$\label{eq:sibang/ms/2010/1} isibang/ms/2010/1 \\ February 12th, 2010 \\ \texttt{http://www.isibang.ac.in/~statmath/eprints}$

Dynamic Random Walks on Motion Groups

C. R. E. RAJA AND R. SCHOTT

Indian Statistical Institute, Bangalore Centre 8th Mile Mysore Road, Bangalore, 560059 India

Dynamic Random Walks on Motion Groups

C. R. E. Raja and R. Schott

Abstract

In this note, we give an original convergence result for products of independent random elements of motion group. Then we consider dynamic random walks which are inhomogeneous Markov chains whose transition probability of each step is, in some sense, time dependent. We show, briefly, how Central Limit theorem and Local Limit theorems can be derived from the classical case and provide new results when the rotations are mutually commuting. To the best of our knowledge, this work represents the first investigation of dynamic random walks on the motion group.

2000 Mathematics Subject Classification: 60G50, 60B15.

Key words. Motion group, dynamic random walk, compact groups.

1 Introduction

The motion group $G = SO(d) \ltimes \mathbb{R}^d$ is the semi-direct product of SO(d), the group of rotations in the space \mathbb{R}^d , and \mathbb{R}^d . This group plays a special role in the study of random walks on Lie groups [8]. A Central Limit Theorem on motion groups has been proved by Roynette [10] and Baldi, Bougerol, Crepel [1] gave a Local Limit Theorem on Motion groups. Random walks on homogeneous spaces of the motion group have been studied by Gallardo and Ries [2]. The main novelty of this paper is in the dynamic model of random walks which we define on the motion group. The theory of dynamic random walks has been done by Guillotin-Plantard in a commutative setting [4, 5, 6]. So far, dynamic random walks have been considered on Heisenberg groups, the dual of SU(2) [7] and Clifford algebras [12]. Needless to say, there is much work to do. This paper is another (small) step/attempt in extending this theory to non-commutative algebraic structures. Recently, random walks on the motion group have been proposed as algorithmic ingredients for searching in peer-to-peer wireless networks [3]. The organization is as follows: Section II contains basic definitions, known limit theorems, as well as a new convergence theorem for product of random elements of the motion group. Dynamic random walks are considered in Section III, we recall known results, show how to derive some limit theorems from the classical case and investigate more deeply when the rotations form an abelian group. It may be noted that these are the first examples of non-discrete dynamic random walks. Section IV provides some concluding remarks and further research aspects.

2 Motion group

2.1 Basic definitions and known results

The composition law of the motion group $G = SO(d) \ltimes \mathbb{R}^d$ is given by:

$$(R_1, T_1).(R_2, T_2) = (R_1 \circ R_2, T_1 + R_1(T_2))$$

Remember that $R_1 \circ R_2$ is the rotation of angle $\Theta_1 + \Theta_2$ if Θ_1 (resp. Θ_2) is the rotation angle of R_1 (resp. R_2). More generally:

$$(R_1, T_1)(R_2, T_2)\dots(R_n, T_n) = (R_1 \circ R_2 \dots \circ R_n, T_1 + R_1(T_2) + R_1 \circ R_2(T_3) + \dots + R_1 \circ R_2 \dots \circ R_{n-1}(T_n))$$

where (R_i, T_i) are G-valued random variables. Let

$$S_n = T_1 + R_1(T_2) + R_1 \circ R_2(T_3) + \dots + R_1 \circ R_2 \dots \circ R_{n-1}(T_n)$$

 S_n gives the position after n steps. S_n is the sum of n (not necessarily independent) random variables.

The following Central Limit Theorem has been proven in [10]:

Theorem 2.1 Assume that R_i , (resp. T_i), $i \in \{1, 2, ..., n, ...\}$ are *n* independent random variables with common law μ (resp ν), that the support of μ generates SO(d)and that ν has a second order moment. Then $\frac{S_n}{\sqrt{n}}$ converges in law to the Gaussian distribution $N(0, \theta I_d)$ when *n* goes to infinity. I_d stands for the $d \times d$ dimensional identity matrix and θ is a positive constant.

Remark: This theorem tells us, intuitively, that $\frac{S_n}{\sqrt{n}}$ becomes rotation invariant when n goes to infinity and that S_n behaves asymptotically as a random walk on \mathbb{R}^d which is rotation invariant. In other words:

$$S_n \sim_{n \to \infty} Y_1 + Y_2 + \ldots + Y_n$$

where $Y_i, i \in \{1, 2, ..., n\}$ are n independent and identically distributed random variables.

The following Local Limit Theorem has been proven in [1], we formulate it below in a simple way: **Theorem 2.2** Let $P_n(O, M)$ be the probability that the random walks on G reaches the point M of \mathbb{R}^d in n steps when staring from the point O, then::

$$P_n(O, M) = P(S_n = M) \sim_{n \to \infty} \frac{K}{n^{d/2}}$$

where K is a positive constant (independent of n).

2.2 A convergence theorem

Let O(d) be the group of orthogonal linear transformations on \mathbb{R}^d $(d \ge 1)$ and K be a compact subgroup of O(d) and $G = K \ltimes \mathbb{R}^d$ be a group of motions of \mathbb{R}^d . Let $Y_i = (R_i, T_i)$ be independent random variables. Let $S_n = T_1 + R_1 T_2 + \ldots + R_1 R_2 \ldots R_{n-1} T_n$ and $X_n = R_1 R_2 \ldots R_{n-1} T_n$ with $R_0 = 1$.

Theorem 2.3 Assume the following:

- 1. $R_1 \ldots R_n$ converges in law to ω_K , the normalized Haar measure on K;
- 2. \mathbb{R}^d has no non-zero K-invariant vectors;
- 3. X_n has second moment;
- 4. $E(T_n)$ is bounded.

Then

$$\frac{S_n}{b_n} \to 0 \quad a.s$$

for any sequence (b_n) such that $\sum \frac{E(||X_n - E(X_n)||^2)}{b_n^2} < \infty$.

Proof We recall that a random vector T in \mathbb{R}^d is said to have finite expectation if there is a vector $v \in \mathbb{R}^d$ such that $\langle v, u \rangle = E(\langle T, u \rangle)$ for any $u \in \mathbb{R}^d$ and in this case we write E(T) = v. Also if R is a random rotation on \mathbb{R}^d , then E(R) is a operator on \mathbb{R}^d defined by

$$\langle E(R)u, v \rangle = E(\langle Ru, v \rangle)$$

for any two vectors $u, v \in \mathbb{R}^d$.

It follows that $E(X_n) = E(R_1 R_2 \dots R_{n-1}) E(T_n)$ for all $n \ge 1$. For $u, v \in \mathbb{R}^d$,

$$\langle E(R_1R_2\ldots R_n)u,v\rangle = \int \langle R_1R_2\ldots R_nu,v\rangle d\omega = \int \langle T(u),v\rangle \rho_n(dT)$$

where ρ_n is the distribution of $R_1R_2...R_n$. Since $R_1R_2...R_n$ converges in law to ω_K , we get that $E(R_1R_2...R_{n-1}) \to P_K$ where P_K is the projection onto K-fixed vectors.

We first claim that $E(X_n) \to 0$. Since \mathbb{R}^d has no K-invariant vectors, $P_K = 0$. Now since $E(T_n)$ is bounded, if v is a limit point of $E(T_n)$, let $E(T_{k_n}) \to v$. Then since $E(R_1R_2...R_{n-1}) \to 0$ in the operator topology, $E(X_{k_n}) \to 0$. Thus, $E(X_n) \to 0$.

Let $u \in \mathbb{R}^d$. Take $Z_n = \langle X_n - E(X_n), u \rangle$. Then $E(Z_n) = 0$. Z_n are independent real random variables with finite second moments. Then

$$\frac{1}{b_n} \sum_{i=1}^n Z_i \to 0 \quad a.s.$$

for any constant (b_n) such that $\sum_{0}^{\infty} \frac{Var(Z_n)}{b_n^2} < \infty$ (cf. [13]). This implies that

$$\frac{1}{b_n} \sum_{i=1}^n (X_i - E(X_i)) \to 0 \ a.s.$$

We have shown that $E(X_n) \to 0$ and hence $\frac{1}{b_n} \sum_{i=1}^n E(X_i) \to 0$. Thus,

$$\frac{1}{b_n} \sum_{i=1}^n X_i \to 0 \quad a.s.$$

The conditions in Theorem 2.3 are verified if we take R_i to be iid with the support of the common law is aperiodic (that is, support is not contained in a coset of a proper normal subgroup) and T_i to be dynamic random walk with $b_n = \frac{1}{n^{\alpha}}$ for any $\alpha > \frac{1}{2}$. Thus, under these assumptions we get that

$$\frac{1}{n^{\alpha}}(T_1 + R_1 T_2 + \ldots + R_1 R_2 \ldots R_{n-1} T_n) \to 0 \quad a.s$$

3 Dynamic random walks

3.1 Preliminaries and known results

Let $S = (E, \mathcal{A}, \rho, \tau)$ be a dynamical system where (E, \mathcal{A}, ρ) is a probability space and τ is a transformation defined on E. Let $d \geq 1$ and h_1, \ldots, h_d be functions defined on E with values in $[0, \frac{1}{d}]$. Let $(T_i)_{i\geq 1}$ be a sequence of independent random vectors with values in \mathbb{Z}^d . Let $x \in E$ and $(e_j)_{1\leq j\leq d}$ be the unit coordinate vectors of \mathbb{Z}^d . For every $i \geq 1$, the law of the random vector T_i is given by

$$P(T_i = z) = \begin{cases} h_j(\tau^i x) & \text{if } z = e_j \\ \frac{1}{d} - h_j(\tau^i x) & \text{if } z = -e_j \\ 0 & \text{otherwise} \end{cases}$$

We write

$$S_0 = 0, \ S_n = \sum_{i=1}^n T_i \text{ for } n \ge 1$$

for the \mathbb{Z}^d -random walk generated by the family $(T_i)_{i\geq 1}$. The random sequence $(S_n)_{n\geq 0}$ is called a dynamic \mathbb{Z}^d -random walk.

It is worth remarking that if the functions h_j are constant then we have the classical random walks but if these functions are all not constant, $(S_n)_{n \in \mathbb{N}}$ is a non-homogeneous Markov chain.

Let $C_1(S)$ denote the class of functions $f \in L^1(E,\mu)$ satisfying the following condition (H_1) :

$$\left|\sum_{i=1}^{n} \left(f(\tau^{i}x) - \int_{E} f(x) d\rho(x) \right) \right| = o\left(\frac{\sqrt{n}}{\log(n)}\right)$$

Let $C_2(S)$ denote the class of functions $f \in L^1(E,\mu)$ satisfying the following condition (H_2) :

$$\sup_{x \in E} \left| \sum_{i=1}^{n} \left(f(\tau^{i}x) - \int_{E} f(x) d\rho(x) \right) \right| = o\left(\sqrt{n}\right)$$

A Central Limit Theorem:

Theorem 3.1 Assume that for every $j, l \in \{1, ..., d\}, h_j \in C_2(S), h_j h_l \in C_2(S)$ and $\int_E h_j d\rho = \frac{1}{2d}$. Then, for every $x \in E$, the sequence of processes $(\frac{1}{\sqrt{n}}S_{[nt]})_{t\geq 0}$ weakly converges in the Skorohod space $\mathcal{D} = \mathcal{D}([0, \infty[)$ to the d-dimensional Brownian motion

$$B_t = (B_t^{(1)}, \dots, B_t^{(d)})$$

with zero mean and covariance matrix At.

The proof of this theorem is in [7].

A Local Limit Theorem:

Theorem 3.2 Let $h_j \in C_1(S)$, $h_j h_l \in C_1(S)$ and $\int_E h_j d\rho = \frac{1}{2d}$. Then, for almost every $x \in E$, $P(S_{2n} = 0)$, the probability that starting from the point O, the random walks comes back to O in 2n steps, has the following asymptotic behavior:

$$P(S_{2n} = 0) \sim \frac{2}{\sqrt{det(A)}(4\pi n)^{d/2}}$$

as $n \to \infty$

The proof of this theorem is also in [7].

3.2 Dynamic random walks on the motion group

Recall that we consider the random walk

$$S_n = T_1 + R_1(T_2) + R_1 \circ R_2(T_3) + \dots + R_1 \circ R_2 \dots \circ R_{n-1}(T_n)$$

where $T_i, i \in \mathbb{N}$ are dynamic random variables as defined above and we now define dynamic random rotations R_i .

If the rotations are classical random variables and translations are dynamic random variables then one can adapt the result in [10] and prove a Central Limit Theorem and a Local Limit Theorem [1] for S_n thanks to the Central Limit Theorem and the Local Limit Theorem for dynamic random walks [7] given in the above section. We do not write explicitly these theorems because these formulation is almost the same as in [1], [10]. Similar Central Limit Theorem and Local Limit Theorem hold true under Lindenberg condition on the translations T_i .

If both rotations and translations are dynamic random walks the problem is still open.

We consider now the 2-dimensional case. It is a known that SO(2) is a compact abelian group (isomorphic to U(1)) and for any irrational number $\theta \in \mathbb{R}$, $e^{2\pi i\theta}$ generates a dense subgroup of SO(2). Using this fact we prove that the convolution product $\mu_1 * \mu_2 * \ldots * \mu_n$ of dynamic measures corresponding to dynamic rotations R_1, \ldots, R_n converges weakly to the Haar measure of SO(2).

Let θ be an irrational number and R_j be random rotations on \mathbb{R}^2 defined by

$$P(R_j = z) = \begin{cases} f(\tau^j x) & \text{if } z = e^{2\pi i\theta} \\ 1 - f(\tau^j x) & \text{if } z = e^{-2\pi i\theta} \\ 0 & \text{otherwise} \end{cases}$$

and $f: E \to [0, 1]$ satisfies $f(1 - f) \in \mathcal{C}_2(S)$ where E and $\mathcal{C}_2(S)$ are as in 3.1 with d = 1.

If f is an indicator function taking values 0 and 1, then it can be easily seen that R_i degenerate and hence the product $R_1 \ldots R_n$ does not converge (in law) as the set $\{e^{2\pi i k\theta} \mid k \geq 1\}$ is dense in SO(2). This forces us to assume that f is not a indicator function. In this case, we have the following:

Theorem 3.3 Almost surely $R_1R_2...R_n$ converges in law to the Haar measure on SO(2).

In order to prove the above result we need to recall some details on the dual of compact abelian groups and Fourier transform of probability measures on compact groups.

Dual of compact groups: For a compact abelian group K, continuous homomorphisms from K into SO(2) are known as characters and characters form a (locally compact abelian) group which is denoted by \hat{K} and is called the dual group of K: cf. [9] for details on duality of locally compact abelian groups. For each integer n, the map $z \mapsto z^n$ defines a character on SO(2) and defines an isomorphism between the group \mathbb{Z} of integers with the dual of SO(2) (cf. 23.27 (a) of [9]). It is known that if K_1, \ldots, K_d are compact abelian groups, then the dual of $K_1 \times \ldots \times K_d$ is isomorphic to $\hat{K}_1 \times \ldots \times \hat{K}_d$ (cf. 23.18 of [9]).

Fourier transform: Let K be a compact abelian group and μ be a probability measure on K. Then the Fourier transform of μ , denoted by $\hat{\mu}$ is a function on \hat{K} and is defined by

$$\hat{\mu}(\chi) = \int \chi(x) d\mu(x)$$

for all $\chi \in \hat{K}$. It is known that μ is the normalized Haar measure on K if and only if

$$\hat{\mu}(\chi) = \begin{cases} 0 & \text{if } \chi \text{ is non} - \text{trivial} \\ 1 & \text{if } \chi \text{ is trivial} \end{cases}$$

and if X_n are K-valued random variables with Fourier transform f_n , then X_n converges in law to a K-valued random variable X if and only if f_n converges to the Fourier transform of X pointwise (cf. [11]).

Proof of Theorem 3.3 Let k be any non-zero integer. It is sufficient to claim that

$$\prod_{j=1}^{n} |\int e^{2\pi i k x} d\mu_j| \to 0$$

as $n \to \infty$.

$$\begin{aligned} |\int e^{2\pi i k x} d\mu_j|^2 &= |e^{2\pi i k \theta} f(\tau^j x) + e^{-2\pi i k \theta} (1 - f(\tau^j x))|^2 \\ &= |\cos(2\pi k \theta) + i \sin(2\pi k \theta) (f(\tau^j x) - 1 + f(\tau^j x))|^2 \\ &= \cos^2(2\pi k \theta) + \sin^2(2\pi k \theta) (1 - 2f(\tau^j x))^2 \\ &= 1 - 4 \sin^2(2\pi k \theta) f(\tau^j x) (1 - f(\tau^j x)) \end{aligned}$$

Suppose $f(\tau^j x)(1-f(\tau^j x)) \not\to 0$. Then $1-4\sin^2(2\pi k\theta)f(\tau^j x)(1-f(\tau^j x)) \not\to 1$ and hence $\prod_{j=1}^n |\int e^{2\pi i k\theta} d\mu_j| \to 0$. Thus, it is sufficient to show that $f(\tau^j x)(1-f(\tau^j x)) \not\to 0$. If $f(\tau^j x)(1 - f(\tau^j x)) \to 0$, then

$$\frac{1}{n}\sum_{1}^{n} f(\tau^{j}x)(1 - f(\tau^{j}x)) \to 0 = \int f(x)(1 - f(x))d\rho(x)$$

and hence f is an indicator function. This is a contradiction. Thus proving the result.

Let K be a compact connected abelian subgroup of SO(d), for instance one may take K to be the maximal torus in SO(d). In this situation one can define dynamic random walks in many ways and we will now consider two forms of dynamic random walks on K. The first one is the following: $a \in K$ is such that the closed subgroup generated by a is K (see 25.15, [9] for existence of such a) and R_j are random variables taking values in K defined by

$$P(R_j = x) = \begin{cases} f(\tau^j x) & \text{if } x = a\\ 1 - f(\tau^j x) & \text{if } x = a^{-1}\\ 0 & \text{otherwise} \end{cases}$$

and $f: E \to [0, 1]$ satisfies $f(1 - f) \in \mathcal{C}_2(S)$ where E and $\mathcal{C}_2(S)$ are as in Section 3.

In the situation we have the following as a consequence of Theorem 3.3.

Theorem 3.4 Almost surely $R_1R_2...R_n$ converges in law to the Haar measure on K.

Proof For any non-trivial character χ on K, the element $\chi(a)$ in SO(2) corresponds to an irrational rotation, hence we get from Theorem 3.3 that $\chi(R_1R_2...R_n)$ converges in law to the Haar measure on SO(2) which is $\chi(\omega_K)$. This implies that $(R_1R_2...R_n)$ converges in law to the Haar measure on K

Remark The Corollary 3.4 could be proved for any monothetic compact connected abelian group in a similar way but for simplicity and for the purpose of the article we restrict our attention to compact connected abelian subgroups of SO(d): a topological group K is called monothetic if K contains an element a such that the closed subgroup generated by a is K itself (cf. 9.2 of [9] for monothetic compact groups).

We will now consider the second form of dynamic random walks on K. Let v_1, \dots, v_r be a basis for the Lie algebra of K and exp be the exponential map of the Lie algebra of K into K. Let $e_k = \exp(v_k)$ $1 \le k \le r$. Let R_j be the random variables taking values in K defined by

$$P(R_j = x) = \begin{cases} f_k(\tau^j x) & \text{if } x = e_k \\ \frac{1}{r} - f_k(\tau^j x) & \text{if } x = e_k^{-1} \\ 0 & \text{otherwise} \end{cases}$$

and f_k are functions from E taking values in $[0, \frac{1}{r}]$ where E is as in Section 3. We further assume that k-the coordinate of v_k is irrational so that e_k is an irrational rotation by an angle θ_k and all other coordinates of v_k are 0. We further assume that 1 and θ_k are independent over \mathbb{Q} .

In this situation also we have the following which could be proved as in Theorem 3.4

Theorem 3.5 Almost surely $R_1R_2...R_n$ converges in law to the Haar measure on K.

As an application of the results proved in the previous section and the above results on compact groups we get the following:

Theorem 3.6 Let (R_j, T_j) be dynamic random walk on $K \ltimes \mathbb{R}^d$ where R_j is the dynamic random walk on K given in Theorem 3.4 or Theorem 3.5 and T_j is dynamic random walk on \mathbb{R}^d defined in 3.1. Then for $\alpha > \frac{1}{2}$,

$$\frac{1}{n^{\alpha}}(T_1 + R_1 T_2 + \ldots + R_1 R_2 \ldots R_{n-1} T_n) \to P_K(v_0) \quad a.s$$

where P_K is the projection onto the K-invariant vectors in \mathbb{R}^d and $v_0 = (2E(h_j|\mathcal{I}) - \frac{1}{d})_{1 \leq j \leq d}$.

Proof We first assume that \mathbb{R}^d has no non-zero K-invariant vectors. Condition (1) of Theorem 2.3 follows from Theorems 3.4 and 3.5. Let $X_n = R_1 R_2 \dots R_{n-1} T_n$. Then $E(\langle X_n, u \rangle^2) = \int \langle R_1 R_2 \dots R_{n-1} T_n, u \rangle^2 d\omega$ is finite as T_n takes only finitely many values and rotations preserve the norm. Thus, Condition (3) of Theorem 2.3 is verified and condition (4) is easy to verify. Hence

$$\frac{1}{n^{\alpha}}(T_1 + R_1 T_2 + \ldots + R_1 R_2 \ldots R_{n-1} T_n) \to 0 \quad a.s$$

In general, let V be the space of K-invariant vectors in \mathbb{R}^d . Let P_K be the orthogonal projection onto V and Q be the projection onto the orthogonal complement of V. Then for any $v \in \mathbb{R}^d$, $v = P_K(v) + Q(v)$ and $Q(\mathbb{R}^d)$ has no non-zero K-invariant vector. Since both V and $Q(\mathbb{R}^d)$ are K-invariant, we get that $P_K(R(v)) = P_K(v)$ Q(R(v)) = R(Q(v)) for any $v \in \mathbb{R}^d$ and $R \in K$. Now the result follows from the previous case and Theorem 2.1 of [7]

4 Concluding remarks

We have proved a new convergence result for classical random walks on the motion group. Our results for the dynamic case are still partial and we are planning to characterize recurrent and transient random walks (in this model) on the motion group and the corresponding homegeneous spaces. So far, dynamic random walks have only been considered on Heisenberg groups, the dual of SU(2) [7], the motion group and Clifford algebras [12]. A more general study of dynamic random walks on Lie groups, homogeneous spaces and quantum groups is still to be done. This is a challenging research project.

Acknowledgement This work is done as a part of the IFIM project on "Dynamic random walks on algebraic structures and applications in quantum computing". The authors would like to thank IFIM for its support.

References

- [1] P. Baldi, P. Bougerol and P. Crepel, Théorème central limite local sur les extensions compactes de \mathbb{R}^d , Annales de l'I.H.P., section B, 14, 1, 99-111, 1978.
- [2] L. Gallardo and V. Ries, Marches aléatoires sur les espaces homogènes du groupe des déplacements de \mathbb{R}^n , Astérisque 74, 123-138, 1980.
- [3] C.Gkantsidis, M. Mihail and A. Saberi, Random walks in peer-to-peer networks: algorithms and evaluation, Performance Evaluation, 63, 3, 241-263, 2006.
- [4] N. Guillotin, Asymptotics of a dynamic random walk in a random scenery I. A law of large numbers, Annales de l'I.H.P., section B, 36, 2, 127-151, 2000.
- [5] N. Guillotin, Asymptotics of a dynamic random walk in a random scenery II. A functional limit theorem, Markov Processes and Related Fields, 5, 2, 201-218, 1999.
- [6] N. Guillotin-Plantard, Dynamic Z^d-random walks in a random scenery: A strong law of large numbers, J. Theoret. Prob., 14, 1, 241-260, 2001.
- [7] N. Guillotin-Plantard and R. Schott, Dynamic Random walks, Theory and Applications, Elsevier (2006).
- [8] Y. Guivarc'h, M. Keane and B. Roynette, Marches aléatoires sur les groupes de Lie, Lecture Notes in Mathematics 624, Springer Verlag (1977).
- [9] E. Hewitt and K. A. Ross, Abstract harmonic analysis. Vol. I. Structure of topological groups, integration theory, group representations. Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 115. Springer-Verlag, Berlin-New York, 1979.

- [10] B. Roynette, Théorème central limite pour le groupe des déplacements de \mathbb{R}^d , Annales de l'I.H.P., section Bx, 10, 4, 391-398, 1974.
- [11] W. Rudin, Fourier analysis on groups. Interscience Tracts in Pure and Applied Mathematics, No. 12 Interscience Publishers (a division of John Wiley and Sons), New York-London 1962.
- [12] R. Schott and G. S. Staples, Dynamic random walks in Clifford algebras, Adv. Pure Appl. Math 1, 2010 (in press).
- [13] A. N. Shiryayev, Probability, Springer Verlag (1984).

C. R. E. Raja, Stat-Math Unit, 8th Mile Mysore Road, Bangalore 560 009, India. e-mail: creraja@isibang.ac.in

R. Schott, IECN and Loria, Univesité Henri Poincaré, BP 239, Nancy, Rrance. e-mail: schott@loria.fr