On intersections of ideals in Banach spaces

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Abstract. The notion of an ideal in a Banach space was introduced by Godefroy, Kalton and Saphar in [5]. In this short note we are interested in studying finite intersections of ideals in Banach spaces. We show that for a Banach space \( X \), if in the bidual \( X^{**} \), every ideal of finite codimension is the intersection of ideals of codimension one, then the same property holds in \( X \). For a Banach space whose dual is isometric to \( L^1(\mu) \) for a positive measure \( \mu \) (the so called \( L^1 \)-predual spaces), we show that any ideal of finite codimension is a finite intersection of ideals of codimension one. These results extend a recent result of Bandyopadhyay and Dutta [1] proved for continuous function spaces.

1. Introduction

Let \( X \) be a Banach space. A closed subspace \( Y \subset X \) is said to be an ideal if there is a linear projection \( P : X^* \to X^* \) such that \( \ker(P) = Y^\perp \) and \( \|P\| = 1 \). This notion was introduced in [5]. We consider a Banach space as canonically embedded in its bidual \( X^{**} \). It is well known that the canonical projection \( Q : X^{***} \to X^{***} \) defined by \( Q(\Lambda) = \Lambda|X \) is a linear projection of norm one with \( \ker(Q) = X^\perp \). Thus \( X \) under the canonical embedding is an ideal in \( X^{**} \). Hence a prime motivation for the study of ideals is to exploit the structure that is inherent in the natural duality of \( X \) and its bidual (for example the Principle of local reflexivity) in the abstract set-up. See [9]. Clearly if \( Y \subset X \) is the range of a projection of norm one then, \( Y \) is an ideal in \( X \). As noted in Section 2 of [9], if \( Y \subset X \) is an ideal and \( Y \) is isometric to the range of a projection of norm one in some dual space, then \( Y \) is the range of a projection of norm one in \( X \). In particular any weak*-closed ideal in a dual space is the range of a projection of norm one.
Ideals can also be seen as a generalization of the well known notion of a $M$-ideal studied by E. M. Alfsen and E. G. Effros, see [4] for the details. Let $K$ be a compact Hausdorff space and let $C(K)$ denote the space of continuous functions, equipped with the supremum norm. It is known that $M$-ideals here are precisely closed algebraic ideals, where as ideals in our sense are precisely subspaces that are $L^1$-preduals (see Proposition 1 in [9]). Thus the natural embedding of $C([0, 1])$ in the space of bounded sequences $\ell^\infty$ is an ideal and as $C([0, 1])$ is separable, it is not even complemented in $\ell^\infty$. It is also known that intersection of finitely many $M$-ideals is again a $M$-ideal, see Proposition I.1.11 in [4]. However examples from [3] show that even in finite dimensional spaces the intersection of ranges of two projections of norm one, need not be the range of a projection of norm one.

We next state a relevant portion of Theorem 1.1 in [1] that has motivated this work. This result in turn extends some earlier work of M. Baronti and P. L. Papani, see, [2],[3].

**Theorem 1.** Let $K$ be a compact Hausdorff space. Let $Y \subset C(K)$ be a subspace of codimension $n$. Then $Y$ is the range of a projection of norm one if and only if there exists $n$ distinct isolated points $t_1, ..., t_n$ in $K$ and a basis $\mu_1, ..., \mu_n$ in $Y^\perp$ so that for $1 \leq i \leq n$,

$$
\|\mu_i\| \leq 2|\mu_i(t_i)| \text{ and } Y = \cap_{1 \leq i \leq n} ker(\mu_i).
$$

Implicit in Theorem 1 is that $ker(\mu_i)$ is the range of a projection of norm one as also any finite intersection of the kernels. Motivated by this one can ask the question, in which Banach spaces $X$, ideals of finite codimension can be expressed as a finite intersection of ideals of codimension one. The space $C(K)$ is a well known example of a $L^1$-predual space. See [8] and [6] for more information on these spaces.

We first show that in $C(K)$ any ideal of finite codimension is the intersection of ideals of codimension one. We also show that for any Banach space $X$, if $X^{**}$ has this property then so does $X$. Since for any $L^1$-predual space $X$, $X^{**}$ is isometric to a $C(K)$ space, this allows us to conclude that any $L^1$-predual space has this property.
2. Main Results

We start with an extension of Theorem 1.

**Proposition 2.** Let $K$ be a compact Hausdorff space and let $Y \subset C(K)$ be an ideal of codimension $n$. There exists $\mu_1, \ldots, \mu_n \in Y^\perp$ such that $\ker(\mu_i)$ is an ideal for all $i$ and $Y = \cap_i \ker(\mu_i)$.

**Proof.** Since $Y$ is an ideal, it follows from a result of Lima, [7] (see also page 597 in [9]) that $Y^{\perp\perp}$ is the range of a norm one projection in $C(K)^{**}$. Note that by the Gelfand-Naimark theorem, $C(K)^{**}$ is isometric to $C(K')$ for some compact set $K'$. Also $Y^{\perp\perp}$ is a weak$^*$-closed subspace of codimension $n$. Therefore by Theorem 1, there exists $\mu_1, \ldots, \mu_n \in C(K)^*$ such that $\ker(\mu_i)$ in $C(K')$ is the range of a projection of norm one and $Y^{\perp\perp} = \cap_i \ker(\mu_i)$. Now consider $\ker(\mu_i)$ in $C(K)$. As $\ker(\mu_i)^{\perp\perp} = \ker(\mu_i)$, we get by the result of Lima again that $\ker(\mu_i)$ is an ideal in $C(K)$ for all $i$. It is easy to see that $Y = \cap_i \ker(\mu_i)$. \hfill $\square$

**Theorem 3.** Let $X$ be a Banach space such that in $X^{**}$, any ideal of finite codimension is an intersection of ideals of codimension one. Then any ideal of finite codimension in $X$ is the intersection of ideals of codimension one. If $X$ is a $L^1$-predual space, any ideal of finite codimension is a finite intersection of ideals of codimension one.

**Proof.** Suppose in $X^{**}$ every ideal of finite codimension is an intersection of ideals of codimension one. Let $Y \subset X$ be an ideal of finite codimension. By the result of Lima ([7]), $Y^{\perp\perp}$ is the range of a projection of norm one and hence is a weak$^*$-closed ideal of finite codimension. Thus by hypothesis, there is a family $\{x_j^*\} \subset X^*$ such that $\ker(x_j^*)$ is an ideal and $Y^{\perp\perp} = \cap_j \ker(x_j^*)$. Since $\ker(x_j^*)$ is a weak$^*$-closed subspace, as remarked in the Introduction, it is the range of a projection of norm one. Also if we consider $\ker(x_j^*)$ in $X$, we have that $\ker(x_j^*)^{\perp\perp} = \ker(x_j^*)$ (now considered in $X^{**}$), for all $j$. Therefore by duality it follows that $Y$ is an intersection of ideals of codimension one.
If \( X^* \) is isometric to \( L^1(\mu) \), then it is well known that \( X^{**} \) is isometric to \( C(K) \) for some compact Hausdorff space \( K \). Thus the conclusion follows from the above arguments and the Proposition.

\[ \square \]

The following corollary is now easy to deduce.

**Corollary 4.** Let \( X \) be a \( L^1 \)-predual space. Let \( Y \subset X \) be an ideal of codimension \( n \). Then for \( 1 \leq k < n \), there is an ideal \( Z \) of codimension \( k \) such that \( Y \subset Z \subset X \).

**Remark 5.** We do not know how to interpret the conditions in Theorem 1 of [1] to ensure intersection of ideals is an ideal. When \( K \) has no isolated points, there are no subspaces of finite codimension that are ranges of projections of norm one. Unlike this, ideals of finite codimension (that need not be algebraic ideals) in \( C(K) \) are easy to produce. As noted in [10], page379, for any \( \mu_1, ..., \mu_n \in C(K)^* \) with \( \| \mu_i \| \leq 1 \) for \( 1 \leq i \leq n \) and for any \( x_1, ..., x_n \in K \), \( V = \{ f \in C(K) : f(x_i) = \mu_i(f), \ 1 \leq i \leq n \} = \bigcap_i \ker(\mu_i - \delta(x_i)) \) is a \( L^1 \)-predual space. Thus by Proposition 1 in [9] \( V \) is an ideal of finite codimension.

**Remark 6.** If \( X \) is a \( L^1 \)-predual and \( Y \subset X \) is a \( M \)-ideal of finite codimension, then it can be shown that there exists finitely many extreme points \( x_1^*, ..., x_k^* \) of the dual unit ball of \( X \) such that \( Y = \bigcap_i \ker(x_i^*) \) (see [4] Section II.5). Since for any extreme point \( x^* \) of the dual unit ball, \( \ker(x^*) \) is a \( M \)-ideal, we have that any \( M \)-ideal of finite codimension in a \( L^1 \)-predual space, is an intersections of \( M \)-ideals of codimension one. In general a Banach space may have \( M \)-ideals of finite codimension but no \( M \)-ideal of codimension one. Taking \( X = \ell^2 \bigoplus_{\infty} \ell^2(2) \) (direct sum has the maximum norm), we see that \( \ell^2 \) is a \( M \)-ideal (in fact \( M \)-summand) of codimension 2. Using Theorem I.1.8 of [4], it can be seen that \( X \) has no \( M \)-ideals of codimension one.
Not many examples of Banach spaces are known where intersections of ideals is an ideal. Since being an ideal is transitive, we can see that if in $X$ the intersection of ideals is an ideal, then the same holds for any ideal in $X$.

**Proposition 7.** In $\ell^1$ the intersection of any family of ideals is an ideal.

**Proof.** Let $Y \subset \ell^1$ be an ideal. Since $Y^{**} = Y^{\perp} \subset (\ell^1)^{**}$ is the range of a projection of norm one, we get that $Y^{**}$ and hence $Y$ is an abstract $L$-space (see [6] Section 17). In particular $Y$ is the range of a projection of norm one in its bidual. Therefore $Y$ is the range of a norm one projection. As remarked in [3], the intersection of ranges of projections of norm one in $\ell^1$, is the range of a projection of norm one. Thus the conclusion follows.

□

**References**