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Abelian Surfaces, Kummer Surfaces and the non-Archimedian Hodge-$\mathcal{D}$-conjecture

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Abstract

We construct new elements in the higher Chow group $CH^2(A,1)$ of a principally polarized Abelian surface over a non Archimedean local field, which generalize an element constructed by Collino [Col97]. These elements are constructed using a generalization, due to Birkenhake and Wilhelm [BW03], of a classical construction of Humbert. They can be used to prove the non-Archimedean Hodge-$\mathcal{D}$-conjecture - namely, the surjectivity of the boundary map in the localization sequence - in the case when the Abelian surface has good and ordinary reduction.

1 Introduction

The aim of this paper is to prove the non-Archimedean Hodge-$\mathcal{D}$-conjecture for Abelian surfaces. This conjecture asserts that the boundary map in the localization sequence of higher Chow groups is surjective. If an S-integral version of the Beilinson conjectures were known this would be a consequence of them – but since they seem a little out of reach at the moment it is of interest to prove this weaker statement.

The conjecture is the following –

Conjecture 1.1. *Let X be a projective scheme over a global field K. Let p be a prime in $\mathcal{O}_K$ and we think of X as a variety over K_p, the completion at p. Let $\mathcal{X}$ be a model over the ring of integers $\mathcal{O}_{K_p}$ of K_p and let $\mathcal{X}_p$ be the special fibre. We assume $\mathcal{X}_p$ is smooth - that is - X has good reduction at p.*

For $m,n \geq 1$ let

$$\Sigma_X^{m,n} := \{\mathrm{Ker}\{CH^m(\mathcal{X},n-1) \longrightarrow CH^m(X,n-1)\}.$$

Then the map –

$$CH^m(X,n)\otimes\mathbb{Q} \xrightarrow{\partial} CH^{m-1}(\mathcal{X}_p,n-1)\otimes\mathbb{Q}$$

is surjective and $\Sigma_X^{m,n}$ is finite.

In fact, one can formulate this conjecture more generally for primes p of semi-stable reduction [Sre08] - but in this paper we will only deal with primes of good reduction. We can also consider this conjecture for X over a global field itself - looking at the boundary map for *all* primes - but that is a much harder question.

In the local case surjectivity of ∂ suffices to show Σ_X is finite, but in the global case it only shows that Σ_X is torsion.

Several cases of this conjecture are known –

- When $m=n=1$ and $\mathcal{X} = \mathrm{Spec}(\mathcal{O})$ is the ring of integers of a global or local field, this follows from the finiteness of class number in the global case, and is trivially true in the local case.
- Mildenhall [Mil92] and Flach [Fla92] independently showed that this map is surjective in the case when $\mathcal{X}$ is a self-product of elliptic curves over $\mathbb{Q}$ and $m=2, n=1$ and p is any prime of good reduction. However, Mildenhall showed that Σ_X is finite only when X is a self product of CM elliptic curves. Finiteness of Σ_X in general is still not known.
- Spiess [Spi99] showed that this map is surjective in the case when $\mathcal{X}$ is a product of two elliptic curves over the ring of integers of a local field of characteristic p where p is a prime of good reduction.

- In [Sre08] we showed that this is true for $\mathcal{X}$ is a semi-stable model of a product of (non-isogenous) elliptic curves over a local field of characteristic p where p a prime of semi-stable reduction.
- When $n > 1$ and p a prime of good reduction this would be a consequence of a conjecture of Parshin and Soule which asserts that the higher Chow groups of a smooth projective variety over a finite field is torsion. In particular, in the cases when their conjecture is known, this conjecture follows.
- One can also formulate a function field variant of this conjecture - in fact this is used to formulate the Beilinson conjectures in that setup. A special case is discussed in [Sre10].

The 'Archimedean' Hodge-$\mathcal{D}$-conjecture of Beilinson asserts that the regulator map to Deligne cohomology is surjective [Jan88].

$$CH^m(X,n)\otimes\mathbb{R}\xrightarrow{r_{\mathcal{D}}} H^{2m-n}_{\mathcal{D}}(X,\mathbb{R}(m)).$$

This is false in general but was proved for $K3$ surfaces and Abelian surfaces by Chen and Lewis [CL05]. It is still expected to be true if X is defined over a number field. In fact Asakura and Saito [AS07] show that a non-Archimedean version of this is false over a p-adic field as well - for certain generic surfaces. In this case too they expect the conjecture to be true for varieties defined over global fields. This is why in the statement of the conjecture one has to assume X is defined over a global field.

Since the boundary map ∂ is a non-Archimedean version of the Beilinson regulator map sometimes we refer to it as the non-Archimedean regulator map. Our conjecture – which amounts to surjectivity of the boundary map – is sometimes referred to as the *non-Archimedean Hodge-$\mathcal{D}$-conjecture.*

We prove the following theorem –

Theorem 1.2. *Let A be simple, principally polarized, Abelian surface over a local field $\mathbb{K}_p$. Let $\mathcal{A}$ be the Néron model of A over $\mathcal{O}_{\mathbb{K}_p}$ where p is an odd prime of good ordinary reduction. Let $\mathcal{A}_p$ be the special fibre. Let*

$$\Sigma_A = \mathrm{Ker}\{CH^2(\mathcal{A}) \longrightarrow CH^2(A)\}$$

Then,

- *Σ_A is a finite p-group.*
- *The boundary map*

$$CH^2(A,1)\otimes\mathbb{Q}\xrightarrow{\partial} CH^1(\mathcal{A}_p)\otimes\mathbb{Q}$$

  is surjective.

Our theorem is hence a non-Archimedean version of the theorem of Chen and Lewis [CL05].

The outline of the paper is as follows – from Spiess [Spi99], Section 4, it suffices to prove surjectivity of the boundary map – as that implies finiteness of Σ_A and he shows that it is a p-group as a consequence of the finiteness. In order to prove surjectivity, we first describe the structure of the Néron-Severi group of the Abelian surface. To get a geometric understanding of the extra cycle in the case when the Picard number is two, we have to go to the associated Kummer surface and Kummer plane. This is where we use the assumption on odd characteristic. We use a theorem of Birkenhake and Wilhelm which describes the extra cycle in terms of rational curves on the Kummer plane.

We then use a slight generalization of the work of Bogomolov-Hassett-Tschinkel [BHT10] to deform a sum of curves in the special fibre of the associated Kummer $K3$ surface to the generic fibre. This is where we have to assume that the reduction is ordinary.

We then use this deformed curve to construct an indecomposable higher Chow cycle. This cycle suffices to prove surjectivity of the boundary map. Finally we discuss some applications and some related questions.

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2 Notation

- A – a principally polarized Abelian surface.
- $\mathcal{L}_0$ – the line bundle representing the principal polarization.
- K_A – the Kummer surface of A.
- $\tilde{K}_A$ – the associated $K3$ surface.
- $K\mathbb{P}_A$ - the associated Kummer Plane.
- ϕ – the map from $A \longrightarrow K_A$ induced by $\mathcal{L}_0^2$.
- $\mathcal{C}_0$ – the line bundle corresponding to $\phi(D)$ for $D \in |\mathcal{L}_0|$
- $\mathcal{F}_0$ – the hyperplane section on K_A. $\mathcal{F}_0 = \mathcal{C}_0^2$.
- π – the map from $K_A \longrightarrow K\mathbb{P}_A$.
- ρ – the blow-up which gives the minimal resolution of singularities – $\tilde{K}_A \cdots \to K_A$
- Q' – the cycle on the Kummer plane corresponding to the extra endomorphism of A.
- Q_1 and Q_2 – the two components of $\pi^{-1}(Q')$ in K_A.
- $\tilde{Q}_i$ – the strict transform of Q_i.
- Q – the deformation of $\tilde{Q}_1 + \tilde{Q}_2$.
- D_i – the curve $\phi^{-1}(Q_i)$ in A representing the extra endomorphism.

3 Abelian surfaces

3.1 The Hodge-$\mathcal{D}$-conjecture for Abelian surfaces

Let A be an Abelian surface over a p-adic field $\mathbb{K}_p$ with finite residue field. In this paper we will always assume that A is principally polarized by a line bundle $\mathcal{L}_0$ - and write $(A, \mathcal{L}_0)$ when we wish to stress that fact. Let $\mathcal{A}$ be a model over the ring of integers $\mathcal{O}_{\mathbb{K}_p}$ with special fibre $\mathcal{A}_p$. We assume A has good ordinary reduction at p, so the special fibre $\mathcal{A}_p$ is smooth.

One has a map

$$\ldots \longrightarrow CH^2(A,1) \otimes \mathbb{Q} \xrightarrow{\partial} CH^1(\mathcal{A}_p) \otimes \mathbb{Q} \longrightarrow \ldots$$

coming form the localization sequence for higher Chow groups. The conjecture above asserts that the map ∂ is surjective.

In order to prove this conjecture one has to first understand the right hand side - namely the Chow group of the special fibre, and then construct the higher Chow cycles that bound the cycles in the special fibre. In the next section we describe the Chow group of the special fibre.

3.2 The Néron-Severi group of an Abelian surface

We want to understand the group $CH^1(\mathcal{A}_p) \otimes \mathbb{Q}$. As $CH^1_{hom}(\mathcal{A}_p) \otimes \mathbb{Q} = 0$, this is the same as the rational Néron-Severi group

$$CH^1(\mathcal{A}_p) \otimes \mathbb{Q} \simeq NS(\mathcal{A}_p) \otimes \mathbb{Q}.$$

It is well known that the Néron-Severi group can be identified as the part of the endomorphism algebra, $\mathrm{End}_{\mathbb{Q}}(\mathcal{A}_p)$, fixed by the Rosati involution $\dagger$

$$NS(\mathcal{A}_p) \otimes \mathbb{Q} \simeq \mathrm{End}_{\mathbb{Q}}(\mathcal{A}_p)^{\dagger}.$$

From Tate's theorem on the description of the endomorphism algebra [Tat66] one knows that the algebra contains a CM field of degree 4. In particular, the endomorphism algebra contains a real quadratic field. On this field, the Rosati involution acts trivially so the rank of the Néron-Severi group is at least two.

We would like to get an explicit understanding of the generators of this Néron-Severi group. To a certain extent it suffices to understand the case when the rank is 2 as the higher rank cases - namely rank 3 and 4 - can be thought of as instances when one has real multiplication by 2 or 3 different real quadratic fields. The rank 5 case does not exist and rank 6 corresponds to supersingular reduction. So we will only deal with the case when the rank is 2. In this case the Néron-Severi can be identified with elements of the real quadratic field. Let $\mathbb{Q}(\sqrt{\Delta})$ be this real quadratic field.

The principal polarization $\mathcal{L}_0$ of A is represented by a genus 2 curve C. In fact $A = J(C)$, the Jacobian of C. This curve reduces to give one of the generators of $NS(\mathcal{A}_p)$. To get an understanding of the second generator, one has to do a little more work. For this, we have to look at Kummer surface associated to A.

3.3 The Kummer surface and the Kummer plane

All the statements in this section are classical and can be found in [BW03], for example.

3.3.1 The Kummer surface

Let A be an Abelian surface. The *Kummer surface* of A is defined to be the hypersurface in $\mathbb{P}^3$

$$K_A = \phi_{\mathcal{L}_0^2}(A)$$

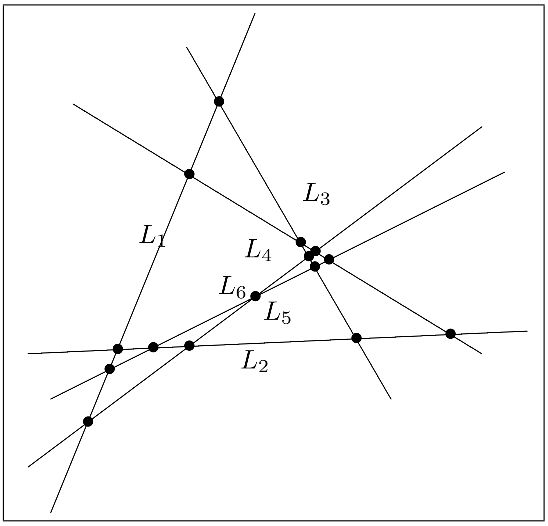
where $\phi = \phi_{\mathcal{L}_0^2}$ is the map

$$\phi_{\mathcal{L}_0^2} : A \longrightarrow \mathbb{P}^3$$

induced by the square of the principal polarization. Equivalently, this can be identified with $A/\{\pm 1\}$ – so the map $A \xrightarrow{\phi} K_A$ is a double cover ramified at the sixteen 2-torsion points of A. It is well known, see [BW03], for example, that the blow up of K_A at these 16 points is a $K3$ surface $\rho : \tilde{K}_A \longrightarrow K_A$.

3.3.2 The Kummer Plane

Let $\pi : \mathbb{P}^3 \backslash \{0\} \longrightarrow \mathbb{P}^2$ be the projection with centre 0, where $0 = \phi(0)$. The map π restricted to K_A is a double cover of $\mathbb{P}^2$ ramified at six lines $L_1, \dots, L_6$. These six lines are tangent to a conic Q. The six lines meet at 15 points $\{q_{ij}\}$ where $q_{ij} = (L_i \cap L_j)$. These points are the images of the non-zero 2-torsion points under the map $\pi \circ \phi$. The collection $K\mathbb{P}_A = (\mathbb{P}^2, L_1, \dots, L_6)$ is called the *associated Kummer plane* of the Abelian surface A.



The six lines and fifteen points on $\mathbb{P}^2$

The situation is summarized in the diagram below –

$$\begin{tikzcd}
 & \tilde{K}_A \arrow[d, dashed, "\rho"] & \\
A \arrow[r, "\phi", "{[2:1]}"'] & K_A \arrow[r, "\pi", "{[2:1]}"'] & K\mathbb{P}_A
\end{tikzcd}$$

3.3.3 Humbert's Theorem and its generalizations

A classical theorem of Humbert [BW03] states that an Abelian surface $A = J(C)$ has real multiplication by $\mathbb{Z}(\frac{1+\sqrt{5}}{2})$ if and only if there is a conic Q' on the Kummer plane which passes through 5 of the 15 points $\{q_{ij}\}$ *and* is tangent to one of the other lines. Further, there is a curve D on A such that

$$Q' = \pi \circ \phi(D)$$

and D and C generate the part of the rational Néron-Severi group coming from $\mathbb{Z}(\frac{1+\sqrt{5}}{2})$.

Birkenhake and Wilhelm [BW03] generalize this theorem and provide a geometric characterization of the cycles in the Néron-Severi group of all Abelian surfaces with real mutiplication. In order to describe their theorem we need some definitions.

The *Humbert invariant* $\Delta(\mathcal{L})$ of a line bundle $\mathcal{L}$ is defined to be

$$\Delta(\mathcal{L}) = (\mathcal{L}.\mathcal{L}_0)^2 - 2\mathcal{L}^2.$$

where (.) is the intersection pairing on $\mathrm{Pic}(A)$. This is the negative of the intersection pairing on the primitive part of the Néron-Severi group. Hence there is a line bundle of non-zero Humbert invariant if and only if the Picard number is > 1. In fact one has –

Theorem 3.1 (Humbert)**.** *An Abelian surface has real multiplication by and order of discriminant Δ in $\mathbb{Q}(\sqrt{\Delta})$ if and only if there is a line bundle $\mathcal{L}_\Delta$ of Humbert invariant Δ in* $\mathrm{Pic}(A)$.

An immediate consequence of the theorem of Birkenhake and Wilhelm is the following –

Theorem 3.2 (Birkenhake-Wilhelm)**.** *Let $(A, \mathcal{L}_0)$ be a principally polarized Abelian surface with a line bundle $\mathcal{L}_\Delta$ of invariant Δ. Then there exists a* **rational curve** *Q' on the Kummer plane $K\mathbb{P}_A$ which passes through some of the points q_{ij} and is tangent to some of the lines L_j. Further, $Q' = \pi \circ \phi(D)$ where D is a curve on A which lies in the linear system of divisors of a line bundle $\mathcal{L}$ of the form $\mathcal{L}_0^a \otimes \mathcal{L}_\Delta^b$ with $b \neq 0$.*

In particular, if the Picard number is 2, the curves C and D generate the rational Néron-Severi group. Their theorem is much more precise - it gives a and b precisely and the degree of the curve and the number of points that it passes through – in terms of the numbers determining Δ. We will use the more precise version later on, but at the moment, for our purposes, it suffices to note that there is a *rational curve* on $K\mathbb{P}_A$ which represents the extra cycle. We now lift this cycle to the Kummer surface. We have the following lemma, which is proved by Jakob [Jak94] in a special case –

Lemma 3.3. *$\pi^{-1}(Q') = Q_1 \cup Q_2$, where Q_1 and Q_2 are rational curves on K_A.*

Proof. Birkenhake-Wilhelm [BW03] show that there is a curve Q_1 such that the map $\pi : Q_1 \longrightarrow Q'$ is birational. Hence the curve Q_1 is rational. Since the map π is a double cover, $\pi^{-1}(Q) = Q_1 \cup Q_2$ where Q_2 is another rational curve.

□

Since π is a double cover, it induces an involution ι on K_A. Under this involution one has

$$\iota(Q_1) = Q_2.$$

Since π is ramified over the lines L_i, $\pi^{-1}(L_i)$ is fixed under ι and we will abuse notation to denote the line $\pi^{-1}(L_i)$ in K_A by L_i as well. This involution also acts on the Néron-Severi group of the Abelian surface. We know that $\mathbb{Q}(\sqrt{\Delta})$ can be identified with a subgroup of $NS(A) \otimes \mathbb{Q}$ and with respect to this identification, ι can be though of as the non-trivial Galois conjugation.

The work of Birkenhake and Wilhelm is in the complex situation but their work is purely algebraic and carries through, *mutatis mutandis*, to the case of Abelian surfaces over finite fields as long as the characteristic is not 2.

3.3.4 The principal part of the Néron-Severi group of $(X, \mathcal{F})$

We define the *principal part* of the Néron-Severi group of a pair $(X, \mathcal{F})$, where X is a surface and $\mathcal{F}$ a line bundle on X, to be the subgroup of the Néron-Severi which is the saturation of the subgroup generated by the class of the line bundle – that is

$$\mathbb{Q}[\mathcal{F}] \cap NS(X) = \{x \in NS(X) | \lambda x = \mu[\mathcal{F}]\}$$

for some $\lambda, \mu \in \mathbb{Z}$.

The principal part of the Néron-Severi group of $(A, \mathcal{L}_0)$ is generated by the class of the principal polarization, $\mathcal{L}_0$. However, this is no longer the case for $(K_A, \mathcal{F}_0)$. The map to $\mathbb{P}^3$ which defines K_A comes from $\mathcal{L}_0^2$. Hence $[\mathcal{F}_0]$ - the class of the hyperplane section - does not generate the principal part of the Néron-Severi of K_A – it is generated by the class $[\mathcal{C}_0] = \phi([\mathcal{L}_0])$.

The Néron-Severi group of $\mathbb{P}^2 \simeq \mathbb{Z}$. So the pull-back of a cycle in $K\mathbb{P}_A$ lies in the principal part of $(K_A, \mathcal{F}_0)$. The involution ι acts trivially on the classes of such cycles and the class of a cycle in K_A lies in the principal part if and only if its class is fixed by ι.

4 Rational curves on $K3$ surfaces

Our next step is to deform the rational curves above to the generic fibre. For this we use a modification of an argument of Bogomolov-Hassett-Tschinkel to deform a sum of rational curves on a special fibre to a nodal rational curve on the generic fibre.

The existence of rational curves on $K3$ surfaces is something that has been studied for a long time. A conjecture, attributed to Mumford, states that there are infinitely many rational curves on an arbitrary $K3$

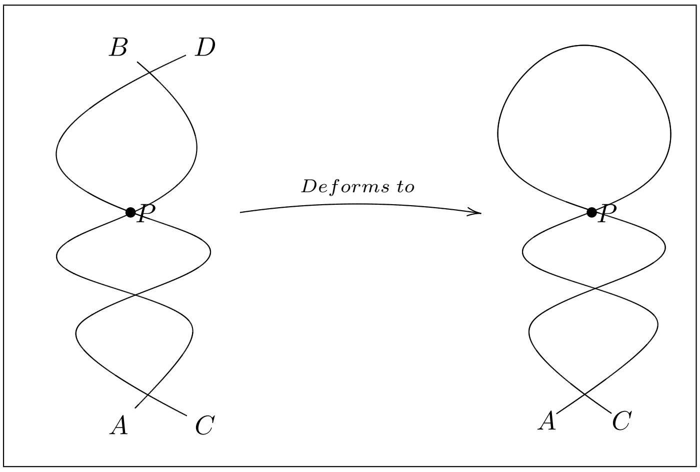
surface. The existence of even a single rational curve on a $K3$ surfaces is not always trivial. In a recent preprint, Bogomolov, Hassett and Tschinkel [BHT10] proved a mixed characteristic generalization of a result of Mori and Mukai [MM83] which can be used to construct infinitely many rational curves on a $K3$ of Picard number 1. Using a result of Ogus [Ogu79], Proposition 1.6, one can generalize their argument and one has the following theorem.

Theorem 4.1 (Bogomolov-Hassett-Tschinkel [BHT10], Theorem 18 + Ogus [Ogu79], Prop. 1.6)**.** *Let (S_0, f_0) be an ordinary $K3$ surface over a finite field k of characteristic p, where f_0 is a line bundle on S. Suppose*

$$C = C_1 + \cdots + C_r$$

is a connected union of rational curves $C_i \subset S_0$ such that $[C] = f_0$. Assume that the C_i are distinct. Let (S, f) be a $K3$ surface over the Witt vectors $W(\bar{k})$ reducing to (S_0, f_0). Then there is a (nodal) rational curve $R \subset S$ defined over a finite extension of $W(k)$ such that R reduces to C and all irreducible components of R are rational.

Bogomolov, Hassett and Tschinkel assume further that f_0 is ample. They use this to say that Σ_f, the formal versal deformation space corresponding to $K3$ surfaces with a line bundle f, is an irreducible divisor in the versal deformation space of S_0. However, the proposition of Ogus shows that that statement holds for more general f_0 – in fact for all f_0 which lift. The rest of the argument of [BHT10] then carries through in this case as well to prove the more general statement.



A schematic representation of the union of rational curves deforming to an irreducible nodal curve in the generic fibre.

5 Elements of the Higher Chow Group

Let X be a surface over a global or local field K. The group $CH^2(X,1)$ has the following presentation [Ram89]. It is generated by formal sums of the type

$$\sum_i (C_i, f_i)$$

where C_i are curves on X and f_i are $\bar{K}$-valued functions on the C_i satisfying the cocycle condition

$$\sum_i \operatorname{div} f_i = 0.$$

Relations in this group are give by the tame symbol of pairs of functions on X.

There are some *decomposable* elements of this group coming from the product structure

$$\bigoplus_{L/K} CH^1(X_L) \otimes CH^1(X_L, 1) \longrightarrow \bigoplus_{L/K} CH^2(X_L, 1) \overset{\oplus N_{L/K}}{\longrightarrow} CH^2(X, 1)$$

where L runs through all finite extensions of K and $N_{L/K}$ is the norm map. A theorem of Bloch [Blo86] says that $CH^1(X_L, 1)$ is simply L^* where L is the field of definition of X_L – so such an element looks like a sum of elements of the type (C, a), where C is a curve on X_L and a is in L^*. The group of *indecomposable* elements of $CH^2(X, 1)$ is the quotient of $CH^2(X, 1)$ by the subgroup of decomposable elements.

The group $CH^2(X, 1) \otimes \mathbb{Q}$ is the same as the $\mathcal{K}$-cohomology group $H^1_{Zar}(X, \mathcal{K}_2) \otimes \mathbb{Q}$ and the motivic cohomology group $H^3_{\mathcal{M}}(X, \mathbb{Q}(2))$.

5.1 The boundary map

Let X be as above and $\mathcal{X}$ a model of X over the ring of integers with special fibre $\mathcal{X}_p$ at a prime p. We assume $\mathcal{X}_p$ is smooth - that is, X has good reduction at p. The boundary map

$$\partial : CH^2(X, 1) \longrightarrow CH^1(\mathcal{X}_p)$$

is defined as follows

$$\partial \left(\sum_i (C_i, f_i) \right) = \sum_i \mathrm{div}_{\bar{C}_i}(f_i)$$

where $\bar{C}_i$ denotes the closure of C_i in the semi-stable model $\mathcal{X}$ of X. From the cocycle condition, the 'horizontal divisors' namely, the closure of $\sum_i \mathrm{div}_{C_i}(f_i)$, cancel out – so the boundary is supported on the special fibre.

For a decomposable element of the form (D, a) the boundary map is particularly simple to compute

$$\partial((D, a)) = \mathrm{ord}_p(a)\mathcal{D}_p.$$

where $\mathcal{D}_p$ is the special fibre of the closure of D in $\mathcal{X}$. In particular, a cycle in the special fibre which is not the restriction of a cycle in the generic fibre cannot appear in the boundary of a decomposable element.

5.2 A new element in the higher Chow group of an Abelian Surface

Let $(A, \mathcal{L}_0)$ be an Abelian surface over $\mathbb{K}_p$ as before. We assume

$$NS(A) \otimes \mathbb{Q} \simeq \mathbb{Q}$$

and that p is a prime of good, ordinary reduction. Further, we assume that

$$NS(\mathcal{A}_p) \otimes \mathbb{Q} \simeq \mathbb{Q}(\sqrt{\Delta})$$

for some $\Delta > 0$. In this section we will construct an an element Ξ_A of $CH^2(A, 1)$ such that the image $\partial(\Xi_A)$ is the extra cycle in the Néron-Severi group of $\mathcal{A}_p$. In particular, it is an *indecomposable* element in the higher Chow group.

5.2.1 Lifting to the generic fibre.

Let K_A be the Kummer Surface of A. Let $\mathcal{F}_0$ denote the hyperplane section on K_A – so $\phi^*(\mathcal{F}_0) = \mathcal{L}_0^2$. Let $(\tilde{K}_A, \tilde{\mathcal{F}}_0)$ denote the $K3$ surface obtained by blowing up the sixteen nodal points on $(K_A, \mathcal{F}_0)$ and

$$\rho : \tilde{K}_A \longrightarrow K_A$$

denote the birational map from $\tilde{K}_A$ to K_A. Let $\rho^*(\mathcal{F})$ denote the total transform of a line bundle $\mathcal{F}$ on K_A and $\tilde{\mathcal{F}}_0$ denote the strict transform of $\mathcal{F}_0$. Let $\tilde{\mathcal{C}}_0$ denote the strict transform of $\mathcal{C}_0$.

Let Q' be the rational curve on $K\mathbb{P}_{\mathcal{A}_p}$ corresponding to the extra cycle in the Néron-Severi of $\mathcal{A}_p$ determined by [BW03]. Let Q_1, Q_2 and $\tilde{Q}_1, \tilde{Q}_2$ the rational curves lying above this curve in $K_{\mathcal{A}_p}$ and $\tilde{K}_{\mathcal{A}_p}$ respectively. $\tilde{Q}_i$ is the strict transform of Q_i under the birational map ρ. Let D_1 and D_2 denote the pullbacks of Q_1 and Q_2 to $\mathcal{A}_p$.

Lemma 5.1. *The cycle $\tilde{Q}_1 + \tilde{Q}_2 \in |\tilde{\mathcal{C}}_0^N|$ for some $N \in \mathbb{Z}$.*

Proof. Since $\mathrm{Pic}(\mathbb{P}^2) \simeq \mathbb{Z}$, $\pi^*(Q)$ lies in the principal part - which is $\mathbb{Z}\mathcal{C}_0$. Hence the strict transform $\widetilde{\pi^*(Q)}$ lies in $\mathbb{Z}\tilde{\mathcal{C}_0}$. However, $\pi^{-1}(Q)) = Q_1 \cup Q_2$. Hence its strict transform is $\tilde{Q}_1 \cup \tilde{Q}_2$ and one has

$$\tilde{Q}_1 + \tilde{Q}_2 \in |\tilde{\mathcal{C}_0^N}|$$

for some $N \in \mathbb{Z}$.

□

To compute N we have to use the more precise version of the theorem of Birkenhake and Wilhelm [BW03]. To prove their theorem they have to distinguish several different cases of Δ –

$$\Delta = \begin{cases} I. & 8d^2 + 9 - 2k \\ II. & 8d(d+1) + 9 - 2k \\ III. & 8d^2 + 8 - 2k \\ IV. & 8d(d+1) + 12 - 2k \end{cases} \tag{1}$$

where $d \geq 1$ and $k \in \{4, 6, 8, 10, 12\}$. One has the following result.

Lemma 5.2. *Let N and d be as above. Then $N = 4d + 1$*

Proof. This follows from Theorems 7.1 – 7.4 of [BW03] – which correspond to the four cases of Δ. The proof is identical in all cases so we will do it in case I - where $\Delta = 8d^2 + 9 - 2k$. From Theorem 7.1 of [BW03] one has that the pull-back D_1 of Q_1 lies in $\mathcal{L}_0^{2d} \otimes \mathcal{L}_\Delta$. The pull-back D_2 of Q_2 lies in the conjugate line bundle, $\mathcal{L}_0^{2d+1} \otimes \mathcal{L}_\Delta^{-1}$. Therefore their sum lies in $\mathcal{L}_0^{4d+1}$.

To compare this with N we use Lemma 3.1 of [BW03]. One has, since $\deg(\mathcal{F}_0) = 4$,

$$\begin{aligned} \deg(Q_1 + Q_2) &= (\mathcal{F}_0^{N/2}.\mathcal{F}_0) = N/2(\mathcal{F}_0.\mathcal{F}_0) = 2N \\ &= (\mathcal{L}_0^{4d+1}.\mathcal{L}_0) = 8d + 2 \end{aligned}$$

so $N = 4d + 1$.

□

We would like to apply the theorem of Bogomolov-Hassett-Tschinkel+Ogus to this situation. For that we need the curve $\tilde{Q}_1 + \tilde{Q}_2$ to be connected. To ensure this is the case, we analyze the points of intersection of Q_1 and Q_2 under the blow-up . The curves Q_1 and Q_2 intersect at several points on $K_{\mathcal{A}_p}$ – in fact, one has

Lemma 5.3. *$x \in Q_1 \cap Q_2$ if and only if $x \in L_i \cap Q_1$ for some i. Further –*

- *x lies in the image of the two torsion if and only if $x \in L_i \cap L_j \cap Q_1 (\cap Q_2)$ for some i and j.*
- *The other points of $L_i \cap Q_j$ which are not the images of two torsion points appear with even multiplicity.*

Proof. If x lies on $L_i \cap Q_1$, then, as ι fixes L_i, $\iota(x) = x$. Hence x lies on $\iota(Q_1) = Q_2$. Conversely, if $x \in Q_1 \cap Q_2$, then so is $\iota(x)$. So both x and $\iota(x)$ lie on Q_1. This contradicts the fact that the map $\pi : Q_1 \to Q'$ is of degree 1.

For the second and third statements we refer to [BW03], Lemma 6.1.

□

From [BW03] we know there are k or $k - 1$ points on Q' which are the images of the 2-torsion points of $\mathcal{A}_p$. These points lie on $L_i \cap L_j \cap Q'$, hence their pre-image under π lie on $Q_1 \cap Q_2$. Under the blow-up to $\tilde{K}_{\mathcal{A}_p}$ they need not be points of intersection of $\tilde{Q}_1 \cap \tilde{Q}_2$ any longer. The remaining points on $L_i \cap Q'$ are points of even multiplicity. Hence we need to show that either there is a point of intersection which does not come from the two torsion points or a two torsion point continues to lie on the intersection after the blow-up.

Lemma 5.4. *$\tilde{Q}_1 \cup \tilde{Q}_2$ is connected.*

Proof. From [BW03], we know that the degree of Q' is either $2d$, in cases I and III, or $2d+1$, in cases II and IV of (1). Since there are six lines L_i,

$$(\cup_{i=1}^{6} L_i).Q' = \begin{cases} 12d \text{ in cases I and III} \\ 12d+6 \text{ in cases II and IV.} \end{cases}$$

The number of points coming from 2 torsion is $(k-1)$ in cases I and III and k in the other two cases. Each such point lies on two lines hence should be counted twice. There are two cases –
Case 1 – If the multiplicity at the two torsion points is 1 - so the curves $\tilde{Q}_1$ and $\tilde{Q}_2$ do not meet at the those points. In this case the number is is $12d-2(k-1)$, $12d+6-2k$, $12d-2k$ or $12d+6-2(k-1)$ depending on the case of Δ. Each of these points appears with even multiplicity so in fact the number of actual points is at least half of this. This would be less than 1 only for small d and large k. However, is this case one can repeat the argument with $m^2\Delta$ in the place of Δ since an abelian surface which contains a line bundle of invariant Δ also contains line bundles of invariant $m^2\Delta$ for any $m \in \mathbb{Z}$. Hence there is a point of intersection P which does not lie in the image of the two-torsion and hence continues to lie on $\tilde{Q}_1 \cap \tilde{Q}_2$. Hence $\tilde{Q}_1 \cup \tilde{Q}_2$ is connected.
Case 2 – If the multiplicity of a two-torsion point is greater than 1. In this case under the blow up to $\tilde{K}_{\mathcal{A}_p}$ the point still lies on the intersection of $\tilde{Q}_1$ and $\tilde{Q}_2$ hence the $\tilde{Q}_1 \cup \tilde{Q}_2$ is connected.

□

Hence $\tilde{Q}_1 \cup \tilde{Q}_2$ satisfy the hypothesis of the theorem of Bogomolov-Hassett-Tschinkel + Ogus (Theorem 4.1). Therefore there exists a nodal rational curve $\tilde{Q}$ on K_A such that it splits into a sum of these two rational curves mod p

$$\tilde{Q}_p = \tilde{Q}_1 \cup \tilde{Q}_2.$$

This curve $\tilde{Q}$ is defined over some finite extension $\mathbb{M}_p$ of $\mathbb{K}_p$.

5.2.2 The higher Chow cycle

We would like to use the curve $\tilde{Q}$ to construct a higher Chow cycle. A standard construction of a higher Chow cycle can be made using a nodal rational curve - one uses a function on the normalization which has support on the infinitely near points of the node. The co-cycle condition is satisfied as all the points map to the same point and hence the divisor cancels out. However, in our case, it is not guaranteed that $\tilde{Q}$ has an ordinary node – but we show that if that is not the case, some blow-up of $\tilde{Q}$ has a node. For this we have to understand the singularities coming from the deformation of the points of intersection of $\tilde{Q}_1$ and $\tilde{Q}_2$.

Let P be such a point of intersection of $\tilde{Q}_1 \cap \tilde{Q}_2$. . This is a singularity of the curve $\tilde{Q}_1 \cup \tilde{Q}_2$. Locally, the singularity looks like $y^2 = x^{2m}$ for some m. We would like to see what such a point can deform to in $\tilde{Q}$ under the theorem of [BHT10]+[Ogu79].

Using [Har10], [Theorem 14.1] and Elkik [Elk74], [Theoreme 8], one can see that the versal deformation space of the singularity is defined by

$$y^2 = \prod_i (x-a_i)^{n_i} = F(x)$$

where $\sum_i n_i a_i = 0$ and $\sum_i n_i = 2m$ – so $F(x)$ is a monic polynomial of degree $2m$ with no degree $2m-1$ term. This has singularities at a_i if $n_i > 1$.

The arithmetic genus of $\tilde{Q}$ is the same as that of $\tilde{Q}_1 \cup \tilde{Q}_2$. Hence the contribution to the arithmetic genus from a singular point P on $\tilde{Q}_1 \cup \tilde{Q}_2$ and the points it can deform to in $\tilde{Q}$ has to be the same. This puts restrictions on the possible singularities that can arise in $\tilde{Q}$ from deforming P.

Lemma 5.5. *The only singularities of $\tilde{Q}$ that can arise from deforming the singularity at a point P on $\tilde{Q}_1 \cap \tilde{Q}_2$ are higher order nodes – that is, singularities which are locally like the singularity at the origin of $y^2 = x^{2r}$ for some r.*

Proof. The contribution to the arithmetic genus from a singularity P is

$$\delta_P = \sum_Q \frac{m_Q(m_Q-1)}{2}$$

where the sum is over all the infinitely near points Q of P including P (that is, points lying over P in the strict transform of the curve) with multiplicity m_Q. If one is looking at the singularity of multiplicity 2 at the origin of $y^2 = x^r$ then one has

$$\delta_{(0,0)} = \begin{cases} r/2 & \text{if } r \text{ is even} \\ (r-1)/2 & \text{if } r \text{ is odd} \end{cases}$$

as the blow up of $y^2 = x^r$ gives a singularity of type $y^2 = x^{r-2}$ - so shifts it by 2. The contribution from this blow up – and every subsequent blow up – is $2(2-1)/2 = 1$. One repeats the process till the point is non-singular - which happens when the penultimate stage is ordinary node or a cusp. For an ordinary double point $y^2 = x^2$ and for a cusp – $y^2 = x^3$ one has $\delta_{0,0} = 1$ and from this the result follows.

Hence the contribution from the singularity at a_i is $n_i/2$ if n_i is even and $(n_i - 1)/2$ if n_i is odd. The contribution from the singularity at $(0,0)$ of $\tilde{Q}_1 \cup \tilde{Q}_2$ is $2m/2 = m$. So if the singularity deforms to a (higher order) cusp - where n_i is odd, then since $\sum n_i = 2m$, the contribution from singularities cannot add up to m. Hence this singularity cannot deform to a cusp on $\tilde{Q}$.

□

Remark 5.6. *For example, the possibilities for the singular points in the deformation of the curve $y^2 = x^4$ with a singularity at $(0,0)$ are –*

- *a. 4 smooth points – $y^2 = (x-a)(x-b)(x-c)(x-d)$.*
- *b. 1 ordinary node and two smooth points – $y^2 = (x-a)^2(x-b)(x-c)$.*
- *c. 1 cusp and a smooth point – $y^2 = (x-a)^3(x-b)$.*
- *d. 2 ordinary nodes. – $y^2 = (x-a)^2(x-b)^2$.*
- *e. a tacnode - that is, the trivial deformation. $y^2 = x^4$.*

However, the contribution to the arithmetic genus of $\tilde{Q}$ by this singularity is 2 while the sum of the contributions is 0 in case a, 1 in case b and c and 2 in case d. Hence the only possibility are cases d and e – the point P has to deform to a sum of two nodes or remain constant. In case e the blow up of $\tilde{Q}$ at P gives an ordinary node in the fibre over P.

The upshot of all this is that, after blowing-up, if necessary, one of the singularities of $\tilde{Q}$ can be assumed to be an ordinary node. Let P be one of these points. Since it is a ramification point of $\tilde{Q}$ it is defined over some finite extension $\mathbb{L}_p$ of $\mathbb{M}_p$.

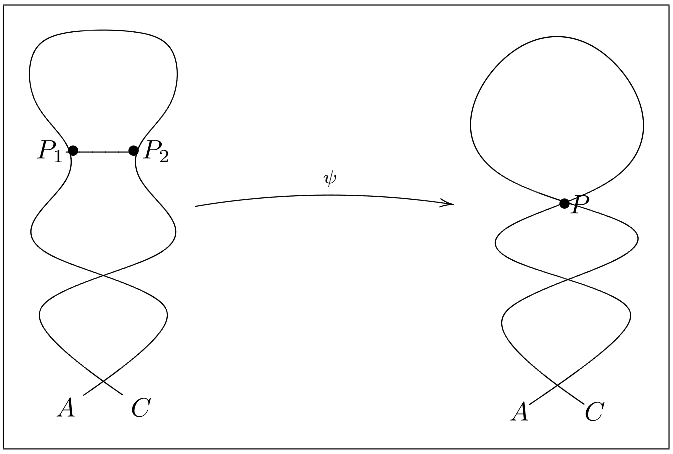
Let $\psi : N_P \longrightarrow \tilde{Q}$ denote the (chain of) blow ups of $\tilde{Q}$ which resolve the singularity at P. The fibre over P is a chain of exceptional curves E_{P^j}, $1 \leq j \leq r$ for some r. There are two points P_1 and P_2 of E_{P^r} in the closure of $\tilde{Q}$ in N_P which lie over P. As N_P is rational, we can find a function f_P on N_P with divisor

$$\mathrm{div}(f_P) = P_1 - P_2.$$

Then $\psi_*(N_P, f_P)$ is an element of $CH^2(\tilde{K}_{A_{\mathbb{L}_p}}, 1)$ as, by construction,

$$\mathrm{div}\, \psi_*(f_P) = \psi(P_1) - \psi(P_2) = P - P = 0$$

on $\tilde{Q}$. Let $\Xi_{\tilde{K}_{A_{\mathbb{L}_p}}}$ be this element.



A schematic representation of the blow up of a nodal curve

One can push forward $\Xi_{\tilde{K}_{A_{\mathbb{L}_p}}}$ under the map ρ to get an element $\Xi_{K_{A_{\mathbb{L}_p}}}$ of $CH^2(K_{A_{\mathbb{L}_p}}, 1)$ and pull it back under ϕ and apply the norm to get an element

$$\Xi_A = N_{\mathbb{L}_p/\mathbb{K}_p}(\phi^*(\Xi_{K_A})) \in CH^2(A, 1).$$

5.2.3 Indecomposability and the boundary.

Let D_1 and D_2 be pull backs of the cycles Q_1 and Q_2. These correspond to conjugate elements of $\mathbb{Q}(\sqrt{\Delta})$. Let D denote the pull back of $\rho_*(\tilde{Q})$. This lies in the linear system of $\mathcal{L}_0^N$ and is defined over $\mathbb{K}_p$.

Theorem 5.7. *The element Ξ_A is an indecomposable element of $CH^2(A, 1)$. Further, the boundary under the map ∂ is the extra cycle in the special fibre $\mathcal{A}_p$.*

$$\partial(\Xi_A) = mD_{1p}$$

for some $m \neq 0$, up to a decomposable element of $CH^2(A, 1)$.

Proof. The cycles are defined over $\mathbb{L}_p$ so we work over that. To show indecomposability it suffices to show the boundary is a non-zero multiple of the cycle D_1 as D_1 does not deform to a cycle on the generic fibre – hence cannot be the boundary of a decomposable element.

We will compute the divisor of f_P on the closure of N_P. The fibre of N_P mod p is $\tilde{Q}_1^P \cup \tilde{Q}_2^P \cup \bigcup_{j=1}^r E_{Pj}$, where $\tilde{Q}_1^P$ and $\tilde{Q}^P$ are the strict transforms of the $\tilde{Q}_i$ under the blow-up ψ and E_{Pj} are the exceptional fibres of the blow ups.

Let $\mathcal{H} = \overline{\operatorname{div}(f_P)}$ be the horizontal divisor. The boundary is of the form

$$\partial((N_P, f_P)) = \operatorname{div}(\bar{f}_P) = \mathcal{H} + a\tilde{Q}_1^P + b\tilde{Q}_2^P + \sum_{j=1}^r c_j E_{Pj}$$

for some rational numbers a, b and c_i. A decomposable element of the form (N_P, p) has boundary

$$\partial((N_P, p)) = \tilde{Q}_1^P + \tilde{Q}_2^P + \sum_{j=1}^r E_{Pj}$$

Hence by adding $-b(N_P, p)$ to (N_P, f_P) we can assume $b = 0$. From the intersection theory of arithmetic surfaces – described in Lang [Lan88], chapter III, for instance – we have that

$$(\operatorname{div}(\bar{f}_P).D) = 0$$

for *all* divisors D supported in the special fibre. Hence, we can compute a, b and the c_j by intersecting with the divisors $\tilde{Q}_1^P$, $\tilde{Q}_2^P$ and E_{Pj} and using what we know about their intersection numbers. We know

- $(E_{P^j}.E_{P^k}) \begin{cases} -1 & |j-k| = 0. \\ 1 & |j-k| = 1. \\ 0 & |j-k| > 1. \end{cases}$
- $(E_{P^r}.\tilde{Q}_i^P) = 1$ for $i \in \{1,2\}$.
- $(E_{P^j}.\tilde{Q}_i^P) = 0$ for $j \neq r$ and $i \in \{1,2\}$.
- $(\mathcal{H}.\tilde{Q}_1^P) = 1$.
- $(\mathcal{H}.\tilde{Q}_2^P) = -1$.
- $(\mathcal{H}.E_{P^j}) = 0$ for all $1 \leq j \leq r$.

So we have

$$0 = (\mathrm{div}(\bar{f}_P).\tilde{Q}_1^P) = (\mathcal{H}.\tilde{Q}_1^P) + a(\tilde{Q}_1^P.\tilde{Q}_1^P) + \sum_{j=1}^{r} c_j(E_{P^j}.\tilde{Q}_1^P) = 1 + a(\tilde{Q}_1^P)^2 + c_r$$

$$0 = (\mathrm{div}(\bar{f}_P).\tilde{Q}_2^P) = (\mathcal{H}.\tilde{Q}_2^P) + a(\tilde{Q}_1^P.\tilde{Q}_2^P) + \sum_{j=1}^{r} c_j(E_{P^j}.\tilde{Q}_2^P) = -1 + a(\tilde{Q}_1^P.\tilde{Q}_2^P) + c_r$$

Equating the two shows

$$a = \frac{2}{((\tilde{Q}_1^P.\tilde{Q}_2^P) - (\tilde{Q}_1^P.\tilde{Q}_1^P))}$$

Since a is a finite number, it is non-zero. The precise value can be computed in terms of the invariants determining Δ.

Applying ρ_* and ϕ^* and $N_{\mathbb{L}_p/\mathbb{Q}_p}$ to the boundary shows that

$$\partial(\Xi_A) = \lambda.N_{\mathbb{L}_p/\mathbb{Q}_p}(\phi^*(\rho_*(\psi_*(\tilde{Q}_1^P)))).$$

However, the right hand side is simply a multiple of the extra cycle D_1.

□

5.3 The Hodge-$\mathcal{D}$-conjecture for Abelian surfaces.

Combining the above result with earlier results we have

Theorem 5.8. *Let A be an abelian surface over a local field $\mathbb{K}_p$. Let $\mathcal{A}$ be a model over the ring of integers and $\mathcal{A}_p$ the special fibre. Assume A has good ordinary reduction at p. Then the map*

$$CH^2(A,1) \otimes \mathbb{Q} \xrightarrow{\partial} CH^1(\mathcal{A}_p) \otimes \mathbb{Q}$$

is surjective.

Proof. We have to do a case by case analysis - depending on the Picard number of A, the Picard number of the special fibre $\mathcal{A}_p$ and the type of reduction. A consequence of the Tate conjecture is that the Picard number of the special fibre is even - so we only have to consider the cases when it is 2,4 or 6. Let $\rho(X)$ denote the Picard number of X.

- Case 0 - If $\rho(\mathcal{A}_p) = 6$ then $\mathcal{A}_p$ is isogenous to a product of isogenous supersingular curves - so the reduction is not ordinary.
- Case 1 - A simple, $\rho(A) = 1$ and $\rho(\mathcal{A}_p) = 2$. This case is the situation described above. Hence we can use the element constructed above.

- Case 2 - A simple, $\rho(A) = 1$ and $\rho(\mathcal{A}_p) = 4$. In this case $\mathrm{End}(\mathcal{A}_p)$ contains several real quadratic fields. If C_1, C_2 and C_3 are the generators of the primitive part of the Néron-Severi then they correspond to different rational curves on $\mathbb{P}^2$ by the work of Birkenhake and Wilhelm. Hence the construction above will give three elements of the higher Chow group of A.
- Case 3 - A is isogenous to a product of elliptic curves. We can use the work of Spiess [Spi99] or Collino [Col97] to construct the required elements.
- Case 4 - A simple, $\rho(A) = 2$ and $\rho(\mathcal{A}_p) = 2$. In this case, since all the cycles in the special fibre lift to the generic fibre, one can use decomposable elements of the higher Chow group to bound them.
- Case 5 - A simple. $\rho(A) = 2$ and $\rho(A_p) = 4$. We can use the argument as in Case 2 to obtain the required elements.

□

Remark 5.9. *1. We needed to use ordinarity in order to use the theorem of Bogomolov-Hassett-Tschinkel + Ogus. Perhaps that can be generalized to work in all the cases where the reduction is not super-singular. In the case of super-singular reduction, the special fibre is unirational and that allows the possibility of the cycle being deformed within the special fibre.*

2. This theorem can likely be extended to work in the case of semi-stable reduction. A special case was studied in [Sre08].

3. Curiously, the case when $\mathrm{End}(A_p) = \mathbb{Z}(\frac{1+\sqrt{5}}{2})$ *is not answered by our argument.*

5.4 Some applications

5.4.1 Torsion in co-dimension two

One immediate consequence of the construction of these higher Chow cycles is the following. Let Σ_X be the group

$$\Sigma_X := \mathrm{Ker}(CH^2(\mathcal{X}) \longrightarrow CH^2(X))$$

where $\mathcal{X}$ is a semi-stable model of X. Then for X an Abelian surface A with ordinary reduction over a p-adic field, Σ_A is torsion. This is because the long exact localization sequence gives

$$\ldots \longrightarrow CH^2(A,1) \xrightarrow{\partial} CH^1(\mathcal{A}_p) \longrightarrow CH^2(\mathcal{A}) \longrightarrow CH^2(A) \longrightarrow 0$$

hence the group Σ_A is the same as the cokernel of the image of ∂. As we have shown that the $\partial \otimes \mathbb{Q}$ is surjective, this implies $\Sigma_A \otimes \mathbb{Q} = 0$ – hence Σ_A is torsion.

There are a lot of consequences of the finiteness of Σ_A – they are described in the paper of Spiess [Spi99]. For example, one has

Corollary 5.10. *Let A be an Abelian surface over K with good, ordinary, reduction. Let $\mathcal{A}_p$ denote its reduction mod p. Then*

- $CH_0(A)(non-p)$ *is finite and isomorphic to* $CH_0(\mathcal{A}_p)(non-p)$.
- *The group Σ_A is a p-group.*
- *For every integer $n \neq 0$ which is prime to p the cycle map*

$$cl_n : CH^2(A)/n \longrightarrow H^4(A, \mathbb{Z}/n(2))$$

  is injective.

Proof. [Spi99] - Section 4. This is a consequence of the finiteness of Σ_A and the fact that the Tate conjecture is known for Abelian surfaces over a finite field. □

5.4.2 Relations between CM cycles

The work of Mori and Mukai [MM83] can be used in a similar manner to deform sums of rational curves to nodal rational curves. An argument similar to the one above can then be used to construct indecomposable higher chow cycles in the generic Abelian surface over the Siegel modular threefold which degenerate over Humbert surfaces. This is a generalization of the work of Collino [Col97] where he constructs a higher Chow cycle which degenerates over the moduli of products of elliptic curves - namely the Humbert surface of invariant 1. In [Sre01], Collino's elements were used to construct relations between CM cycles - certain codimension two cycles in the CM fibres over modular and Shimura curves. These new elements can be used to get more relations. This in more detail will be the subject of another paper.

5.4.3 $K3$ surfaces

Since the key point of our construction was done over $K3$ surfaces and the result we used holds over fairly general $K3$ surfaces, we expect that out construction could be used to prove the non-Archimedean Hodge-$\mathcal{D}$-conjecture for such $K3$ surfaces.

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