mathcal{S}_q, \|.\|_q)$, $(\tilde{A}U_{-1}+\tilde{B}U_{+1})$ is also bounded linear operator on $(\mathcal{S}_q, \|.\|_q)$. $\square$

Proof of Lemma (2.3).
Since $\phi,\psi \in \mathcal{S}$, we have $\phi = \sum\limits_{n=0}^{\infty}\phi_n h_n$, $\psi = \sum\limits_{n=0}^{\infty}\psi_n h_n$.
Now,

$$\begin{aligned}
\partial\phi &= \sum_{n=0}^{\infty}\phi_n(\partial h_n) \\
&= \sum_{n=0}^{\infty}\phi_n\left[\sqrt{\frac{n}{2}}h_{n-1} - \sqrt{\frac{n+1}{2}}h_{n+1}\right] \\
&= \sum_{n=-1}^{\infty}\phi_{n+1}\sqrt{\frac{n+1}{2}}h_n - \sum_{n=1}^{\infty}\phi_{n-1}\sqrt{\frac{n}{2}}h_n \\
&= \sum_{n=0}^{\infty}\phi_{n+1}\sqrt{\frac{n+1}{2}}h_n - \sum_{n=0}^{\infty}\phi_{n-1}\sqrt{\frac{n}{2}}h_n \qquad (2.9)
\end{aligned}$$

Similar expression is true for $\partial\psi$.

Therefore, $\langle \phi, \partial\psi\rangle_q = \sum_{n=0}^{\infty}(2n+1)^{2q}\phi_n\left[\sqrt{\frac{n+1}{2}}\psi_{n+1} - \sqrt{\frac{n}{2}}\psi_{n-1}\right]$......(*)

and

$$\begin{aligned}
\langle \partial\phi, \psi\rangle_q &= \sum_{n=0}^{\infty}(2n+1)^{2q}\psi_n\left[\sqrt{\frac{n+1}{2}}\phi_{n+1} - \sqrt{\frac{n}{2}}\phi_{n-1}\right] \\
&= \sum_{n=1}^{\infty}(2n-1)^{2q}\phi_n\psi_{n-1}\sqrt{\frac{n}{2}} - \sum_{n=-1}^{\infty}(2n+3)^{2q}\phi_n\psi_{n+1}\sqrt{\frac{n+1}{2}} \\
&= \sum_{n=0}^{\infty}(2n-1)^{2q}\phi_n\psi_{n-1}\sqrt{\frac{n}{2}} - \sum_{n=0}^{\infty}(2n+3)^{2q}\phi_n\psi_{n+1}\sqrt{\frac{n+1}{2}} \\
&= \sum_{n=0}^{\infty}(2n+1)^{2q}\phi_n\left[\psi_{n-1}\sqrt{\frac{n}{2}}\left(\frac{2n-1}{2n+1}\right)^{2q}\right] \\
&\quad - \sum_{n=0}^{\infty}(2n+1)^{2q}\phi_n\left[\psi_{n+1}\sqrt{\frac{n+1}{2}}\left(\frac{2n+3}{2n+1}\right)^{2q}\right] \text{......(**)}
\end{aligned}$$

Using (*) and (**), we get the result. □

Proof of Lemma (2.2).

For $n \in \mathbb{N}\cup\{0\}$, $|a_n| = \sqrt{\frac{n}{2}}\left|\left[\left(\frac{2n-1}{2n+1}\right)^{2q} - 1\right]\right|$.

To find an upper bound of a_n's, we follow the method in Lemma (2.2) of [2]. Choose an analytic branch of $z \mapsto z^{2q}$ in a domain containing the positive real axis and then we can define

$$f(z) := \left(\frac{2-z}{2+z}\right)^{2q} - 1$$

in a neighbourhood of 0, say in a ball of radius $\delta > 0$, i.e. $B(0,\delta)$.

Since $f(0) = 0$, $\exists$ an analytic function g defined on $B(0,\delta)$ such that $f(z) = zg(z)$, $\forall z \in B(0,\delta)$. But on the compact set $\overline{B(0,\frac{\delta}{2})}$ the function g is bounded, say by some constant $R > 0$.

Fix a positive integer N such that $\frac{1}{N} < \frac{\delta}{2}$. Then $\forall n > N$,

$$|a_n| = \sqrt{\frac{n}{2}}\left|f\left(\frac{1}{n}\right)\right| \leq \frac{1}{\sqrt{2n}}\left|g\left(\frac{1}{n}\right)\right| \leq \frac{R}{\sqrt{2n}}.$$

Then taking $M := \sup\{|a_1|, \sqrt{2}|a_2|, \cdots, \sqrt{N}|a_N|, \frac{R}{\sqrt{2}}\}$, we have

$$|a_n| \leq \frac{M}{\sqrt{n}},\ \forall n > 0.$$

From this inequality required bound can be obtained.

Proof for b_n's are similar. □

The operator T_∂ has the following important property that will be needed in the next section.

Lemma 2.4. *The map $\langle \partial(\cdot), T_\partial(\cdot)\rangle_q : \mathcal{S}\times\mathcal{S}\to\mathbb{R}$ defined by*

$$(\phi,\psi)\mapsto \langle \partial\phi, T_\partial\psi\rangle_q,\ \forall \phi,\psi\in\mathcal{S}$$

is a bounded bilinear form in $\|\|_q$ and hence extends to a bounded bilinear form on $(\mathcal{S}_q,\|\|_q)\times(\mathcal{S}_q,\|\|_q)$.

Proof. For $\phi,\psi\in\mathcal{S}$,

$$\begin{aligned}
&= \langle \partial\phi, T_\partial\psi\rangle_q\\
&= \sum_{n=0}^{\infty}(2n+1)^{2q}\langle \partial\phi, h_n\rangle\langle T_\partial\psi, h_n\rangle\\
&= -\sum_{n=0}^{\infty}(2n+1)^{2q}\langle \phi, \partial h_n\rangle\langle \tilde{A}U_{-1}+\tilde{B}U_{+1}\psi, h_n\rangle\\
&= -\sum_{n=0}^{\infty}(2n+1)^{2q}\langle \phi, \sqrt{\frac{n}{2}}h_{n-1}-\sqrt{\frac{n+1}{2}}h_{n+1}\rangle\langle (\tilde{A}U_{-1}+\tilde{B}U_{+1})\psi, h_n\rangle\\
&= -\sum_{n=0}^{\infty}(2n+1)^{2q}\left(\sqrt{\frac{n}{2}}\phi_{n-1}-\sqrt{\frac{n+1}{2}}\phi_{n+1}\right)(a_n\psi_{n-1}+b_n\psi_{n+1})
\end{aligned}$$

From Lemma (2.2), we have that $a_n\sim O(\frac{1}{\sqrt{n}}), b_n\sim O(\frac{1}{\sqrt{n}})$.
Now using the Cauchy-Schwarz inequality, we get $\exists C>0$, such that

$$|\langle \partial\phi, T_\partial\psi\rangle_q|\le C\|\phi\|_q\|\psi\|_q.$$

This completes the proof. □

Multi dimensional version of the above results can be formulated as follows:

For any fixed integer $i, 1\le i\le d$ consider the sequences:

$$a_{n,i} := \sqrt{\frac{n_i}{2}}\left[\frac{(2k+d-2)^{2q}-(2k+d)^{2q}}{(2k+d)^{2q}}\right],$$

$$b_{n,i} := \sqrt{\frac{n_i+1}{2}}\left[\frac{(2k+d)^{2q}-(2k+d+2)^{2q}}{(2k+d)^{2q}}\right].$$

where $n=(n_1,\cdots,n_d)$ is a multi-index with $|n|=k\ge 0$.
Let $\{f_i : 1\le i\le d\}$ denote the standard basis for $\mathbb{R}^d$. Define linear operators $\tilde{A}_i, \tilde{B}_i, U_{-f_i}, U_{+f_i}$ on $\mathcal{S}(\mathbb{R}^d)$ by the formal expressions: for $\psi=\sum_{k=0}^{\infty}\sum_{|n|=k}\psi_n h_n\in\mathcal{S}$,

$$\tilde{A}_i\phi := \sum_{k=0}^{\infty}\sum_{|n|=k}a_{n,i}\psi_n h_n,\ \tilde{B}_i\phi := \sum_{k=0}^{\infty}\sum_{|n|=k}b_{n,i}\psi_n h_n. \qquad (2.10)$$

$$U_{+f_i}\phi := \sum_{k=0}^{\infty}\sum_{|n|=k}\psi_{n+f_i}h_n,\ U_{-f_i}\phi := \sum_{k=0}^{\infty}\sum_{|n|=k}\psi_{n-f_i}h_n. \qquad (2.11)$$

Theorem 2.5. *Each of $\tilde{A}_i, \tilde{B}_i, U_{-f_i}, U_{+f_i}$ is a bounded operator on $(\mathcal{S}_q(\mathbb{R}^d), \|.\|_q)$ hence extends to $(\mathcal{S}_q(\mathbb{R}^d), \|.\|_q)$ as bounded linear operators. Furthermore, for any $1 \leq i \leq d$ and for any $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$,*

$$\langle \partial_i \phi, \psi \rangle_q + \langle \phi, \partial_i \psi \rangle_q = \langle \phi, (\tilde{A}_i U_{-f_i} + \tilde{B}_i U_{+f_i}) \psi \rangle_q$$

and hence we have

$$\partial_i^* = -\partial_i + T_{\partial_i} \text{ on } \mathcal{S},$$

where $T_{\partial_i} = \tilde{A}_i U_{-f_i} + \tilde{B}_i U_{+f_i}$.

3 The Monotonicity Inequality

Let $\{e_i : 1 \leq i \leq r\}$ denote the standard ONB in $\mathbb{R}^r$. Let $\sigma = (\sigma_{ij})$ be a constant $d \times r$ matrix with $(a_{ij}) = (\sigma\sigma^t)_{ij}$ and $b = (b_1, ..., b_d) \in \mathbb{R}^d$. For $\phi \in \mathcal{S}$, we define

$$\left.\begin{array}{l} L\phi := \frac{1}{2}\sum_{i,j=1}^d a_{ij}\partial_{ij}^2\phi - \sum_{i=1}^d b_i\partial_i\phi, \\ A_i\phi := -\sum_{j=1}^d \sigma_{ji}(\partial_j\phi) \\ A\phi = (A_1\phi, \ldots, A_r\phi) \end{array}\right\}$$

So that for $l \in \mathbb{R}^r$,

$$A\phi(l) := -\sum_{i=1}^r \sum_{j=1}^d \sigma_{ji}(\partial_j\phi)l_i = \sum_{i=1}^r A\phi(e_i)l_i.$$

The following result has already been established in [2].

Theorem 3.1. *For every $q \in \mathbb{R}, \exists$ a constant $C = C(q, d, (\sigma_{ij}), (b_j)) > 0$, such that*

$$2\langle \phi, L\phi \rangle_q + \|A\phi\|_{HS(q)}^2 \leq C.\|\phi\|_q^2 \tag{3.1}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, where $\|A\phi\|_{HS(q)}^2 := \sum_{i=1}^r \|A_i\phi\|_q^2$.
Furthermore, by density arguments the above inequality can be extended to all $\phi \in \mathcal{S}_{q+1}(\mathbb{R}^d)$.

Proof of Theorem (3.1) for the case $d = 1$.
Let $\phi \in \mathcal{S}$.
For the sake of convenience, we shall denote
$L_2\phi := \frac{1}{2}a\partial^2\phi$ and $L_1\phi := -b\partial\phi$, so that $L\phi = L_1\phi + L_2\phi$.
Observe that,

$$\begin{aligned} \langle \phi, \partial\psi \rangle_q + \langle \partial\phi, \psi \rangle_q &= \langle T_\partial \phi, \psi \rangle_q \\ &\leq \|T_\partial\|_q \|\psi\|_q \|\psi\|_q. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \phi, L_1\psi \rangle_q + \langle L_1\phi, \psi \rangle_q &= -b\left[\langle \phi, \partial\psi \rangle_q + \langle \partial\phi, \psi \rangle_q\right] \\ &\leq (|b|\|T_\partial\|_q)\|\phi\|_q\|\psi\|_q. \end{aligned}$$

Taking $\phi = \psi$, we obtain

$$\begin{aligned} 2\langle \phi, L_1\phi \rangle_q &= \langle \phi, L_1\phi \rangle_q + \langle L_1\phi, \phi \rangle_q \\ &\leq \left(\sum_i |b_i|\|T_\partial\|_q\right)\|\phi\|_q^2 \text{ ....}(*) \end{aligned} \tag{3.2}$$

Now consider,

$$\begin{aligned}2\langle\phi, L_2\phi\rangle_q &= a\langle\phi, \partial^2\phi\rangle_q\\ &= -a\langle\partial\phi, \partial\phi\rangle_q + a\langle T_\partial\phi, \partial\phi\rangle_q,\ \text{(using Theorem (2.1))}\end{aligned}$$

But $a = \sum_{k=1}^{r}\sigma_k^2$.

$$\begin{aligned}\therefore -a\langle\partial\phi, \partial\phi\rangle_q &= -\sum_{k=1}^{r}\langle\sigma_k\partial\phi, \sigma_k\partial\phi\rangle_q\\ &= -\sum_{k=1}^{r}\langle A\phi(e_k), A\phi(e_k)\rangle_q\\ &= -\|A\phi\|_{HS(q)}^2\end{aligned}$$

$$\therefore 2\langle\phi, L_2\phi\rangle_q + \|A\phi\|_{HS(q)}^2 = a\langle T_\partial\phi, \partial\phi\rangle_q. \tag{3.3}$$

By Lemma (2.4), we have

$$|a\langle T_\partial\phi, \partial\phi\rangle_q| \le C\|\phi\|_q^2,\ \forall\phi\in\mathcal{S} \tag{3.4}$$

for some constant $C > 0$.
Combining (*) and (**), proof is complete. □

Remark 3.2. From inequalities (3.2) and (3.4), it is clear that the constant C in the Monotonicity Inequality actually depends on the upper bound of $|\sigma_{ij}|$ and $|b_i|$.

4 The Case of Variable Coefficients

Let α be a C^∞ $\mathbb{R}^d$-valued function on $\mathbb{R}^d$ with bounded derivatives satisfying the linear growth condition, i.e.

$$\|\alpha(x)\| := \left(\sum_{i=1}^{d}|\alpha_i(x)|^2\right)^{\frac{1}{2}} \le M(1+|x|),\ \forall x\in\mathbb{R}^d. \tag{4.1}$$

Let $M_\alpha : \mathcal{S}\to\mathcal{S}$ denote the multiplication operator defined by $\phi\mapsto\alpha\phi$. By duality, we can extend this operator to $M_\alpha : \mathcal{S}'\to\mathcal{S}'$ as follows: for $\psi\in\mathcal{S}'$,

$$\langle M_\alpha\psi, \phi\rangle := \langle\psi, M_\alpha\phi\rangle = \langle\psi, \alpha\phi\rangle,\ \forall\phi\in\mathcal{S}. \tag{4.2}$$

Unless specifically mentioned, we shall take q to be an integer.
Following the proof of Proposition (3.2) in [8], one can show the following:

$$\|M_\alpha\psi\|_q \le \|\psi\|_{q+1},\ \forall\psi\in\mathcal{S}_{q+1}.$$

Therefore, $M_\alpha : \mathcal{S}_{q+1}\to\mathcal{S}_q$ is a bounded linear operator. Since $\mathcal{S}\subset\mathcal{S}_q$, $M_\alpha : \mathcal{S}_q\to\mathcal{S}_q$ is a densely defined closed linear operator, with $Dom(M_\alpha)\supset\mathcal{S}_{q+1}$.

Let M_α^* denote the Hilbert space adjoint of $M_\alpha : \mathcal{S}_q \to \mathcal{S}_q$. Then, $M_\alpha^* : Dom(M_\alpha^*) \subset \mathcal{S}_q \to \mathcal{S}_q$ with

$$Dom(M_\alpha^*) := \{\psi \in \mathcal{S}_q : \phi \mapsto \langle \psi, M_\alpha \phi\rangle_q \text{ is a continuous linear operator on } Dom(M_\alpha)\}.$$

Note that $\mathcal{S} \subset Dom(M_\alpha^*)$.

We list a few known results regarding the Monotonicity Inequality for variable coefficients, which is stated below.
Suppose $\sigma = (\sigma_{ij})$, $i = 1, .., d$, $j = 1, .., r$ and $b = (b_1, .., b_d)$, where σ_{ij} and b_i are $\mathcal{C}^\infty$ functions on $\mathbb{R}^d$ with bounded derivatives satisfying

$$\begin{aligned} \|\sigma(x)\| + \|b(x)\| :&= \left(\sum_{i=1}^{d}\sum_{j=1}^{r}|\sigma_{ij}(x)|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{d}|b_i(x)|^2\right)^{\frac{1}{2}} \\ &\le M(1+|x|)\ \forall x \in \mathbb{R}^d \end{aligned} \tag{4.3}$$

for some $M > 0$ and $|x|^2 := \sum_{i=1}^d x_i^2$.
Let $\mathcal{E}'$ denote the space of distributions with compact support.

Definition 4.1 (L^*, A^*)**.** Define $A^* : \mathcal{E}' \to \mathcal{L}(\mathbb{R}^r, \mathcal{E}')$ and $L^* : \mathcal{E}' \to \mathcal{E}'$ by

$$\begin{cases} A^*\phi = (A_1^*\phi, .., A_r^*\phi), \\ A_i^*\phi = -\sum_{k=1}^d \partial_k(\sigma_{ki}\phi), \\ L^*\phi = \frac{1}{2}\sum_{i,j=1}^d \partial_{ij}^2\left((\sigma\sigma^t)_{ij}\phi\right) - \sum_{i=1}^d \partial_i(b_i\phi) \end{cases} \tag{4.4}$$

Note that in [8], it was shown that under the linear growth condition on σ, b we have the following: given a positive real number p, for any $q \in \mathbb{R}$ satisfying $q > [p]+4$, the operators $L^* : \mathcal{S}_{-p}\cap\mathcal{E}'(K) \to \mathcal{S}_{-q}\cap\mathcal{E}'(K)$ and $A^* : \mathcal{S}_{-p}\cap\mathcal{E}'(K) \to \mathcal{L}(\mathbb{R}^r, \mathcal{S}_{-q}\cap\mathcal{E}'(K))$ are bounded operators, where $\mathcal{E}'(K) \subset \mathcal{E}'$ denotes the space of distributions whose support is contained in a compact set K.
Consider the following linear stochastic partial differential equation:

$$\begin{cases} dY_t = L^*Y_t\,dt + A^*Y_t\,dB_t, \\ Y_0 = \psi \end{cases} \tag{4.5}$$

where $\psi \in \mathcal{E}'$. It was shown in [8] that the explicit solutions for such equations when $supp(\psi)$ is contained in some compact set, can be constructed by considering the flow corresponding to

$$\begin{cases} dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt, \\ X_0 = x \end{cases} \tag{4.6}$$

It was also shown in [8] that $\psi_t = EY_t$ solves the initial value problem:

$$\begin{cases} \frac{\partial \psi_t}{dt} = L^*\psi_t, \\ \psi_0 = \psi \end{cases} \tag{4.7}$$

Assuming that the pair of operators L^*, A^* satisfy the Monotonicity Inequality, i.e. $\exists$ a constant $C = C(p, d, (\sigma_{ij}), (b_j)) > 0$, such that

$$2\langle\phi, L^*\phi\rangle_{-q} + \|A^*\phi\|_{HS(-q)}^2 \le C\|\phi\|_{-q}^2 \tag{4.8}$$

for all $\phi \in \mathcal{S}_{-p} \cap \mathcal{E}'$, one can prove the uniqueness of the solutions of equations (4.5) and (4.7) (see [1], [8]).
Note that when $\sigma = Id, b_i(x) = x_i \,\forall i, 1 \leq i \leq d$, the associated diffusion is the Ornstein-Uhlenbeck diffusion.

We are now going to prove the Monotonicity Inequality for the pair of operators L^*, A^* for specific σ, b. We prove the one dimensional version of the result.

Theorem 4.2. *Let $\sigma = (\sigma_i)$ be a constant function. Let $b(x) := b_0 + b_1 x$ for fixed $b_0, b_1 \in \mathbb{R}$. Fix any positive real number q. Then the monotonicity inequality for L^*, A^* holds, i.e. $\exists$ a positive constant $C = C(q, \sigma, b)$, such that*

$$2\langle \phi, L^*\phi\rangle_{-q} + \|A^*\phi\|^2_{HS(-q)} \leq C\|\phi\|^2_{-q} \tag{4.9}$$

for all $\phi \in \mathcal{S}$.

For the proof of Theorem (4.2), we need the following Lemma:

Lemma 4.3. *Let $q \in \mathbb{R}$. If $b(x) = x,\ \forall x \in \mathbb{R}$, then we have*

$$\langle \phi, M_b\psi\rangle_q = \langle M_b\phi, \psi\rangle_q + \langle T_x\phi, \psi\rangle_q,\ \forall \phi, \psi \in \mathcal{S}$$

where $T_x = (\tilde{A}U_{-1} - \tilde{B}U_{+1})$ is a bounded operator on $(\mathcal{S}_q, \|\cdot\|_q)$, with the operators $\tilde{A}, \tilde{B}, U_{+1}, U_{-1}$ as in Lemma (2.1). This also shows

$$M_b^* = M_b + T_x \text{ on } \mathcal{S}.$$

Proof. Note that $bh_n = \sqrt{\frac{n}{2}}h_{n-1} + \sqrt{\frac{n+1}{2}}h_{n+1}$ for all $n \geq 0$. The proof follows as in the case of T_∂ in Theorem (2.1). $\square$

Using the recurrence relations for $xh_n(x)$, it is easy to check that $\partial M_b : \mathcal{S}_{q+1} \to \mathcal{S}_q$ (where $b(x) \equiv x$ and $q \in \mathbb{R}$) is a bounded linear operator and hence using density arguments we have

Corollary 4.4. *Under the assumptions of Theorem (4.2), equation (4.9) holds for all $\phi \in \mathcal{S}_{-q+1}$.*

Proof of Theorem (4.2). We use the same approach as in Theorem (3.1). We shall denote
$L_2^*\phi := \frac{1}{2}\partial^2(a\phi)$ and $L_1^*\phi := -\partial(b\phi)$, where $a = \sum_{k=1}^r \sigma_k^2$.
Observe that if we take σ, b to be constants, the inequalities

$$2\langle \phi, L_2^*\phi\rangle_{-q} + \|A^*\phi\|^2_{HS(-q)} \leq C.\|\phi\|^2_{-q},\ \forall \phi \in \mathcal{S} \tag{4.10}$$

$$2\langle \phi, L_1^*\phi\rangle_{-q} \leq C.\|\phi\|^2_{-q},\ \forall \phi \in \mathcal{S} \tag{4.11}$$

follows from Theorem (3.1).
Combining the inequalities involving L_1^*, L_2^*, A^* we get the result.

Next we show that taking $b(x) = x$, L_1^* inequality works out.
For $\phi, \psi \in \mathcal{S}$,

$$\begin{aligned}\langle \phi, \partial(b\psi)\rangle_{-q} &= \langle (b + T_x)(-\partial + T_\partial)\phi, \psi\rangle_{-q},\\ &= -\langle b\partial\phi, \psi\rangle_{-q} + \langle bT_\partial\phi, \psi\rangle_{-q} - \langle T_x\partial\phi, \psi\rangle_{-q} + \langle T_xT_\partial\phi, \psi\rangle_{-q}\end{aligned}$$

But, $\partial(b\phi) = \phi + b\partial\phi$, $T_x^* = -T_x$ and $\langle bT_\partial\phi, \psi\rangle_{-q} = \langle T_\partial\phi, b\psi\rangle_{-q} + \langle T_\partial\phi, T_x\psi\rangle_{-q}$. Therefore,

$$\langle\phi, \partial(b\psi)\rangle_{-q} = \langle\phi, \psi\rangle_{-q} - \langle\partial(b\phi), \psi\rangle_{-q} + \langle T_\partial\phi, b\psi\rangle_{-q} + \langle\partial\phi, T_x\psi\rangle_{-q}.$$

Taking $\phi = \psi$ gives,

$$2\langle\phi, \partial(b\phi)\rangle_{-q} = \|\phi\|_{-q}^2 + \langle T_\partial\phi, b\phi\rangle_{-q} + \langle\partial\phi, T_x\phi\rangle_{-q}. \tag{4.12}$$

To verify the L_1^* inequality, we only need to check

$$\langle T_\partial\phi, b\phi\rangle_{-q} \le C\|\phi\|_{-q}^2 \tag{4.13}$$

$$\langle\partial\phi, T_x\phi\rangle_{-q} \le C\|\phi\|_{-q}^2 \tag{4.14}$$

The proofs of (4.13) and (4.14) are completed as in Lemma (2.4), noting that both the operators T_∂, T_x may be written in terms of $\tilde{A}, \tilde{B}, U_{+1}, U_{-1}$, except for a possible change of sign.
Checking the inequality for $b(x) := b_0 + b_1 x$ for fixed $b_0, b_1 \in \mathbb{R}$ is routine. $\square$

Remark 4.5. For the multi dimension case, we have the following:
Given $\phi \in \mathcal{S}(\mathbb{R}^d)$, observe that for $i \neq j$, $\partial_i(x_j\phi) = x_j(\partial_i\phi)$ and hence we have a relation similar to (4.12):

$$2\langle\phi, \partial_i(x_j\phi)\rangle_{-q} = \langle T_{\partial_i}\phi, x_j\phi\rangle_{-q} + \langle\partial_i\phi, T_{x_j}\phi\rangle_{-q}, \tag{4.15}$$

where $T_{x_j} = \tilde{A}_j U_{-f_j} - \tilde{B}_j U_{+f_j}$, with operators on right hand side as in Theorem (2.5).
From this we conclude that we can take any permutation of $x_1, \cdots, x_d$ as the functions $b_1, \cdots, b_d$ and the result is still true. Furthermore, even if we take linear combinations of x_j's as our b_i's, the bounds work out.

Multi dimensional version of the Theorem (4.2) is as follows:

Theorem 4.6. *Let σ be a constant function. Let $b = (b_1, \cdots, b_d)$ with each $b_i(x) := d_i + \sum_{j=1}^d c_{ij}x_j$ for fixed $d_i, c_{ij} \in \mathbb{R}$. Fix any positive real number q. Then the monotonicity inequality for L^*, A^* holds, i.e. $\exists$ a positive constant $C = C(q, d, (\sigma_{ij}), (b_j))$, such that*

$$2\langle\phi, L^*\phi\rangle_{-q} + \|A^*\phi\|_{HS(-q)}^2 \le C.\|\phi\|_{-q}^2 \tag{4.16}$$

for all $\phi \in \mathcal{S}_{-q+1}(\mathbb{R}^d)$.

References

[1] L. Gawarecki, V. Mandrekar, and B. Rajeev, *Linear stochastic differential equations in the dual of a multi-Hilbertian space*, Theory Stoch. Process. **14** (2008), no. 2, 28–34. MR 2479730 (2010e:60135)

[2] ———, *The monotonicity inequality for linear stochastic partial differential equations*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **12** (2009), no. 4, 575–591. MR 2590157 (2011c:60199)

[3] K. Itō, *Foundations of stochastic differential equations in infinite-dimensional spaces*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 47, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1984. MR 771478 (87a:60068)

[4] G. Kallianpur and J. Xiong, *Stochastic differential equations in infinite-dimensional spaces*, Institute of Mathematical Statistics Lecture Notes—Monograph Series, 26, Institute of Mathematical Statistics, Hayward, CA, 1995, Expanded version of the lectures delivered as part of the 1993 Barrett Lectures at the University of Tennessee, Knoxville, TN, March 25–27, 1993, With a foreword by Balram S. Rajput and Jan Rosinski. MR 1465436 (98h:60001)

[5] N. V. Krylov and B. L. Rozovskiĭ, *Stochastic evolution equations*, Current problems in mathematics, Vol. 14 (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1979, pp. 71–147, 256. MR 570795 (81m:60116)

[6] E. Pardoux, *Stochastic partial differential equations and filtering of diffusion processes*, Stochastics **3** (1979), no. 2, 127–167. MR 553909 (81b:60059)

[7] B. Rajeev, *Translation Invariant Diffusions in $\mathcal{S}'$*, [Preprint] (2012).

[8] B. Rajeev and S. Thangavelu, *Probabilistic representations of solutions of the forward equations*, Potential Anal. **28** (2008), no. 2, 139–162. MR 2373102 (2009e:60139)

[9] S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, Mathematical Notes, vol. 42, Princeton University Press, Princeton, NJ, 1993, With a preface by Robert S. Strichartz. MR 1215939 (94i:42001)

[10] François Trèves, *Topological vector spaces, distributions and kernels*, Dover Publications Inc., Mineola, NY, 2006, Unabridged republication of the 1967 original. MR 2296978 (2007k:46002)