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Analogues of the Wiener-Tauberian and Schwartz theorems for radial functions on symmetric spaces

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Analogues of the Wiener-Tauberian and Schwartz theorems for radial functions on symmetric spaces

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Abstract

We prove a Wiener-Tauberian theorem for the L^1 spherical functions on a semisimple Lie group of arbitrary real rank. We also establish a Schwartz type theorem for complex groups. As a corollary we obtain a Wiener-Tauberian type result for compactly supported distributions.

Keywords: Wiener-Tauberian theorem, Schwartz theorem, ideals, Schwartz space.

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1 Introduction

Two celebrated theorems from classical analysis dealing with translation invariant subspaces are the Wiener-Tauberian theorem and the Schwartz theorem. Let $f \in L^1(\mathbb{R})$ and $\tilde{f}$ be its Fourier transform. Then the celebrated Wiener-Tauberian theorem says that the ideal generated by f is dense in $L^1(\mathbb{R})$ if and only if $\tilde{f}$ is a nowhere vanishing function on the real line.

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The result due to L. Schwartz says that, every closed translation invariant subspace V of $C^\infty(\mathbb{R})$ is generated by the exponential polynomials in V. In particular, such a V contains the function $x \to e^{i\lambda x}$ for some $\lambda \in \mathbb{C}$. Interestingly, this result fails for $\mathbb{R}^n$, if $n \geq 2$. Even though an exact analogue of the Schwartz theorem fails for $\mathbb{R}^n$ $n \geq 2$, it follows from the well known theorem of Brown-Schreiber-Taylor [BST] that, if $V \subset C^\infty(\mathbb{R}^n)$ is a closed subspace which is translation and rotation invariant then V contains a ψ_s for some $s \in \mathbb{C}$ where

$$\psi_s(x) = \frac{J_{\frac{n}{2}-1}(s|x|)}{(s|x|)^{\frac{n}{2}-1}} = \int_{S^{n-1}} e^{isx.w}\ d\sigma(w).$$

Here $J_{\frac{n}{2}-1}$ is the Bessel function of the first kind and order $n/2-1$ and σ is the unique, normalized rotation invariant measure on the sphere S^{n-1}. It also follows from the work in [BST] that V contains all the exponentials $e^{z.x}$, if $z = (z_1, z_2, \ldots z_n) \in \mathbb{C}^n$ satisfies $z_1^2 + z_2^2 + \cdots + z_n^2 = s^2$.

Our aim in this paper is to prove analogues of these results in the context of non compact semisimple Lie groups.

Notation and preliminaries: For any unexplained terminology we refer to [H]. Let G be a connected non compact semisimple Lie group with finite center and K a fixed maximal compact subgroup of G. Fix an Iwasawa decomposition $G = KAN$ and let $\mathbf{a}$ be the Lie algebra of A. Let $\mathbf{a}^*$ be the real dual of $\mathbf{a}$ and $\mathbf{a}^*_{\mathbb{C}}$ its complexification. Let ρ be the half sum of positive roots for the adjoint action of $\mathbf{a}$ on $\mathbf{g}$, the Lie algebra of G. The Killing form induces a positive definite form $< .,>$ on $\mathbf{a}^* \times \mathbf{a}^*$. Extend this form to a bilinear form on $\mathbf{a}^*_{\mathbb{C}}$. We will use the same notation for the extension as well. Let W be the Weyl group of the symmetric space G/K. Then there is a natural action of W on $\mathbf{a}, \mathbf{a}^*, \mathbf{a}^*_{\mathbb{C}}$ and $< .,. >$ is invariant under this action.

For each $\lambda \in \mathbf{a}^*_{\mathbb{C}}$, let φ_λ be the elementary spherical function associated with λ. Recall that φ_λ is given by the formula

$$\varphi_\lambda(x) = \int_K e^{(i\lambda-\rho)(H(xk))}\, dk \quad x \in G.$$

See [H] for more details. It is known that $\varphi_\lambda = \varphi_{\lambda'}$ if and only if $\lambda' = \tau\lambda$ for some $\tau \in W$. Let l be the dimension of $\mathbf{a}$ and F denote the set (in $\mathbb{C}^l$)

$$F = \mathbf{a}^* + iC_\rho \quad \text{where } C_\rho = \text{ convex hull of } \{s\rho : s \in W\}.$$

Then it is a well known theorem of Helgason and Johnson that φ_λ is bounded if and only if $\lambda \in F$.

Let $I(G)$ be the set of all complex valued spherical functions on G, that is

$$I(G) = \{f : \ f(k_1xk_2) = f(x) : k_1, k_2 \in K, x \in G\}.$$

Fix a Haar measure dx on G and let $I_1(G) = I(G) \cap L^1(G)$. Then it is well known that $I_1(G)$ is a commutative Banach algebra under convolution and that the maximal ideal space of $I_1(G)$ can be identified with F/W.

For $f \in I_1(G)$, define its spherical Fourier transform, $\hat{f}$ on F by

$$\hat{f}(\lambda) = \int_G f(x)\ \varphi_{-\lambda}(x)\ dx.$$

Then $\hat{f}$ is a W invariant bounded function on F which is holomorphic in the interior F^0 of F, and continuous on F. Also $\widehat{f*g} = \hat{f}\hat{g}$ where the convolution of f and g is defined by

$$f * g(x) = \int_G f(xy^{-1})\ g(y)\ dy.$$

Next, we define the L^1- Schwartz space of K-biinvariant functions on G which will be denoted by $S(G)$. Let $x \in G$. Then $x = k\ expX,\ k \in K,\ X \in \mathbf{p}$, where $\mathbf{g} = \mathbf{k} + \mathbf{p}$

is the Cartan decomposition of the Lie algebra $\mathbf{g}$ of G. Put $\sigma(x) = \|X\|$, where $\|.\|$ is the norm on $\mathbf{p}$ induced by the Killing form. For any left invariant differential operator D on G and any integer $r \geq 0$, we define for a smooth K-biinvariant function f

$$p_{D,r}(f) = \sup_{x \in G} \ (1 + \sigma(x))^r \ |\varphi_0(x)|^{-2} \ |Df(x)|$$

where φ_0 is the elementary spherical function corresponding to $\lambda = 0$. Define

$$S(G) = \{f : \ p_{D,r}(f) < \infty \text{ for all } D, r\}.$$

Then $S(G)$ becomes a Fréchet space when equipped with the topology induced by the family of semi norms $p_{D,r}$.

Let $P = P(\mathbf{a}^*_{\mathbb{C}})$ be the symmetric algebra over $\mathbf{a}^*_{\mathbb{C}}$. Then each $u \in P$ gives rise to a differential operator $\partial(u)$ on $\mathbf{a}^*_{\mathbb{C}}$. Let $Z(F)$ be the space of functions f on F satisfying the following conditions:

(i) f is holomorphic in F^0 (interior of F) and continuous on F,

(ii) If $u \in P$ and $m \geq 0$ is any integer, then

$$q_{u,m}(f) = \sup_{\lambda \in F^0} \ (1 + \|\lambda\|^2)^m \ |\partial(u) f(\lambda)| < \infty,$$

(iii) f is W invariant .

Then $Z(F)$ is an algebra under pointwise multiplication and a Fréchet space when equipped with the topology induced by the seminorms $q_{u,m}$.

If $a \in Z(F)$ we define the "wave packet" ψ_a on G by

$$\psi_a(x) = \frac{1}{|W|} \int_{\mathbf{a}^*} \ a(\lambda) \ \varphi_\lambda(x) \ |c(\lambda)|^{-2} \ d\lambda,$$

where $c(\lambda)$ is the well known Harish-Chandra c-function. By the Plancherel theorem due to Harish-Chandra we also know that the map $f \to \hat{f}$ extends to a unitary map from $L^2(K\backslash G/K)$ onto $L^2(\mathbf{a}^*, |c(\lambda)|^{-2}d\lambda)$. We are now in a position to state a result of Trombi-Varadarajan [TV].

Theorem 1.1 *(i) If $f \in S(G)$ then $\hat{f} \in Z(F)$.*

(ii) If $a \in Z(F)$ then the integral defining the "wave packet" ψ_a converges absolutely and $\psi_a \in S(G)$. Moreover, $\hat{\psi}_a = a$.

(iii) The map $f \to \hat{f}$ is a topological linear isomorphism of $S(g)$ onto $Z(F)$.

The plan of this paper is as follows: in the next section we prove a Wiener-Tauberian theorem for $L^1(K\backslash G/K)$ assuming more symmetry on the generating family of functions. In the final section we establish a Schwartz type theorem for complex semisimple Lie groups. As a corollary we also obtain a Wiener-Tauberian type theorem for compactly supported distributions on G/K.

2 A Wiener-Tauberian theorem for $L^1(K\backslash G/K)$

In [EM], Ehrenpreis and Mautner observed that an exact analogue of the Wiener-Tauberian theorem is not true for the commutative algebra of K-biinvariant functions on the semisimple Lie group $SL(2, \mathbb{R})$. Here K is the maximal compact subgroup $SO(2)$. However, in the same paper it was also proved that an additional "not too rapidly decreasing condition" on the spherical Fourier transform of a function suffices to prove an analogue of the Wiener-Tauberian theorem. That is, if f is a K-biinvariant integrable function on $G = SL(2, \mathbb{R})$ and its spherical Fourier transform $\hat{f}$ does not vanish anywhere on the maximal ideal space (which can be identified with a certain strip on the complex plane) then the function f generates a dense subalgebra of $L^1(K\backslash G/K)$ provided $\hat{f}$ does not vanish too fast at ∞. See [EM] for precise statements.

There have been a number of attempts to generalize these results to $L^1(K\backslash G/K)$ or $L^1(G/K)$ where G is a non compact connected semisimple Lie group with finite center. Almost complete results have been obtained when G is a real rank one group. We refer the reader to [BW], [BBHW] [RS98] and [S88] for results on rank one case. See also [RS97]for a result on the whole group $SL(2, I\!R)$.

In [S80], it is proved that under suitable conditions on the spherical Fourier transform of a single function f an analogue of the Wiener-Tauberian theorem holds for $L^1(K\backslash G/K)$, with no assumptions on the rank of G. Recently, the first named author improved this result to include the case of a family of functions rather than a single function (see[N]). One difference between rank one results and higher rank results has been the precise form of the "not too rapid decay condition". In [S80] and [N] this condition on the spherical Fourier transform of a function is assumed to be true on the whole maximal domain, while for rank one groups it suffices to have this condition on $\mathbf{a}^*$ (see [BW] and [RS98] (An important corollary of this is that, in the rank one case one can get a Wiener- Tauberian type theorem for a wide class of functions purely in terms of the non vanishing of the spherical Fourier transform in a certain domain without having to check any decay conditions, see [MRSS], Theorem 5.5). In the first part of this paper we show that such a stronger result is true for higher rank case as well provided we assume more symmetry on the generating family of functions, and again as a corollary we get a result of the type alluded to in the parenthesis above.

If $dim\mathbf{a}^* = l$, then $\mathbf{a}^*_{\mathbb{C}}$ may be identified with $\mathbb{C}^l$ and a point $\lambda \in \mathbf{a}^*_{\mathbb{C}}$ will be denoted $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_l)$. Let B_R denote the ball of radius R centered at the origin in $\mathbf{a}^*$ and F_R denote the domain in $\mathbf{a}^*_{\mathbb{C}}$ defined by

$$F_R = \{\lambda \in \mathbf{a}^*_{\mathbb{C}} : \ \|Im(\lambda)\| < R\}.$$

For $a > 0$, let I_a denote the strip in the complex plane defined by

$$I_a = \{z \in \mathbb{C} : \ |Imz| < a\}.$$

Now, suppose that f is a holomorphic function on F_R and f depends only on $(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_l^2)^{\frac{1}{2}}$. Then it is easy to see that

$$g(s) = f(\lambda_1, \lambda_2, \ldots \lambda_l)$$

where $s^2 = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_l^2$ defines an even holomorphic function on I_R and vice versa.

We will need the following lemmas. Let $A(I_a)$ denote the collection of functions g with the properties:

(i) g is even, bounded and holomorphic on I_a,

(ii) g is continuous on $\bar{I}_a$,

(iii) $\lim_{|s|\to\infty} g(s) = 0$.

Then $A(I_a)$ with the supremum norm is a Banach algebra under pointwise multiplication.

Lemma 2.1 *Let $\{g_\alpha : \ \alpha \in I\}$ be a collection of functions in $A(I_a)$. Assume that there exists no $s \in \bar{I}_a$ such that $g_\alpha(s) = 0 \ \forall \alpha \in I$. Further assume that there exists $\alpha_0 \in I$ such that g_{α_0} does not decay very rapidly on $\mathbb{R}$, i.e,*

$$\limsup_{|s|\to\infty} \ |g_{\alpha_0}(s)| \ e^{ke^{|s|}} > 0$$

on $\mathbb{R}$ for all $k > 0$. Then the closed ideal generated by $\{g_\alpha : \ \alpha \in I\}$ is whole of $A(I_a)$.

Proof: Let ψ be a suitable biholomorphic map which maps the strip I_a onto the unit disc (see [BW]). Let $h_\alpha(z) = g_\alpha(\psi(z))$. Then $h_\alpha \in A_0(D)$, where $A_0(D)$ is the collection of even holomorphic functions h on the unit disc, continuous up to the boundary and $h(i) = h(-i) = 0$. The not too rapid decay condition on $I\!R$ is precisely what is needed to apply the Beurling-Rudin theorem to complete the proof. We refer to [BW] (see the proof of Theorem 1.1 and Lemma 1.2) for the details.

Let p_t denote the K-biinvariant function defined by $\hat{p}_t(\lambda) = e^{-t\langle\lambda,\lambda\rangle}$. It is easy to see that $p_t \in S(G)$.

Lemma 2.2 *Let $J \subset L^1(K\backslash G/K)$ be a closed ideal. If $p_t \in J$ for some $t > 0$, then $J = L^1(K\backslash G/K)$.*

Proof: This follows from the main result in [N] or [S80].

Before we state our main theorem we define the following: We say that a function $f \in L^1(K\backslash G/K)$ is *radial* if the spherical Fourier transform $\hat{f}(\lambda)$ is a function of $(\lambda_1^2 + \lambda_2^2 + \cdots \lambda_l^2)^{\frac{1}{2}}$. Notice that, if the group G is of real rank one, then the class of *radial* functions is precisely the class of K-biinvariant functions in $L^1(G)$. When the group G is complex, it is possible to describe the class of *radial* functions (see next section). The following is our main theorem in this section:

Theorem 2.3 *Let $\{f_\alpha : \ \alpha \in I\}$ be a collection of radial functions in $L^1(K\backslash G/K)$. Assume that the spherical transform $\hat{f}_\alpha$ extends as a bounded holomorphic function to the bigger domain F_R, where $R > \|\rho\|$ with $\lim_{|\lambda|\to\infty} \ \hat{f}_\alpha(\lambda) = 0$ for all α and that there exists no $\lambda \in F_R$ such that $\hat{f}_\alpha(\lambda) = 0$ for all α. Further assume that there exists an α_0 such that $\hat{f}_{\alpha_0}$ does not decay too rapidly on $\mathbf{a}^*$, i.e,*

$$\limsup_{|\lambda|\to\infty} |\hat{f}_{\alpha_0}(\lambda)| \ exp(ke^{|\lambda|}) > 0$$

for all $k > 0$ on $\mathbf{a}^*$. Then the closed ideal generated by $\{f_\alpha : \ \alpha \in I\}$ is all of $L^1(K\backslash G/K)$.

Proof: Since f_α is radial, each $\hat{f}_\alpha$ gives rise to an even bounded holomorphic function $g_\alpha(s)$ on the strip I_R. If $|\rho| < a < R$, then the collection $\{g_\alpha(s), \alpha \in I\}$ satisfies the hypotheses in Lemma 2.1 on the domain I_a. It follows that the family $\{g_\alpha\}$ generates $A(I_a)$. In particular, we have a sequence

$$h_1^n(s)g_{\alpha_1(n)}(s) + h_2^n(s)g_{\alpha_2(n)}(s) + \cdots + h_k^n(s)g_{\alpha_k(n)}(s) \to e^{-\frac{s^2}{2}}$$

uniformly on $\bar{I}_a$, where $g_{\alpha_j(n)}$ are in the given family and $h_j^n(s) \in A(I_a)$.

Notice that each h_j^n can be viewed as a holomorphic function on the domain F_a contained in $\mathbf{a}^*_{\mathbb{C}}$ which depends only on $(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_l^2)^{\frac{1}{2}}$. Since h_j^n are bounded and $|\rho| < a$ it can be easily verified that $e^{-\frac{\langle\lambda,\lambda\rangle}{2}}h_j^n(\lambda) \in Z(F)$. Again, an application of the Cauchy integral formula says that

$$e^{-\frac{\langle\lambda,\lambda\rangle}{2}}h_1^n(\lambda)\hat{f}_{\alpha_1(n)}(\lambda) + e^{-\frac{\langle\lambda,\lambda\rangle}{2}}h_2^n(\lambda)\hat{f}_{\alpha_(n)}(\lambda) + \cdots e^{-\frac{\langle\lambda,\lambda\rangle}{2}}h_k^n(\lambda)\hat{f}_{\alpha_k(n)}(\lambda)$$

converges to $e^{-\langle\lambda,\lambda\rangle}$ in the topology of $Z(F)$ (see the proof of Theorem 1.1 in [BW]). By Theorem 1.1 this simply means that the ideal generated by $\{f_\alpha : \alpha \in I\}$ in $L^1(K\backslash G/K)$ contains the function p where $\hat{p}(\lambda) = e^{-\langle\lambda,\lambda\rangle}$. We finish the proof by appealing to Lemma 2.2.

Corollary 2.4 *Let $\{f_\alpha : \alpha \in I\}$ be a family of radial functions satisfying the hypotheses in Theorem 2.3. Then the closed subspace spanned by the left $G-$translates of the above family is all of $L^1(G/K)$.*

Proof: Let J be the closed subspace generated by the left translates of the given family. By Theorem 2.3, $L^1(K\backslash G/K) \subset J$. Now, it is easy to see that J has to be equal to $L^1(G/K)$.

Corollary 2.5 *Let $\{f_\alpha : \alpha \in I\}$ be a family of L^1-radial functions. Assume that each $\hat{f}_\alpha$ extends to a bounded holomorphic function to the bigger domain F_R for some $R > \|\rho\|$. Assume further that* $\lim_{\|\lambda\|\to\infty} \hat{f}_\alpha(\lambda) \to 0$. *If there exists an α_0 such that f_{α_0} is* **not** *equal to a real analytic function almost everywhere, then the left $G-$translates of the above family span a dense subset of $L^1(G/K)$.*

Proof: This follows exactly as in Theorem 5.5 of [MRSS].

3 Schwartz theorem for complex groups

When G is a connected non compact semisimple Lie group of real rank one with finite center, a Schwartz type theorem was proved by Bagchi and Sitaram in [BS79]. Let K be a maximal compact subgroup of G, then the result in [BS79] states the following: Let V be a closed subspace of $C^\infty(K\backslash G/K)$ with the property that $f \in V$ implies $w * f \in V$ for every compactly supported K-biinvariant distribution w on G/K, then V contains an elementary spherical function φ_λ for some $\lambda \in \mathbf{a}^*_{\mathbb{C}}$. This was done by establishing a one-one correspondence between ideals in $C^\infty(K\backslash G/K)$ and that of $C^\infty(\mathbb{R})_{even}$. This also proves that a similar result can not hold for higher rank groups.

Going back to $\mathbb{R}^n$, we notice that if $f \in C^\infty(\mathbb{R}^n)$ is radial, then the translation invariant subspace V_f generated by f is also rotation invariant. It follows from [BST] that V_f contains a ψ_s for some $s \in \mathbb{C}$ where ψ_s is the Bessel function defined in the introduction. Our aim in this section is to prove a similar result for the complex semisimple Lie groups. Our definition of *radiality* is taken from [VV] and it coincides with the definition in the previous section when the function is in $L^1(K\backslash G/K)$.

Throughout this section we assume that G is a complex semisimple Lie group. Let $Exp: \mathbf{p} \to G/K$ denote the map $P \to (expP)K$. Then Exp is a diffeomorphism. If dx denotes the $G-$invariant measure on G/K, then

$$\int_{G/K} f(x)\ dx = \int_{\mathbf{p}} f(ExpP)\ J(P)\ dP, \tag{3.1}$$

where

$$J(P) = det\left(\frac{\sinh adP}{adP}\right).$$

Since G is a complex group, the elementary spherical functions are given by a simple formula:

$$\varphi_\lambda(ExpP) = J(P)^{-\frac{1}{2}} \int_K e^{i\langle A_\lambda, Ad(k)P\rangle}\ dk, \quad P \in \mathbf{p}. \tag{3.2}$$

Here A_λ is the unique element in $\mathbf{a}_{\mathbb{C}}$ such that $\lambda(H) = \langle A, A_\lambda\rangle$ for all $H \in \mathbf{a}_{\mathbb{C}}$.

Let $E(K\backslash G/K)$ be the dual of $C^\infty(K\backslash G/K)$. Then $E(K\backslash G/K)$ can be identified with the space of compactly supported K-biinvariant distributions on G/K. If w is such a distribution then $\hat{w}(\lambda) = w(\varphi_\lambda)$ is well defined and is called the spherical Fourier transform of w. By the Paley-Wiener theorem we know that $\lambda \to \hat{w}(\lambda)$ is an entire function of exponential type. Similarly, $E(\mathbb{R}^l)$ will denote the space of compactly supported distribution on $\mathbb{R}^l$ and $E^W(\mathbb{R}^l)$ consists of the Weyl group invariant ones. From the work in [BS79] we know that the Abel transform

$$S : E(K\backslash G/K) \to E^W(\mathbb{R}^l)$$

is an isomorphism and $\widetilde{S(w)}(\lambda) = \hat{w}(\lambda)$ for $w \in E(K\backslash G/K)$, where $\widetilde{S(w)}(\lambda)$ is the Euclidean Fourier transform of the distribution $S(w)$. We also need the following result from [BS79].

Proposition 3.1 *There exists a linear topological isomorphism T from $C^\infty(K\backslash G/K)$ onto $C^\infty(\mathbb{R}^l)^W$ such that*

$$S(w)(T(f)) = w(f)$$

for all $w \in E(K\backslash G/K)$ and $f \in C^\infty(K\backslash G/K)$. We also have,

$$S(w^{'}) * T(w * f) = T(w^{'} * w * f)$$

for all $w, w^{'} \in E(K\backslash G/K)$ and $f \in C^\infty(K\backslash G/K)$. Moreover,

$$T(\varphi_\lambda) = \frac{1}{|W|} \sum_{\tau \in W} exp(i\langle \tau.\lambda, x\rangle).$$

A K-biinvariant function f is called *radial* if it is of the form

$$f(x) = J(Exp^{-1}x)^{-\frac{1}{2}} u(d(0,x)),$$

where d is the Riemannian distance induced by the the Killing form on G/K and u is a function on $[0,\infty)$. Theorem 4.6 in [VV] shows that this definition of *radiality* coincides with the one in the previous section if the function is integrable. That is, $f \in L^1(K\backslash G/K)$ has the above form if and only if the spherical Fourier transform $\hat{f}(\lambda)$ depends only on $(\lambda_1^2 + \lambda_2^2 \cdots + \lambda_l^2)^{\frac{1}{2}}$. We denote the class of smooth radial functions by $C^\infty(K\backslash G/K)_{rad}$ and $C_c^\infty(K\backslash G/K)_{rad}$ will consists of compactly supported functions in $C^\infty(K\backslash G/K)_{rad}$.

For $f \in C^\infty(K\backslash G/K)$ define

$$f^{\#}(ExpP) = J(P)^{-\frac{1}{2}} \int_{SO(\mathbf{p})} J(\sigma.P)^{\frac{1}{2}}\ f(\sigma.P)\ d\sigma,$$

where $SO(\mathbf{p})$ is the special orthogonal group on $\mathbf{p}$ and $d\sigma$ is the Haar measure on $SO(\mathbf{p})$. Here, by $f(P)$ we mean $f(ExpP)$. Clearly, $f \to f^{\#}$ is the projection from $C^\infty(K\backslash G/K)$ onto $C^\infty(K\backslash G/K)_{rad}$.

Proposition 3.2 *(a) The space* $C^\infty(K\backslash G/K)_{rad}$ *is reflexive.*

(b) The strong dual $E(K\backslash G/K)_{rad}$ *of* $C^\infty(K\backslash G/K)_{rad}$ *is given by*

$$\{w \in E(K\backslash G/K) : \ \hat{w}(\lambda) \text{ is a function of } (\lambda_1^2 + \lambda_2^2 + \cdots \lambda_l^2)^{\frac{1}{2}}\}.$$

(c) The space $C^\infty(K\backslash G/K)_{rad}$ *is invariant under convolution by* $w \in E(K\backslash G/K)_{rad}$*.*

Proof: (a) The space $C^\infty(K\backslash G/K)_{rad}$ is a closed subspace of $C^\infty(K\backslash G/K)$ which is a reflexive Fréchet space.

(b) Define $B_\lambda = \varphi_\lambda^\#$, the projection of φ_λ into $C^\infty(K\backslash G/K)_{rad}$. A simple computation shows that

$$B_\lambda(ExpP) = J(P)^{-\frac{1}{2}} \int_{SO(\mathbf{p})} e^{i\langle A_\lambda, \sigma.P\rangle} \, d\sigma.$$

It is clear that, B_λ as a function of λ depends only on $(\lambda_1^2 + \lambda_2^2 + \cdots \lambda_l^2)^{\frac{1}{2}}$. Now, let $w \in E(K\backslash G/K)$. Define a distribution $w^\#$ by $w^\#(f) = w(f^\#)$. It is easy to see that $w^\#$ is a compactly supported $K-$biinvariant distribution. Clearly, if $w \in E(K\backslash G/K)_{rad}$, then $w = w^\#$. It follows that $\hat{w}(\lambda) = w(\varphi_\lambda) = w(B_\lambda)$. Consequently, $\hat{w}(\lambda)$ is a function of $(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_l^2)^{\frac{1}{2}}$. It also follows that $E(K\backslash G/K)_{rad}$ is reflexive.

(c) Observe that if $w \in E(K\backslash G/K)_{rad}$ and $g \in C_c^\infty(K\backslash G/K)_{rad}$ then $w * g \in C_c^\infty(K\backslash G/K)_{rad}$. This follows from (b) above and Theorem 4.6 in [VV]. Next, if g is arbitrary, we may approximate g with $g_n \in C_c^\infty(K\backslash G/K)_{rad}$.

We are in a position to state our main result in this section. Let V be a closed subspace of $C^\infty(K\backslash G/K)_{rad}$. We say, V is an ideal in $C^\infty(K\backslash G/K)_{rad}$ if $f \in V$ and $w \in E(K\backslash G/K)_{rad}$ implies that $w * f \in V$.

Theorem 3.3 *(a) If* V *is a non zero ideal in* $C^\infty(K\backslash G/K)_{rad}$ *then there exists a* $\lambda \in \mathbf{a}^*_{\mathbb{C}}$ *such that* $B_\lambda \in V$*.*

(b) If $f \in C^\infty(K\backslash G/K)_{rad}$, then the closed left G invariant subspace generated by f in $C^\infty(G/K)$ contains a φ_λ for some $\lambda \in \mathbf{a}^*_{\mathbb{C}}$.

Proof: We closely follow the arguments in [BS79].

(a) Notice that the map

$$S: \; E(K\backslash G/K)_{rad} \to E(\mathbb{R}^l)_{rad}$$

is a linear topological isomorphism. Using the reflexivity of the spaces involved and arguing as in [BS79] we obtain that (as in Proposition 3.1)

$$T: C^\infty(K\backslash G/K)_{rad} \to C^\infty(\mathbb{R}^l)_{rad}$$

is a linear topological isomorphism, where $C^\infty(\mathbb{R}^l)_{rad}$ stands for the space of C^∞ radial functions on $\mathbb{R}^l$ and

$$S(w)(T(f) = w(f) \; \forall w \in E(K\backslash G/K)_{rad}, f \in C^\infty(K\backslash G/K)_{rad}.$$

Another application of Proposition 3.1 implies that we have a one-one correspondence between the ideals in $C^\infty(K\backslash G/K)_{rad}$ and $C^\infty(\mathbb{R}^l)_{rad}$. Here, ideal in $C^\infty(\mathbb{R}^l)_{rad}$ means a closed subspace invariant under convolution by compactly supported radial distributions on $\mathbb{R}^l$. From [BS90] or [BST] we know that any ideal in $C^\infty(\mathbb{R}^l)_{rad}$ contains a ψ_s (Bessel function) for some $s \in \mathbb{C}$. To complete the proof it suffices to show that under the topological isomorphism T the function B_λ is mapped into ψ_s where $s^2 = (\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_l^2)^2$.

Now, we have $S(w)(T(B_\lambda)) = w(B_\lambda)$. Since $w \in E(K\backslash G/K)_{rad}$ we know that $w(B_\lambda)$ is nothing but $w(\varphi_\lambda)$ which equals $(\widetilde{Sw})(\lambda)$. Since S is onto, this implies that $T(B_\lambda) = \psi_s$ where $s^2 = (\lambda_1^2 + \lambda_2^2 + \cdots \lambda_l^2)^{\frac{1}{2}}$.

(b) From [BS79] we know that $T(\varphi_\lambda) = \psi_\lambda$ where $\psi_\lambda(x) = \frac{1}{|W|} \sum_{\tau \in W} \; exp(i\tau\lambda.x)$. Let V_f denote the left G-invariant subspace generated by f. Then $T(V_f)$ surely contains the space

$$V_{T(f)} = \{S(w) * T(f) : \; w \in E(K\backslash G/K)\}.$$

From Proposition 3.2, $T(f)$ is a radial C^∞ function on $I\!R^l$. Hence, from [BST], the translation invariant subspace $X_{T(f)}$, generated by $T(f)$ in $C^\infty(I\!R^l)$ contains a ψ_s for some $s \in \mathbb{C}$ and consequently all the exponentials $e^{iz.x}$ where $z = (z_1, z_2, \ldots z_l)$ satisfies $z_1^2 + z_2^2 + \cdots + z_l^2 = s^2$. Now, it is easy to see that the map $g \to g^W$ where $g^W(x) = \frac{1}{|W|} \sum_{\tau \in W} \; g(\tau.x)$, from $X_{T(f)}$ into $V_{T(f)}$ is surjective. Hence, there exists a $\lambda \in \mathbb{C}^l$ such that $\psi_\lambda \in V_{T(f)}$. Since $T(\varphi_\lambda) = \psi_\lambda$, this finishes the proof.

Our next result is a Wiener-Tauberian type theorem for compactly supported distributions. Let $E(G/K)$ denote the space of compactly supported supported distributions on G/K. If $g \in G$ and $w \in E(G/K)$ then the left $g-$translate of w is the compactly supported distribution ${}^g w$ defined by

$${}^g w(f) = w({}^{g^{-1}} f), \quad f \in C^\infty(G/K)$$

where ${}^x f(y) = f(x^{-1}y)$.

Theorem 3.4 *Let $\{w_\alpha : \; \alpha \in I\}$ be a family of distributions contained in $E(K\backslash G/K)_{rad}$. Then, the left $G-$translates of this family spans a dense subset of $E(G/K)$ if and only if there exists no $\lambda \in \mathbf{a}^*_{\mathbb{C}}$ such that $\hat{w}_\alpha(\lambda) = 0$ for all $\alpha \in I$.*

Proof: We start with the *if* part of the theorem. Let J stand for the closed span of the left $G-$translates of the distributions w_α in $E(G/K)$. It suffices to show that $E(K\backslash G/K) \subset J$. To see this, let $f \in C^\infty(G/K)$ be such that $w(f) = 0$ for all

$w \in E(K\backslash G/K)$. Since J is left $G-$invariant we also have $w(f_g) = 0$ for all $g \in G$, where f_g is the K-biinvariant function defined by

$$f_g(x) = \int_K f(gkx)\ dk.$$

It follows that $f_g \equiv 0$ for all $g \in G$ and consequently $f \equiv 0$.

Next, we claim that if $E(K\backslash G/K)_{rad} \subset J$ then $E(K\backslash G/K) \subset J$. To prove this it is enough to show that

$$\{g * w :\ w \in E(K\backslash G/K)_{rad}, g \in C_c^\infty(K\backslash G/K)\}$$

is dense in $E(K\backslash G/K)$. Notice that, by Proposition 3.2 the map S from $E(K\backslash G/K)$ onto $E(\mathbb{R}^l)^W$ is a linear topological isomorphism which maps $E(K\backslash G/K)_{rad}$ onto $E(\mathbb{R}^l)_{rad}$ isomorphically. Hence, it suffices to prove a similar statement for $E(\mathbb{R}^l)_{rad}$ and $E(\mathbb{R}^l)^W$ which is an easy exercise in distribution theory!

So, to complete the proof of Theorem 3.4 we only need to show that

$$\{g * w_\alpha :\ \alpha \in I, g \in C_c^\infty(K\backslash G/K)_{rad}\}$$

is dense in $E(K\backslash G/K)_{rad}$. If not, consider

$$J_{rad} = \{f \in C^\infty(K\backslash G/K)_{rad} :\ (g * w_\alpha)(f) = 0\ \forall g \in C_c^\infty(K\backslash G/K),\ \alpha \in I\}.$$

The above is clearly a closed subspace of $C^\infty(K\backslash G/K)_{rad}$ which is invariant under convolution by $C_c^\infty(K\backslash G/K)_{rad}$. By Theorem 3.3 we have $B_\lambda \in J_{rad}$ for some $\lambda \in \mathbf{a}_{\mathbb{C}}^*$. It follows that $\hat{w}_\alpha(\lambda) = 0$ for all $\alpha \in I$ which is a contradiction. This finishes the proof.

For the *only if* part, it suffices to observe that if $g \in C_c^\infty(G/K)$ then

$$g * w_\alpha(\varphi_\lambda) = \hat{g^{\#}}(\lambda)\hat{w}_\alpha(\lambda)$$

where $g^{\#}(x) = \int_K \ g(kx)\ dk$.

Remark: A similar theorem for **all** rank one spaces may be derived from the results in [BS90].

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