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SUBGROUPS OF ALGEBRAIC GROUPS WHICH ARE CLOPEN IN THE S-CONGRUENCE TOPOLOGY

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Abstract

Let K be a global field and S be a finite set of places of K which includes all those of archimedean type. Let $\mathbf{G}$ be an algebraic group over K and G_K be its K-rational points. The authors provide a detailed proof of a lemma of Raghunathan which states that (under fairly weak restrictions) the closure of a subgroup of G_K normalized by an S-arithmetic subgroup in the S-congruence topology is also open. This leads to a significant simplification in the proof of one of the principal results in a recent joint paper of the authors.
By applying the lemma to S-arithmetic lattices in K-rank one $\mathbf{G}$ we can provide a lower estimate for the number of subgroups of a given index in such a lattice which are *not* S-congruence. This extends previous results of the first author and Andreas Schweizer.

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Introduction

Let K be a global field and let S be a finite nonempty set of places of K which contains all the archimedean places. Let $\mathbf{G}$ be an algebraic group over K. This note is motivated by the following result [R, 4.3 Lemma].

Raghunathan's Lemma *Suppose that* $\mathbf{G}$ *is connected, simply-connected and* K*-simple with strictly positive* S*-rank. Let* Γ *be an* S*-arithmetic subgroup of* G_K*, the* K*-rational points of* $\mathbf{G}$*. If* N *is any noncentral subgroup of* G_K *which is normalized by* Γ *then the closure of* N *in the* S*-congruence topology is also open.*

Our attention was first drawn to this result because it provides a significant simplification in the proof of one of the principal results in a recent paper [MPSZ]. A result involving a subgroup which is clopen with respect to the S-congruence topology is central to Weisfeiler's celebrated work [W] on the Strong Approximation Theorem. (Pink [P] has extended these to include, for example, global fields of all positive characteristic.) Weisfeiler's starting point is a subgroup of G_K which is both finitely generated and Zariski dense. As we shall see the hypotheses on N ensure that it is Zariski dense. However Raghunathan's Lemma does not follow from [W] since in general such an N is *not* finitely generated. In fact we will apply the Lemma to such subgroups. The proof [R, 4.3 Lemma] provided by Raghunathan is merely a sketch. Given the importance of this result (and the fact that the likely readership of this note will include group-theorists who are not experts in algebraic groups) it seems appropriate to provide a detailed version.
We apply this theorem to the classical case of an S-arithmetic lattice, Λ, in G_{K_v}, where $\mathbf{G}$ has K-rank one, v is fixed and $S = \{v\}$. We prove a result on the ubiquity of finite index subgroups of Λ which are *not* S-congruence. This extends results [MS] of the second author and Andreas Schweizer for the special case where Λ is a so-called *Drinfeld modular group.*
The proof of Raghunathan's Lemma is not elementary. It involves, for example, the *Inverse Function Theorem* from Lie theory. We conclude by showing that for some special (and important) cases the Lemma can be proved using only elementary methods.

1. Raghunathan's lemma

We will use throughout the following notation.

K	a global field;
S	a finite non-empty set of places of K including all archimedean places;
$\mathcal{O}(S)$	the ring of all S-integers in K;
K_v	the completion of K with respect to a nonarchimedean place v;
$\mathcal{O}_v$	the valuation ring of K_v;
$\mathfrak{p}_v$	the maximal ideal of $\mathcal{O}_v$;
F_v	the residue field of $\mathcal{O}_v$;
$\mathbf{G}$	an algebraic group over K;
G_F	the group of F-rational points of $\mathbf{G}$, where $F \geq K$;
$G_{\mathcal{O}(S)}$	the group of S-integral points of $\mathbf{G}$.

We note that K_v is a *local field* and that $\mathcal{O}_v$ is a local ring whose residue field F_v is *finite*. By definition

$$\mathcal{O}(S) = \bigcap_{v \notin S} (K \cap \mathcal{O}_v).$$

After Margulis [Mar, p.60] we will assume that $\mathbf{G}$ is a K-subgroup of $\mathbf{GL}_n$, for some n and we will use this embedding as a standard way of representing $\mathbf{G}$, including its *S-congruence* subgroups. For each non-zero $\mathcal{O}(S)$-ideal $\mathfrak{a}$ we define

$$\mathbf{G}(\mathfrak{a}) = \{X \in G_K : X \equiv I_n \pmod{\mathfrak{a}}\}.$$

The subgroups $\mathbf{G}(\mathfrak{a})$, where $\mathfrak{a} \neq \{0\}$, form the basis of a topology on $\mathbf{G}$ called the *S-congruence topology.* The group G_{K_v} inherits another (more "natural") topology from that of K_v. For this topology the *principal congruence subgroups*, $\mathbf{G}(\mathfrak{p}_v^t)$, where $t \geq 1$, provide a base of neighbourhoods of the identity in G_{K_v}; see [PR, p. 134]. Let X be the *restricted topological product* [PR, p.161] of G_{K_v} with respect to the distinguished (open, compact) subsets, $\mathbf{G}(\mathcal{O}_v)$, where $v \notin S$. By definition X is the set of all sequences $\{x_v\}$, where $v \notin S$, such that

(i) $x_v \in G_{K_v}$, for all $v \notin S$,

(ii) $x_v \in \mathbf{G}(\mathcal{O}_v)$, for all but finitely many v.

Then X is a topological group with a base of neighbourhoods of the identity consisting of all subgroups of the form

$$\prod_{v \notin S} M_v,$$

where each M_v is an open subgroup of G_{K_v} and $M_v = \mathbf{G}(\mathcal{O}_v)$, for all but finitely many $v \notin S$. Now G_K embeds, via the "diagonal map", in X. It is clear that the S-congruence topology on G_K coincides with that induced on its embedding by the topology on X. Let H be any subgroup of G. Then we can identify the closure of H in X with the (profinite) completion of H with respect to its S-congruence topology. We begin by providing a detailed version of the proof of [R, 4.3. Lemma].

Notation. Let H be a subgroup of G_K. We denote the S-closure of H in G_K (or X) by $\bar{H}$ and the Zariski closure of H in $\mathbf{G}$ by $\hat{H}$.

Theorem 1.1 (Raghunathan) *Suppose that* $\mathbf{G}$ *is connected, simply-connected and K-simple with strictly positive S-rank. Let Γ and N be subgroups of G_K for which:*

(i) Γ is S-arithmetic, i.e. commensurable with $G_{\mathcal{O}(S)}$;

(ii) N is noncentral and normalized by Γ.

Then $\bar{N}$ is also open in the S-congruence topology on G_K.

Proof. It suffices to prove that $\bar{N}$ is open in X. We begin by showing that N is Zariski dense in $\mathbf{G}$. Now $\hat{\Gamma}$ normalizes $\hat{N}$. But $\hat{\Gamma} = \mathbf{G}$ by [Mar, 3.2.10, p.64] and $\hat{N}$ is defined over k by [Mar, 2.5.3, p.56]. Hence $\hat{N} = \mathbf{G}$.

The closure of Γ in G_K in the S-congruence topology is open and so $\bar{\Gamma}$ (in X) contains a subgroup of the type

$$\prod_{v \notin S} \bar{\Gamma}_v,$$

where

(i) each $\bar{\Gamma}_v$ is open in G_{K_v},

(ii) $\bar{\Gamma}_v = \mathbf{G}(\mathcal{O}_v)$, for all but finitely many v.

Then, since $\bar{N}$ is normalized by $\bar{\Gamma}$, $\bar{N}$ contains

$$\prod_{v \notin S} [\bar{N}_v, \bar{\Gamma}_v],$$

where $\bar{N}_v$ is the projection of $\bar{N}$ into G_v. It suffices therefore to prove that, for all $v \notin S$,

(a) $[\bar{N}_v, \bar{\Gamma}_v]$ is open in G_{K_v},

(b) $[\bar{N}_v, \bar{\Gamma}_v] \geq \mathbf{G}(\mathcal{O}_v)$, for all but finitely many v.

Part (a). Here the approach is based on Lie theory as in Section 9 of [BT]. Let $L = L(\mathbf{G})$ be the Lie algebra of $\mathbf{G}$ and let

$$L_0 = \sum_{n \in \bar{N}_v} (\mathrm{Ad}(n) - 1)L.$$

Now L_0 is invariant under $\mathrm{Ad}(\bar{N}_v)$. From the above $\bar{N}_v$ is Zariski dense in $\mathbf{G}$ (since it contains N) and so

(i) L_0 is invariant under $\mathrm{Ad}(\mathbf{G})$,

(ii) $(\mathrm{Ad}(g) - 1)x \in L_0$, for all $g \in \mathbf{G},\ x \in L$.

We now make use of use of the hypothesis that $\mathbf{G}$ is simply-connected to conclude that $L_0 = L$. (See [BT, 3.6].) Since L is a finite dimensional vector space of dimension $d = \dim \mathbf{G}$ over the algebraic closure $\tilde{K}$ of K, there exist $n_1, \cdots, n_d \in \bar{N}_v$ such that

$$\sum_{i=1}^{d} (\mathrm{Ad}(n_i) - 1)L = L.$$

Now consider the morphism of K_v-manifolds

$$\phi : G_{K_v} \times \cdots \times G_{K_v} \longrightarrow G_{K_v},$$

defined by

$$\phi((g_1, \cdots, g_d)) = \prod_{i=1}^{d} [n_i, g_i].$$

Let L_v be the Lie algebra of (the K_v-manifold) G_{K_v}. Then the derivative of ϕ at the identity $(e, \cdots, e)$,

$$\mathrm{d}\phi : L_v \times \cdots \times L_v \longrightarrow L_v,$$

is given by

$$\mathrm{d}\phi((l_1, \cdots, l_d)) = \sum_{i=1}^{d} (\mathrm{Ad}(n_i) - 1) l_i.$$

Now from the above and [PR, Lemma 3.1, p.113]

$$\sum_{i=1}^{d} (\mathrm{Ad}(n_i) - 1) L_v = L_v.$$

It follows that $\mathrm{d}\phi$ is *surjective* in which case we may apply the *Inverse Function Theorem* [Se, LG, Chapter III, Section 9]. Then there exist open neighbourhoods of the identities U, V in $G_v \times \cdots \times G_{K_v}$ and G_{K_v}, respectively, and a restriction ϕ_r of ϕ such that

$$\phi_r : U \longrightarrow V$$

is a homeomorphism. Restricting ϕ_r further to (the open set) $U \cap (\bar{\Gamma}_v \times \cdots \times \bar{\Gamma}_v)$ we conclude that $[\bar{N}_v, \bar{\Gamma}_v]$ is open (in G_{K_v}).

Part (b). We may assume without loss of generality that N is generated by the Γ-conjugates of *finitely many* of its elements. It follows that there exists a *finite* set S', containing S, such that, for all $v \notin S'$,

(i) $\Gamma \leq \mathbf{G}(\mathcal{O}_v)$,

(ii) $\bar{\Gamma}_v = \mathbf{G}(\mathcal{O}_v)$,

(iii) $N \leq \mathbf{G}(\mathcal{O}_v)$.

Let $\tilde{N}_v = [\bar{N}_v, \bar{\Gamma}_v]$. Then from the above, for all $v \notin S'$,

(i) $\tilde{N}_v \cap \mathbf{G}(\mathcal{O}_v) \trianglelefteq \mathbf{G}(\mathcal{O}_v)$,

(ii) $[\mathit{\Gamma}, N] \leq \tilde{N}_v \cap \mathbf{G}(\mathcal{O}_v)$.

Recall that F_v is the (finite) residue field of $\mathcal{O}_v$ (i.e. $\mathcal{O}_v/\mathfrak{p}_v$). For each $s \geq 0$, let

$$\mathbf{G}(\mathfrak{p}_v^s) = \{Y \in \mathbf{G}(\mathcal{O}_v) : Y - I_n \in M_n(\mathfrak{p}_v^s)\}.$$

It is known [PR, Proposition 3.20, p.146] that

$$\mathbf{G}(\mathcal{O}_v)/\mathbf{G}(\mathfrak{p}_v) \cong \mathbf{G}(F_v).$$

It is also known [PR, Proposition 7.5, p.406] that, if $|F_v| \geq 4$, then $\mathbf{G}(F_v)$ has no nontrivial, noncentral normal subgroups. We wish to prove that, for all but finitely many $v \notin S'$, the normal subgroup $\tilde{N}_v \cap \mathbf{G}(\mathcal{O}_v)$ does not map into the centre of $\mathbf{G}(F_v)$. Suppose to the contrary that $\tilde{N}_v \cap \mathbf{G}(\mathcal{O}_v)$ is central $(\mathrm{mod}\, \mathbf{G}(\mathfrak{p}_v))$, for infinitely many v. Then, for all these v, $[[N, \mathit{\Gamma}], \mathit{\Gamma}]$ is contained in $\mathbf{G}(\mathfrak{p}_v)$. It follows that

$$[[N, \mathit{\Gamma}], \mathit{\Gamma}] = 1.$$

Now N and $\mathit{\Gamma}$ are Zariski dense and so by [B, Proposition, p.59]

$$[[\mathbf{G}, \mathbf{G}], \mathbf{G}] = 1.$$

This contradicts the fact that $[\mathbf{G}, \mathbf{G}] = \mathbf{G}$. [B, Proposition, p.181]. We deduce that there exists a *finite* set S'', containing S', for which

(i) $(\tilde{N}_v \cap \mathbf{G}(\mathcal{O}_v)).\mathbf{G}(\mathfrak{p}_v) = \mathbf{G}(\mathcal{O}_v)$,

(ii) $\mathbf{G}(\mathcal{O}_v)$ is perfect.

For (ii) see [PrRag, Section 2.3].[1] For each, $v \notin S''$, it follows that

$$\mathbf{G}(\mathcal{O}_v)/\tilde{N}_v \cap \mathbf{G}(\mathcal{O}_v) \cong \mathbf{G}(\mathfrak{p}_v)/\tilde{N}_v \cap \mathbf{G}(\mathfrak{p}_v).$$

Now $[\mathbf{G}(\mathfrak{p}_v^s), \mathbf{G}(\mathfrak{p}_v^t)] \leq \mathbf{G}(\mathfrak{p}_v^{s+t})$ and so, by part (a), $\mathbf{G}(\mathcal{O}_v)/\tilde{N}_v \cap \mathbf{G}(\mathcal{O}_v)$ is solvable. By (ii) then $\tilde{N}_v \cap \mathbf{G}(O_v) = \mathbf{G}(O_v)$. This completes the proof. □

[1]The authors are indebted to Professor Rapinchuk for providing this reference.

The following consequence is immediate.

Corollary 1.2. *With the notation of the Theorem 1.1, there exists* $\mathfrak{q}_0 \neq \{0\}$ *such that*

$$\bar{N} = \bigcap_{\mathfrak{q} \neq \{0\}} N.\mathbf{G}(\mathfrak{q}) = N.\mathbf{G}(\mathfrak{q}_0).$$

The ideal $\mathfrak{q}_0$ is, of course, not unique. It is clear that if Corollary 1.2 holds for $\mathfrak{q}_0$ then it also holds for any nonzero ideal $\mathfrak{q}_0'$ contained in $\mathfrak{q}_0$. In practise it is convenient to choose $\mathfrak{q}_0$ so that the index $|\mathbf{G}(\mathcal{O}_S) : \mathbf{G}(\mathfrak{q}_0)|$ is minimal. In the final section we will show in some detail how N and $\mathfrak{q}_0$ are related in some special cases.

Theorem 1.1, of course, holds trivially for the case where N is commensurable with Γ. For a nontrivial example of N to which it applies consider the case of the classical modular group, i.e. $G = SL_2, K = \mathbb{Q}, S = \{\infty'\}$ and $\Gamma = SL_2(\mathbb{Z})$. Let M be a normal subgroup of finite index in Γ. Then with finitely many exceptions M is a free nonabelian group of finite rank. For such an M take $N = [M, M]$. Then N is free of infinite rank and hence is not S-arithmetic.

2. Arithmetic lattices in rank one groups

Throughout this section we assume that the characteristic of K is nonzero. We fix a (nonarchimedean) place v of k and let $S = \{v\}$. (The simplest example of such an $\mathcal{O}(S)$ is the polynomial ring $\mathbb{F}_q[t]$, where $\mathbb{F}_q$ is the finite field of order q.) In addition to the hypotheses in the statement of Theorem 1.1, we assume that $\mathbf{G}$ is absolutely almost simple and that the K_v-rank of $\mathbf{G}$ is 1.

Let Λ be a nonuniform, S-arithmetic lattice in (the locally compact group) G_{K_v}. By definition

(i) Λ is a discrete subgroup of $G(k_v)$;

(ii) $\mu(G(K_v)/\Lambda)$ is finite, where μ is a Haar measure on $G(K_v)$;

(iii) $G(K_v)/\Lambda$ is not compact;

(iv) Λ is commensurable with $G_{\mathcal{O}(S)}$.

For our purposes, it suffices to assume that Λ is a (finite index) subgroup of $G_{\mathcal{O}(S)}$.

Notation. For each nonzero $\mathcal{O}_S$ ideal $\mathfrak{q}$ let

$$U_\Lambda(\mathfrak{q}) = \langle u \in \Lambda \cap \mathbf{G}(\mathfrak{q}) : u, \textit{unipotent} \rangle.$$

An immediate application of Theorem 1.1 is the following.

Lemma 2.1. *The closure of $U_\Lambda(\mathfrak{q})$ in G_K in the S-congruence topology is also open.*

N.B. It is well-known that in this case $SL_2(\mathcal{O}(S))$ and hence $U_\Lambda(\mathfrak{q})$ are *not* finitely generated. (This extends a classical result for $SL_2(\mathbb{F}_q)$ due to Nagao.)

One important consequence of Lemma 2.1 is that Lemma 5.7 in [MPSZ] is true for all $\mathfrak{q}$ so that, in the terminology of [MPSZ], the *principal result* always holds. This leads to a significant simplification in the proofs of [MPSZ]. Specifically Zel'manov's solution [Z] of the restricted Burnside problem for topological groups is no longer required.

Associated with G_{K_v} is its *Bruhat-Tits building* which in this case is a tree $\mathcal{T}$ (since the K_v-rank of $\mathbf{G}$ is 1). Bass-Serre theory shows how a presentation for Λ can be inferred from its action on $\mathcal{T}$, via the structure of the quotient graph $\Lambda\backslash\mathcal{T}$. In confirming a conjecture of Serre, Lubotzky has shown [L, Theorem 7.5] that Λ contains infinitely many finite subgroups which are not S-congruence, i.e. so-called S-noncongruence subgroups. Our results can be used to provide information on the ubiquity of the S-noncongruence subgroups of Λ.

It is known [L, Theorem 6.1] that the first Betti number of $\Lambda\backslash\mathcal{T}$, $b_1(\Lambda\backslash\mathcal{T})$, is *finite*.

Theorem 2.2. *Let F_r be the free group on r generators, where $r = b_1(\Lambda\backslash\mathcal{T})$ and let $f(r,n)$ denote the number of index n subgroups of F_r. Let $nc(\Lambda, n)$ be the number of S-noncongruence subgroups of index n in Λ. Then there exists a constant $n_0 = n_0(\Lambda)$ such that, if $n > n_0$, then*

$$nc(\Lambda, n) \geq f(r, n).$$

Moreover, if $r \geq 1$, then for all $n > n_0$, there exists at least one normal, S-noncongruence subgroup of index n in Λ.

Proof. Let $\Lambda(\mathfrak{q}) = \Lambda \cap \mathbf{G}(\mathfrak{q})$. Then by Corollary 1.2 and Lemma 2.1

$$\Lambda(\mathfrak{q}_0) \leq \bigcap_{\mathfrak{q} \neq \{0\}} U_\Lambda(\mathcal{O}_S).\Lambda(\mathfrak{q}),$$

for some nonzero $\mathfrak{q}_0$. We choose $\mathfrak{q}_0$ so that $n_0 = |\Lambda : \Lambda(\mathfrak{q}_0)|$ is minimal.

Now let Λ_V be the subgroup of Λ generated by all the stabilizers in Λ of the vertices of $\mathcal{T}$. By standard Bass-Serre theory we have

$$\Lambda/\Lambda_V \cong F_r.$$

In addition, since $U_\Lambda(\mathcal{O}(S))$ is generated by elements of finite order,

$$U_\Lambda(\mathcal{O}(S)) \leq \Lambda_V.$$

Suppose Λ_c is a congruence subgroup of Λ containing Λ_V. Then by the above

$$\Lambda(\mathfrak{q}_0) \leq \Lambda_c,$$

which implies that $|\Lambda : \Lambda_c| \leq n_0$. The first part follows.

For the second part note that when $r \geq 1$ there exists an epimorphism

$$\theta : \Lambda/\Lambda_V \twoheadrightarrow \mathbb{Z}.$$

□

Notes.

(i) In many cases $b_1(\Lambda\backslash\mathcal{T})$ is nonzero. More precisely it is known [MPSZ, Lemma 3.7] that in this situation every arithmetic S-arithmetic lattice contains lattices of the same type with *arbitrarily large* first Betti numbers.

(ii) *The Drinfeld modular group.* For the case where $\mathbf{G} = \mathbf{SL_2}$ (with as above $S = \{v\}$) the group $SL_2(\mathcal{O}(S))$ is a nonuniform S-arithmetic lattice in G_{K_v}. It plays a fundamental role [G] in the theory of Drinfeld modular curves, analogous to that of the modular group $SL_2(\mathbb{Z})$ in the classical theory of modular forms. It is known [MS, Theorem 2.10] precisely when $b_1(SL_2(\mathcal{O}(S))\backslash\mathcal{T})$ is zero. (This happens in only 4 cases.) In addition when $\Lambda = SL_2(\mathcal{O}(S))$ it is known [MS, Theorem 1.2] that Theorem 2.1 holds for all $n \geq 1$, i.e. $n_0 = 1$, equivalently, $\mathfrak{q}_0 = \mathcal{O}(S)$.

3. The case $\mathbf{G} = \mathbf{SL_2}$

In the final section we show in some restricted circumstances it is possible prove an explicit version of Raghunathan's Lemma in an elementary way which does not involve any Lie theory. We revert here to K of any characteristic and any S.

Definition. Let H be a subgroup of $SL_2(\mathcal{O}(S))$. The *order* of H, $o(H)$, is the $\mathcal{O}(S)$-ideal generated by all $h_{12}, h_{21}, h_{11} - h_{22}$, where $(h_{ij}) \in H$.

Definition. For each $\mathcal{O}(S)$-ideal $\mathfrak{q}$ let

$$\phi(\mathfrak{q}) = \begin{cases} 12\mathfrak{q} & , \quad \mathrm{char}(K) = 0 \\ \mathfrak{q}^4 & , \quad \mathrm{char}(K) \neq 0 \end{cases}$$

Lemma 3.1. *Let N be a noncentral normal subgroup of $SL_2(\mathcal{O}(S))$, with $o(N) = \mathfrak{n}(\neq \{0\})$. Let $\mathfrak{q}$ be any nonzero $\mathcal{O}(S)$-ideal $\mathfrak{q}$. If $\mathfrak{n}' = \mathfrak{n} + \mathfrak{q}$, then*

$$SL_2(\phi(\mathfrak{n}')) \leq N.SL_2(\mathfrak{q}).$$

Proof. Since $M = N.SL_2(\mathfrak{q})$ is an S-congruence subgroup whose level

$$o(M) = \mathfrak{n} + \mathfrak{q},$$

we can apply [Mas, Theorems 3.6, 3.10, 3.14]. □

Theorem 3.2. *Let N be a noncentral normal subgroup of $SL_2(\mathcal{O}(S))$. Then*

$$\bar{N} = \bigcap_{\mathfrak{q} \neq \{0\}} N.SL_2(\mathfrak{q}) = N.SL_2(\phi(\mathfrak{n})),$$

where $\mathfrak{n} = o(N)$.

Under further restrictions Theorem 3.2 can be improved. For example, from the results of [Mas] it follows that, if $o(N)$ is prime to 6, then

$$\bar{N} = \bigcap_{\mathfrak{q} \neq \{0\}} N.SL_2(\mathfrak{q}) = N.SL_2(\mathfrak{n}).$$

In particular, if $o(N) = \mathcal{O}(S)$, then $\bar{N} = SL_2(\mathcal{O}(S))$.

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