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# CORRELATED DRAINAGE MODEL

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#### Abstract

In this article we present an example of a random oriented tree model on  $\mathbb{Z}^d$ , that is a forest in d = 3 with positive probability. This is in contrast with the other random tree models in the literature which are a forest only when  $d \ge 4$ .

### 1 Introduction

In this note we consider the 3-dimensional lattice  $\mathbb{Z}^3$  where each vertex is 'open' or 'closed'. The choice of being open or closed is prescribed by a measure  $\mu$  (defined in the next section). An open vertex  $\mathbf{v}$  is connected by an edge to the closest open vertex w such that the 3rd coordinates of  $\mathbf{v}$  and  $\mathbf{w}$  satisfy w(3) = v(3) - 1. In case of non-uniqueness of such a vertex  $\mathbf{w}$ , we choose any one of the closest vertices with equal probability. The random graph so constructed has no loops. Also it is easy to see that if any two paths intersect at a certain level (3rd-coordinate) then they stay together for all levels below.

Our main result (Theorem 2.1) is that under the correlation structure provided by  $\mu$ , this graph is a random oriented forest (i.e. consists of infinitely many random oriented trees) with positive probability. In order to prove that the collection of all paths yields a forest, it suffices to prove that the paths, as described above, starting at two nodes which are sufficiently far apart, do not meet with positive probability.

This result is in contrast to tree-forest dichotomies that usually occur at d = 4 rather than d = 3. We refer the reader to the uniform spanning tree model in [5], the minimal spanning tree model in [6], and the river network considered in [1]. In [1], each vertex is assumed open or closed with probability p independent of each other. The notations and methodology of the proving of the main result are borrowed entirely from [1]. In the next section we state our model precisely along with the main result. Our proof involves a coupling with a system of independent random walks and the idea is borrowed from the proof presented in [1], this is done in Section 3.

## 2 Model and main result

Let  $\Omega = \{0, 1\}^{\mathbb{Z}^3}$  be the configuration space of our model and  $\mathcal{F}$  be the corresponding  $\sigma$ -algebra generated by the finite dimensional cylindrical sets.

Let  $\Omega_2 = \{0,1\}^{\mathbb{Z}^2}$  and  $\mathcal{F}_2$  be the corresponding  $\sigma$ -algebra generated by finite dimensional cylinderical sets. For  $k \geq 1$ , Let

$$A_k = \{ \omega \in \Omega_2 : \omega(z) = 1 \text{ iff } z_1 + z_2 = 0 \mod k \}$$

i.e., it is the configuration corresponding to having 1's at the four vertices of the  $l_1$ -ball of radius k centred at the origin of  $\mathbb{Z}^2$  and replication this throughout the lattice  $\mathbb{Z}^2$ . We define a probability measure  $\mu^{(2)}$  on  $(\Omega_2, \mathcal{F}_2)$  as follows:

$$\mu^{(2)}(A_k) = \mu^{(2)}(T^{i,j}(A_k)) = c \frac{k^{-\beta}}{g(k)},$$

where  $T^{i,j}$  is the shift operator with *i* steps in left and *j* steps in upward direction for  $i, j \in \mathbb{Z}$ ,  $\beta > 1$  to ensure summability, *c* is the normalizing constant, and g(k) = 1 + 2k(k+1) is the number of transformations of  $A_k$ , by translations alone, that are different from each other. Note that this measure puts zero mass on all other configurations.

Following [1], we shall use the notation:  $\mathbf{u} \in \mathbb{Z}^3$ , be represented as

$$\mathbf{u} = (u(1), u(2), u(3)) = (u, u(3)).$$

We replicate  $\mu^{(2)}$  independently over all the  $\mathbb{Z}^2$  layers of  $\mathbb{Z}^3$  to get a measure on  $(\Omega, \mathcal{F})$  so that for  $\omega \in \Omega$  we set

$$\mu(\omega(\mathbf{u}) = 1) = p \text{ (say).}$$

Let  $\{U_{\mathbf{u},\mathbf{v}}:\mathbf{u},\mathbf{u}\in\mathbb{Z}^3, v(3)=u(3)-1\}$  be i.i.d. uniform (0,1] random variables on some probability space  $(\Xi,\mathcal{G},\nu)$ .

Consider the product space  $(\Omega \times \Xi, \mathcal{F} \times \mathcal{G}, \mathbb{P} := \mu \times \nu)$ . For  $(\omega, \xi) \in \Omega \times \Xi$  let  $\mathcal{V}(=\mathcal{V}(\omega, \xi))$  be the random vertex set defined by

$$\mathcal{V}(\omega,\xi) = \{ \mathbf{u} \in \mathbb{Z}^3 : \omega(\mathbf{u}) = 1 \}.$$

Note that if  $\mathbf{u} \in \mathcal{V}(\omega, \xi)$  for some  $\xi \in \Xi$  then  $\mathbf{u} \in \mathcal{V}(\omega, \xi')$  for all  $\xi' \in \Xi$  and thus we say that a vertex  $\mathbf{u}$  is open in a configuration  $\omega$  if  $\mathbf{u} \in \mathcal{V}(\omega, \xi)$  for some  $\xi \in \Xi$ .

For  $\mathbf{u} \in \mathbb{Z}^3$  let

$$N_{\mathbf{u}} = N_{\mathbf{u}}(\omega, \xi) = \left\{ \mathbf{v} \in \mathcal{V}(\omega, \xi) : v(3) = u(3) - 1 \text{ and } \sum_{i=1}^{3} |v(i) - u(i)| = \min\left\{ \sum_{i=1}^{3} |w(i) - u(i)| : \mathbf{w} \in \mathcal{V}(\omega, \xi), w(3) = u(3) - 1 \right\} \right\}.$$

 $N_{\mathbf{u}}$  is non-empty almost surely and that  $N_{\mathbf{u}}$  is defined for all  $\mathbf{u}$ , irrespective of it being open or closed. For  $\mathbf{u} \in \mathbb{Z}^3$  let

$$h(\mathbf{u}) \in N_{\mathbf{u}}(\omega,\xi)$$
 be such that  $U_{\mathbf{u},h(\mathbf{u})}(\xi) = \min\{U_{\mathbf{u},\mathbf{v}}(\xi) : \mathbf{v} \in N_{\mathbf{u}}(\omega,\xi)\}.$  (1)

Again note that for each  $\mathbf{u} \in \mathbb{Z}^3$ ,  $h(\mathbf{u})$  is open, almost surely unique and  $(h(\mathbf{u}))(3) = u(3) - 1$ . On  $\mathcal{V}(\omega, \xi)$  we assign the edge set  $\mathcal{E} = \mathcal{E}(\omega, \xi) := \{ < \mathbf{u}, h(\mathbf{u}) > : \mathbf{u} \in \mathcal{V}(\omega, \xi) \}$ .

Consider that graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  consisting of the vertex set  $\mathcal{V}$  and edge set  $\mathcal{E}$ . For a vertex  $\mathbf{u} \in \mathcal{V}(\omega, \xi)$ , there is exactly one edge 'going down' from  $\mathbf{u}$ , i.e., there is a unique edge  $\langle \mathbf{u}, \mathbf{v} \rangle$  with v(3) = u(3) - 1. Clearly, the edge that is 'going down' is identically distributed irrespective of the initial vertex  $\mathbf{u}$ , and has only two random components since v(3) = u(3) - 1. Therefore, we shall denote this edge by a generic,  $\mathbb{Z}^2$  valued random variable  $\xi$ . That is, for the edge  $\langle \mathbf{u}, \mathbf{v} \rangle$ , which is going down with the origin  $\mathbf{u}$ , we shall define  $\xi$  as the  $\mathbb{Z}^2$  valued random variable  $(\mathbf{u} - \mathbf{v})$ , whose distribution is given by

$$\begin{split} P(|\xi| > k) &= c \sum_{m=k+1}^{\infty} \frac{g(m) - g(k) - 3}{g(m)} m^{-\beta} \\ &= O(\frac{1}{k^{\alpha}}), \end{split}$$

where  $|\cdot|$  denotes the  $l_1$  norm on  $\mathbb{Z}^2$ , and  $\alpha = \beta - 1$ .

Now the uniqueness of the edge  $\langle \mathbf{u}, \mathbf{v} \rangle$  implies that the graph  $\mathcal{G}$  contains no loops almost surely. Hence, the graph  $\mathcal{G}$  consists of only trees. Our main result is

#### Theorem 2.1

 $\mathbb{P}(\mathcal{G} \text{ is a forest consisting of infinitely many disjoint trees }) > 0.$ 

Remark 2.2

- The model can be constructed on  $\mathbb{Z}^d$  for  $d \ge 2$ . Imitating the proof in [1], one can show that it is a tree with positive probability in d = 2 when  $1 < \alpha < 2$ .
- If the measure μ was ergodic as well then result stated in Theorem 2.1 would hold with probability 1. However we were not able to construct such a measure that provides the necessary correlation structure.
- The proof presented in [1] can be applied to our model to prove the non-existence of bi-infinite paths in this model as well.

### 3 Proof of Theorem 2.1

We begin by constructing a coupling of our random oriented tree with a system of independent random walks.

Let  $\{X_1, X_2, \ldots\}$  and  $\{Y_1, Y_2, \ldots\}$  be two independent collections of i.i.d. copies of the random variable  $\xi$ . Consider a path  $\{\mathbf{0} + (\sum_{i=1}^n X_i, -n)\}$  starting at  $\mathbf{0} \in \mathcal{V}_0$ . Next take a vertex  $\mathbf{v} = (v_1, v_2, 0)$ . If the  $l_1$ -ball  $B = \{\mathbf{u} \in \mathbb{Z}^2 : \|\mathbf{u}\|_{l_1} \leq \|X_1\|_{l_1}\}$  is disjoint from the  $l_1$ -ball  $B' = \{\mathbf{u} \in \mathbb{Z}^2 : \|\mathbf{u} - (v_1, v_2)\|_{l_1} \leq \|Y_1\|_{l_1}\}$ , then we fix  $h^1(\mathbf{v}) = \{\mathbf{v} + (Y_1, -1)\}$ , else we shall define  $h^1(\mathbf{v})$  taking into account the configuration inside the ball B. Repeating this procedure at every step, we shall obtain the pair  $\{(h^i(\mathbf{0}), h^i(\mathbf{v})) : 0 \leq i \leq n\}$  is stochastically equivalent to the pair of independent random walks  $\{\mathbf{0} + (\sum_{i=1}^n X_i, -n), \mathbf{v} + (\sum_{i=1}^n Y_i, -n)\}$ . Next we start with two vertices  $\mathbf{u} = (u, 0)$  and  $\mathbf{v} = (v, 0)$  in  $\mathbb{Z}^3$  with  $u, v \in \mathbb{Z}^2$ . Let

Next we start with two vertices  $\mathbf{u} = (\mathbf{u}, 0)$  and  $\mathbf{v} = (\mathbf{v}, 0)$  in  $\mathbb{Z}^3$  with  $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$ . Let  $\omega^{\mathbf{u}}$  and  $\omega^{\mathbf{v}}$  be two independent realizations of the  $\{0, 1\}$  configurations on the infinite lattice  $\mathbb{Z}^2$ , and let  $\{U^{\mathbf{u}} : \mathbf{u} \in \mathbb{Z}^2\}$  and  $\{U^{\mathbf{v}} : \mathbf{u} \in \mathbb{Z}^2\}$  be two independent collections of i.i.d. U([0, 1]) random variables.

Let us define

$$k_{u} = \min\{k : \omega^{u}(z) = 1 \text{ for some } z \in (u + \Delta_{k})\},\$$
  
$$l_{v} = \min\{l : \omega^{v}(z) = 1 \text{ for some } z \in (v + \Delta_{l})\},\$$

where  $(w + \Delta_m)$  is to be interpreted as the  $l_1$ -ball in  $\mathbb{Z}^2$  of radius m around w. Consequently define  $m_v$  as

$$m_{\mathbf{v}} = \min\{m : \text{either } \omega^{\mathbf{v}}(\mathbf{z}) = 1 \text{ for some } \mathbf{z} \in (\mathbf{v} + \Delta_m) \setminus (\mathbf{u} + \Delta_{k_{\mathbf{u}}}) \\ \text{or } \omega^{\mathbf{u}}(\mathbf{z}) = 1 \text{ for some } \mathbf{z} \in (\mathbf{v} + \Delta_m) \cap (\mathbf{u} + \Delta_{k_{\mathbf{u}}}) \}.$$

Next, let us define the sets

$$N_{\mathbf{u}} = \{ \mathbf{z} \in (\mathbf{u} + \Delta_{k_{\mathbf{u}}}) : \omega^{\mathbf{u}}(\mathbf{z}) = 1 \},$$
  

$$N_{\mathbf{v}}^{1} = \{ \mathbf{z} \in (\mathbf{v} + \Delta_{l_{\mathbf{v}}}) : \omega^{\mathbf{v}}(\mathbf{z}) = 1 \},$$
  

$$N_{\mathbf{v}}^{2} = \{ \mathbf{z} \in (\mathbf{v} + \Delta_{m_{\mathbf{v}}}) \setminus (\mathbf{u} + \Delta_{k_{\mathbf{u}}}) : \omega^{\mathbf{v}}(\mathbf{z}) = 1 \}$$
  

$$\cup \{ \mathbf{z} \in (\mathbf{v} + \Delta_{m_{\mathbf{v}}}) \cap (\mathbf{u} + \Delta_{k_{\mathbf{u}}}) : \omega^{\mathbf{u}}(\mathbf{z}) = 1 \}.$$

Now define

- $\phi(\mathbf{u}) \in N_{\mathbf{u}}$  such that  $U^{\mathbf{u}}(\phi(\mathbf{u})) = \min\{U^{\mathbf{u}}(\mathbf{z}) : \mathbf{z} \in N_{\mathbf{u}}\},\$
- $\zeta(\mathbf{v}) \in N_{\mathbf{v}}^1$  such that  $U^{\mathbf{v}}(\zeta(\mathbf{v})) = \min\{U^{\mathbf{v}}(\mathbf{z}) : \mathbf{z} \in N_{\mathbf{v}}^1\},\$
- $\psi(\mathbf{v}) \in N_{\mathbf{v}}^2$  such that  $U^{\mathbf{v}}(\psi(\mathbf{v})) = \min\{U^{\mathbf{v}}(\mathbf{z}) : \mathbf{z} \in N_{\mathbf{v}}^2\}.$

Finally, writing  $\phi^0(\mathbf{u}) = u$ ,  $\phi^n(\mathbf{u}) = \phi(\phi^{n-1}(\mathbf{u}))$ , and similarly  $\zeta^n(\mathbf{v})$  and  $\psi^n(\mathbf{v})$ , one can observe that

$$\{((\phi^{n}(\mathbf{u}), -n), (\zeta^{n}(\mathbf{v}), -n)) : n \ge 0\} \stackrel{d}{=} \{((\mathbf{u} + \sum_{i=1}^{n} X_{i}, -n), (\mathbf{v} + \sum_{i=1}^{n} Y_{i}, -n)) : n \ge 0\}.$$

Moreover,

$$\{((\phi^{n}(\mathbf{u}), -n), (\psi^{n}(\mathbf{v}), -n)) : n \ge 0\} \stackrel{d}{=} \{(h^{n}(\mathbf{u}, 0), h^{n}(\mathbf{v}, 0)) : n \ge 0\}.$$

Note that the above described procedure gives rise to *trees* originating from (u, 0) and (v, 0). Also, observe that  $\{(\phi^n(u), -n)\}$  describes both the random walk and the tree with (u, 0) as the origin, and if  $\Delta_{k_u} \cap \Delta_{m_v} = \emptyset$ , then  $m_v = l_v$ , hence  $\zeta_v = \psi_v$ . Therefore, the random and the tree from originating from (u, 0) are coupled and so are the random walk and the tee originating from (v, 0). In particular,  $m_v = l_v$  when both  $k_u < ||u - v||_{l_1}/2$  and  $m_v < ||u - v||_{l_1}/2$ . Assume  $k_0 = ||u - v||_{l_1}/2$ . Writing P for the measure generated by  $\zeta$  and  $\psi$ , we shall have

$$P(\{\zeta(\mathbf{v}) \neq \psi(\mathbf{v})\}) \leq P(\{\omega^{\mathbf{u}}(\mathbf{z}) = 0, \forall \mathbf{z} \in (\mathbf{u} + \Delta_{k_0})\})$$
$$\cup \{\omega^{\mathbf{v}}(\mathbf{z}) = 0, \forall \mathbf{z} \in (\mathbf{v} + \Delta_{k_0})\})$$
$$\leq 2P(\omega^{\mathbf{u}}(\mathbf{z}) = 0, \forall \mathbf{z} \in (\mathbf{u} + \Delta_{k_0}))$$
$$\leq \frac{c}{k_0^{\alpha}},$$

Using c as a generic constant, we shall have

$$P(\{\zeta(\mathbf{v}) = \psi(\mathbf{v})\}) \ge 1 - \frac{c}{\|\mathbf{u} - \mathbf{v}\|_{l_1}^{\alpha}}.$$
(2)

Now we shall list few estimates related to the two independent random walks defined in the previous section, with the understanding that u = 0 without loss of generality.

Now for some fixed  $\delta > 0$ , define

$$B_{n,\epsilon}(\mathbf{v}) = \{ \zeta^{n^{1/\alpha}}(\mathbf{v}) \in \phi^{n^{1/\alpha}}(0) + (\Delta_{n^{\frac{1}{\alpha^2}(1+\epsilon)}} \setminus \Delta_{n^{\frac{1}{\alpha^2}(1-\epsilon)}}), \\ \| \zeta^i(\mathbf{v}) - \phi^i(0) \|_{l_1} \ge n^{\delta}, \forall i = 1, \dots, n^{1/\alpha} \}, \\ E_{n,\epsilon}(\mathbf{v}) = \{ \| \zeta^i(\mathbf{v}) - \phi^i(0) \|_{l_1} \le n^{\delta}, \text{ for some } i = 1, \dots, n^{1/\alpha} \}, \\ F_{n,\epsilon}(\mathbf{v}) = \{ \zeta^{n^{1/\alpha}}(\mathbf{v}) \notin \phi^{n^{1/\alpha}}(0) + \Delta_{n^{\frac{1}{\alpha^2}(1+\epsilon)}} \}, \\ G_{n,\epsilon}(\mathbf{v}) = \{ \zeta^{n^{1/\alpha}}(\mathbf{v}) \in \phi^{n^{1/\alpha}}(0) + \Delta_{n^{\frac{1}{\alpha^2}(1-\epsilon)}} \}.$$

Clearly,

$$(B_{n,\epsilon}(\mathbf{v}))^c \subseteq E_{n,\epsilon}(\mathbf{v}) \cup F_{n,\epsilon}(\mathbf{v}) \cup G_{n,\epsilon}(\mathbf{v}).$$
(3)

In order to prove that the trees emerging from (0,0) and (v,0) do not meet, we first prove that the corresponding independent random walks do not meet if the starting points are far enough, and consequently, using the coupling argument, we can easily deduce that the trees do not merge.

We shall start with calculating some estimates on the probabilities of the sets  $E_{n,\epsilon}(\mathbf{v}), F_{n,\epsilon}(\mathbf{v})$  and  $G_{n,\epsilon}(\mathbf{v})$ .

$$P(E_{n,\epsilon}(\mathbf{v})) = P\{ \| \sum_{j=1}^{i} X_j - (\mathbf{v} + \sum_{j=1}^{i} Y_j) \|_{l_1} \le n^{\delta}, \text{ for some } i = 1, \dots, n^{1/\alpha} \}$$
  
$$= P\{ \sum_{j=1}^{i} (X_j - Y_j) \in (\mathbf{v} + \Delta_{n^{\delta}}), \text{ for some } i = 1, \dots, n^{1/\alpha} \}$$
  
$$\le P\{ \sum_{j=1}^{i} (X_j - Y_j) \in (\mathbf{v} + \Delta_{n^{\delta}}), \text{ for some } i \ge 1 \}$$
  
$$= P\{ \bigcup_{z \in (\mathbf{v} + \Delta_{n^{\delta}})} (\sum_{j=1}^{i} X_j - \sum_{j=1}^{i} Y_j = z, \text{ for some } i \ge 1 ) \}$$

Now,  $\sum_{j=1}^{i} (X_j - Y_j)$  is aperiodic, isotropic, symmetric random walk with i.i.d. steps. Therefore, using Green's function analysis for stable random walks (cf [3]) we get,

$$\begin{split} P(E_{n,\epsilon}(\mathbf{v})) &\leq \sum_{\mathbf{z} \in (\mathbf{v} + \Delta_n \delta)} P\left(\sum_{j=1}^i (X_j - Y_j) = \mathbf{z}, \text{ for some } i \geq 1\right) \\ &\leq \sum_{\mathbf{z} \in (\mathbf{v} + \Delta_n \delta)} \frac{c \, l(\mathbf{z})}{\|\mathbf{z}\|_{l_1}^2} \\ &\sim \sum_{\mathbf{z} \in (\mathbf{v} + \Delta_n \delta)} \frac{c \, \|\mathbf{z}\|_{l_1}^{\alpha} t(\mathbf{z})}{\|\mathbf{z}\|_{l_1}^2} \\ &\leq \sum_{\mathbf{z} \in (\mathbf{v} + \Delta_n \delta)} \frac{c \, \|\mathbf{z}\|_{l_1}^{\alpha + \delta^*}}{\|\mathbf{z}\|_{l_1}^2} \\ &\leq \frac{c}{n^{(2-\alpha - \delta^*)\gamma - 2\delta}}, \end{split}$$

where  $\delta^*$  is defined as  $|t(\mathbf{z})| < ||\mathbf{z}||_{l_1}^{\delta^*}$  and assuming v to be in  $\Delta_{n^{(1+\epsilon)/2\alpha^2}} \setminus \Delta_{n^{(1-\epsilon)/2\alpha^2}}$ and  $\mathbf{z} \in \mathbf{v} + \Delta_n \delta$ , we conclude that  $||\mathbf{z}|| > n^{(1-\epsilon)/\alpha^2} - n^{\delta} > n^{\gamma}$  for some  $\gamma$  thereby defining  $\gamma$ . Note that  $\epsilon > 0$  is arbitrary and can be chosen as small as we wish. Moreover the fact that  $\delta^* > 0$  can also be chosen arbitrarily small, ensures that  $\beta_1 = (2-\alpha-\delta^*)\gamma - 2\delta > 0$ . Hence, we shall have

$$\sup_{\mathbf{v}\in\Delta_{n^{(1+\epsilon)/2\alpha^2}\setminus\Delta_{n^{(1-\epsilon)/2\alpha^2}}}P(E_{n,\epsilon}(\mathbf{v})) \le \frac{c}{n^{\beta_1}}.$$
(4)

Repeating similar arguments for the set  $F_{n,\epsilon}(\mathbf{v})$  we get,

$$P(F_{n,\epsilon}(\mathbf{v})) = P\{\mathbf{v} + \sum_{j=1}^{n^{1/\alpha}} (X_j - Y_j) \notin \Delta_{n^{\frac{1}{\alpha^2}(1+\epsilon)}} \}$$
$$= P\{\|\mathbf{v} + \sum_{j=1}^{n^{1/\alpha}} (X_j - Y_j)\|_{l_1} > n^{\frac{1}{\alpha^2}(1+\epsilon)} \}.$$

Now by reinforcing the assumption that  $\mathbf{v}\in \Delta_{n^{(1+\epsilon)/2\alpha^2}}\backslash\Delta_{n^{(1-\epsilon)/2\alpha^2}}$  we obtain,

$$P(F_{n,\epsilon}(\mathbf{v})) \leq P\{ \| \sum_{j=1}^{n^{1/\alpha}} (X_j - Y_j) \|_{l_1} > K n^{\frac{1}{\alpha^2}(1+\epsilon)} \}$$
  
$$\leq P\{ \bigcup_{i=1}^{2} (|\mathbf{v} + \sum_{j=1}^{n^{1/\alpha}} (X_j - Y_j)(i)| > K \frac{n^{\frac{1}{\alpha^2}(1+\epsilon)}}{4}) \}$$
  
$$\leq 2 P\{ (|\mathbf{v} + \sum_{j=1}^{n^{1/\alpha}} (X_j - Y_j)(1)| > K \frac{n^{\frac{1}{\alpha^2}(1+\epsilon)}}{4}) \}.$$

Applying Markov's inequality to the above expression with exponent  $(\alpha - \delta^{**})$  we get,

$$P(F_{n,\epsilon}(\mathbf{v})) \leq \frac{d E |\sum_{j=1}^{n^{1/\alpha}} (X_j - Y_j)(1)|^{\alpha - \delta^{**}}}{(K/4)^{\alpha - \delta^{**}} (n^{(1+\epsilon)/\alpha^2})^{\alpha - \delta^{**}}} \\ = \frac{c n^{(\alpha - \delta^{**})/\alpha} E \left| \frac{\sum_{j=1}^{n^{1/\alpha}} (X_j - Y_j)(1)}{n^{1/\alpha}} \right|^{\alpha - \delta^{**}}}{n^{(1+\epsilon)(\alpha - \delta^{**})/\alpha^2}} \\ \leq \frac{c n^{(\alpha - \delta^{**})/\alpha} E \left( \frac{1}{n^{1/\alpha}} \sum_{j=1}^{n^{1/\alpha}} |(X_j - Y_j)(1)|^{\alpha - \delta^{**}} \right)}{n^{(1+\epsilon)(\alpha - \delta^{**})/\alpha^2}} \\ \leq \frac{c n^{(\alpha - \delta^{**})/\alpha} E |Z'(1)|^{\alpha - \delta^{**}}}{n^{(1+\epsilon)(\alpha - \delta^{**})/\alpha^2}}.$$

Here, we used Jensen's inequality to get the third line from the second, implicitly assuming that  $(\alpha - \delta^{**}) > 1$ , implying  $\alpha > 1$ . Now to get the required result, we need

$$\beta_2 = \frac{(1+\epsilon)(\alpha-\delta^{**})}{\alpha^2} - (\alpha-\delta^{**}/\alpha) > 0.$$

This is true only if

$$1 < \alpha < 1 + \epsilon, \tag{5}$$

assuming which, we get,

$$\sup_{\mathbf{v}\in\Delta_{n^{(1+\epsilon)/2\alpha^2}\setminus\Delta_{n^{(1-\epsilon)/2\alpha^2}}}P(F_{n,\epsilon}(\mathbf{v}))\leq\frac{c}{n^{\beta_2}}.$$
(6)

Now we shall estimate the probability corresponding to the set  $G_{n,\epsilon}(\mathbf{v})$ , for any  $\mathbf{v} \in \Delta_{n^{(1+\epsilon)/2\alpha^2}} \setminus \Delta_{n^{(1-\epsilon)/2\alpha^2}}$ .

$$P(G_{n,\epsilon}(\mathbf{v})) = P\left(\mathbf{v} + \sum_{j=1}^{n^{1/\alpha}} (X_j - Y_j) \in \Delta_{n^{\frac{1}{\alpha^2}(1-\epsilon)}}\right)$$

$$= P\left(\|\mathbf{v} + \sum_{j=1}^{n^{1/\alpha}} (X_j - Y_j)\|_{l_1} < n^{\frac{1}{\alpha^2}(1-\epsilon)}\right)$$

$$\leq P\left(\|\sum_{j=1}^{n^{1/\alpha}} (X_j - Y_j)\|_{l_1} < \|\mathbf{v}\|_{l_1} + n^{\frac{1}{\alpha^2}(1-\epsilon)}\right)$$

$$\leq P\left(\|\sum_{j=1}^{n^{1/\alpha}} (X_j - Y_j)\|_{l_1} < K n^{\frac{1}{\alpha^2}(1-\epsilon)}\right)$$

$$\leq P\left(\left(\bigcup_{i=1}^{2} \{\sum_{j=1}^{n^{1/\alpha}} (X_j - Y_j)(i)\} < K n^{\frac{1}{\alpha^2}(1-\epsilon)}\right)\right)$$

$$\leq 2P\left(\frac{1}{n^{\frac{1}{\alpha^2}}}|\sum_{j=1}^{n^{1/\alpha}} (X_j - Y_j)(i)| < K n^{\frac{1}{\alpha^2}(1-\epsilon)}\right)$$

In order to get an upper bound on the above probability, we shall use the stable central limit theorem<sup>1</sup>:

**Theorem 3.1**  $\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} Z_i$  converges in distribution to a stable law with character-istic exponent  $\alpha$  (0 <  $\alpha$  < 2) if and only if the distribution function F of  $Z_i$ 's satisfies the following conditions

- $\frac{F(-x)}{1-F(x)} \rightarrow \frac{c_1}{c_2}$ , for every constant k > 0

$$\frac{1-F(x)-F(-x)}{1-F(kx)-F(-kx)} \to k^{\alpha}$$

Clearly, in our case

$$P(|(X_i - Y_i)| > k) = k^{-\alpha} L_1(k) + k^{-\alpha} L_2(k),$$

where  $L_1(\cdot)$  and  $L_2(\cdot)$  are slowly varying functions, where  $L_1(\cdot) + L_2(\cdot) \downarrow$  as  $k \uparrow \infty$ . In particular  $L_1(k) \downarrow C$  and  $L_1(k) \downarrow 0$  for some constant C. These slowly varying functions determine the rate of convergence to the stable law. Therefore, we have

$$P(G_{n,\epsilon}(\mathbf{v})) \leq L(n^{1/\alpha^2} \frac{k}{n^{\epsilon/\alpha^2}}) + P_{S\alpha S}(\frac{-k}{n^{\epsilon/\alpha^2}}, \frac{k}{n^{\epsilon/\alpha^2}})$$
  
$$\leq L^*(n), \tag{7}$$

where  $L(\cdot)$  is a slowly varying function determined by the rate of convergence of the  $\alpha$ -stable central limit theorem and  $P_{S\alpha S}$  is the probability distribution corresponding to a symmetric  $\alpha$ -stable random variable.

<sup>&</sup>lt;sup>1</sup>We refer the reader to [2] for the proof of the result and on limit distributions of sums of independent random variables.

Now for  $\mathbf{u} \in \mathbb{Z}^3$  writing  $h^n(\mathbf{u}) = (g^n(\mathbf{u}), u_3 - n)$ , consider the set

$$A_{n,\epsilon}(\mathbf{v}) = \left\{ g^{n^{1/\alpha}}(\mathbf{v}) \in g^{n^{1/\alpha}}(\mathbf{0}) + \left(\Delta_{n\frac{1}{\alpha^2}(1+\epsilon)} \setminus \Delta_{n\frac{1}{\alpha^2}(1-\epsilon)}\right), \\ g^i(\mathbf{v}) \neq g^i(\mathbf{0}) \text{ for all } i = 1, \dots, n^{1/\alpha} \right\},$$
(8)

where  $\mathbf{v} = (v, 0)$  and  $\mathbf{0} = (0, 0, 0)$ .

Clearly,

$$A_{n,\epsilon}(\mathbf{v}) \supseteq B_{n,\epsilon}(\mathbf{v}) \cap \{g^i(\mathbf{0}) = \sum_{j=1}^i X_j, g^i(\mathbf{v}) = \mathbf{v} + \sum_{j=1}^i Y_j \,\forall \, 1 \le i \le n^{1/\alpha}\}.$$

Therefore,

$$\begin{split} P(A_{n,\epsilon}(\mathbf{v})) \\ &\geq P\left\{B_{n,\epsilon}(\mathbf{v}) \cap \{g^{i}(\mathbf{0}) = \sum_{j=1}^{i} X_{j}, g^{i}(\mathbf{v}) = \mathbf{v} + \sum_{j=1}^{i} Y_{j} \forall 1 \leq i \leq n^{1/\alpha}\}\right\} \\ &= P\left\{B_{n,\epsilon}(\mathbf{v}) \cap \{g^{i}(\mathbf{0}) = \sum_{j=1}^{i} X_{j}, g^{i}(\mathbf{v}) = \mathbf{v} + \sum_{j=1}^{i} Y_{j} \forall 1 \leq i \leq n^{1/\alpha} - 1\}\right\} \\ &\times P\left\{g^{n^{1/\alpha}}(\mathbf{0}) = \sum_{j=1}^{n^{1/\alpha}} X_{j}, g^{n^{1/\alpha}}(\mathbf{v}) = \mathbf{v} + \sum_{j=1}^{n^{1/\alpha}} Y_{j}\right| \\ &\quad B_{n,\epsilon}(\mathbf{v}) \cap \left\{g^{i}(\mathbf{0}) = \sum_{j=1}^{i} X_{j}, g^{i}(\mathbf{v}) = \mathbf{v} + \sum_{j=1}^{i} Y_{j} \forall 1 \leq i \leq n^{1/\alpha} - 1\}\right\} \\ &\geq P\left\{B_{n,\epsilon}(\mathbf{v}) \cap \left\{g^{i}(\mathbf{0}) = \sum_{j=1}^{i} X_{j}, g^{i}(\mathbf{v}) = \mathbf{v} + \sum_{j=1}^{i} Y_{j} \forall 1 \leq i \leq n^{1/\alpha} - 1\}\right\} \\ &\times (1 - \frac{c}{(n^{\delta})^{\alpha}}), \end{split}$$

where the last inequality is the result of (2) after observing that, given  $B_{n,\epsilon}(\mathbf{v}), g^i(\mathbf{0}) = \sum_{j=1}^{i} X_j$  and  $g^i(\mathbf{v}) = \mathbf{v} + \sum_{j=1}^{i} Y_j$  hold for all  $1 \le i \le n^{1/\alpha} - 1$ , we have  $\|g^{n^{1/\alpha}-1}(\mathbf{0}) - g^{n^{1/\alpha}-1}(\mathbf{v})\|_{l_1} \ge n^{\delta}$ . This argument when used iteratively for  $i = 1, \ldots, n^{1/\alpha} - 1$  yields,

$$P(A_{n,\epsilon}(\mathbf{v})) \geq \left(1 - \frac{c}{(n^{\delta})^{\alpha}}\right)^{n^{1/\alpha}} \times P(B_{n,\epsilon}(\mathbf{v}))$$
$$\geq \left(1 - c \frac{n^{1/\alpha}}{n^{\delta \alpha}}\right) \times P(B_{n,\epsilon}(\mathbf{v})).$$

Using the estimates from (4), (6), and (7), we obtain

$$P(A_{n,\epsilon}(\mathbf{v})) \geq (1-c\frac{n^{1/\alpha}}{n^{\delta \alpha}})(1-L^*(n)),$$

as the rate of slowly varying function dominates all other polynomial rates. Now choosing  $\delta$  such that  $(\delta \alpha - 1/\alpha) > 0$ , and arguing similarly we get

$$P(A_{n,\epsilon}(\mathbf{v})) \geq 1 - L^*(n).$$

Therefore we have proven that for large enough n,

$$\inf_{g^0(\mathbf{v})\in\Delta_{n^{(1+\epsilon)/2\alpha^2}}\setminus\Delta_{n^{(1-\epsilon)/2\alpha^2}}}P(A_{n,\epsilon}(\mathbf{v})) \ge 1 - L^*(n).$$
(9)

Next, choose f(n, i) such that

$$\sum_{i=1}^{\infty} L^*(f(n,i)) < \infty, \tag{10}$$

and f(n, 0) = n.

Now for  $i \ge 1$  and a large enough n, take  $\tau_0 = 1$  and  $\tau_i(n) = 1 + \sum_{j=0}^{i-1} (f(n,j))^{1/\alpha}$ , and with a fixed **v** define

$$B_0 = B_0(\mathbf{v}) = \{g(\mathbf{v}) \in g(\mathbf{0}) + (\Delta_{f(n,0)^{(1+\epsilon)/2\alpha^2}} \setminus \Delta_{f(n,0)^{(1-\epsilon)/2\alpha^2}})\},$$

and using it recursively define

$$\begin{split} B_i &= B_i(\mathbf{v}) \quad = \quad \{g^{\tau_i}(\mathbf{v}) \in g^{\tau_i}(\mathbf{0}) + (\Delta_{f(n,i)^{(1+\epsilon)/2\alpha^2}} \backslash \Delta_{f(n,i)^{(1-\epsilon)/2\alpha^2}}) \\ & \text{and} \ g^j(\mathbf{v}) \neq g^j(\mathbf{0}) \, \forall \, \tau_{i-1} + 1 \leq j \leq \tau_i \}. \end{split}$$

Clearly,

$$P\{g^{j}(\mathbf{v}) \neq g^{j}(\mathbf{0}) \forall j \ge 1\} \ge P\left(\bigcap_{i=0}^{\infty} B_{i}\right)$$
$$= \lim_{i \to \infty} P\left(\bigcap_{j=0}^{i} B_{j}\right)$$
$$= \lim_{i \to \infty} \prod_{l=1}^{i} P\left(B_{l} \middle| \bigcap_{j=0}^{l-1} B_{j}\right) P(B_{0}).$$

Since  $P(B_0) > 0$ , we have that  $P\{g^j(\mathbf{v}) \neq g^j(\mathbf{0}) \forall j \ge 1\} > 0$  if  $\sum_{l=1}^{\infty} (1 - P(B_l | \bigcap_{j=0}^{l-1} B_j)) < \infty$ . Now using the fact that the tree processes generated by  $h^n(\cdot)$  are jointly Markov, we have

$$\begin{split} P\Big(B_l\Big|\bigcap_{j=0}^{l-1}B_j\Big) \\ &\geq \inf_1 P\Big(g^{f(n,l)^{1/\alpha}}(\mathbf{v}_1) \in g^{f(n,l)^{1/\alpha}}(\mathbf{u}_1) + (\Delta_{f(n,l+1)^{(1+\epsilon)/2\alpha^2}} \setminus \Delta_{f(n,l+1)^{(1-\epsilon)/2\alpha^2}}); \\ &\quad g^k(\mathbf{u}_1) \neq g^k(\mathbf{v}_1) \,\forall \, 1 \leq k \leq f(n,l)^{1/\alpha}\Big) \\ &= \inf_2 P(A_{f(n,l),\epsilon}(\mathbf{u})) \\ &\geq 1 - L^*(f(n,l)), \end{split}$$

where  $inf_1$  is denoted as the infimum over the set

$$\{\mathbf{u}_1, \, \mathbf{v}_1 \in \mathbb{Z}^3: \, g^0(\mathbf{v}_1) \in g^0(\mathbf{u}_1) + (\Delta_{f(n,l)^{(1+\epsilon)/2\alpha^2}} \setminus \Delta_{f(n,l)^{(1-\epsilon)/2\alpha^2}})\},$$

and  $inf_2$  is defined as the infimum over the set

$$\{\mathbf{u}:\ g^0(\mathbf{u})\in +(\Delta_{f(n,l)^{(1+\epsilon)/2\alpha^2}}\backslash\Delta_{f(n,l)^{(1-\epsilon)/2\alpha^2}})\},$$

and the last inequality follows from (9). Thus,  $\sum_{l=1}^{\infty} \left(1 - P(B_l | \bigcap_{j=0}^{l-1} B_j)\right) < \infty$ , thereby proving that

$$P\{g^{\mathfrak{I}}(\mathbf{v}) \neq g^{\mathfrak{I}}(\mathbf{0}) \,\forall \, j \ge 1\} > 0.$$

$$(11)$$

This proves that if the original nodes are far enough then the resulting graphs emanating from these nodes remain disjoint, thus forming two separate trees.

Now to complete the proof of Theorem2.1, we need to prove that there the resulting graph composes of infinitely many trees. To prove that, we define,  $k \geq 2$  $D^k(n,\epsilon) = \{(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_k) : \mathbf{u}_i \in \mathbb{Z}^3 \text{ such that } n^{(1-\epsilon)/2\alpha^2} \leq ||g^0(\mathbf{u}_i) - g^0(\mathbf{u}_j)|| \leq n^{(1+\epsilon)/2\alpha^2} \text{ for all } i \neq j\}$ . Next, following the steps of [1] we define the event

$$A(n,\epsilon,\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_k) = \{n^{(1-\epsilon)/\alpha^2} \le \|g^0(\mathbf{u}_i) - g^0(\mathbf{u}_j)\| \le n^{(1+\epsilon)/\alpha^2} \text{ and } g^l(\mathbf{u}_i) \ne g^l(\mathbf{u}_j) \text{ for all } i \ne j\}.$$

Now repeating the calculations leading to the equation (9), we get

$$\inf P(A(n,\epsilon,\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_k): (\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_k) \in D^k(n,\epsilon)) \ge 1 - L^*(n), \quad (12)$$

where the  $\epsilon$  is the same as the one appearing in (9), and the  $L^*(\cdot)$  appearing in the above equation differs from the  $L^*$  in (9) by a constant. Thereafter, again mimicing the arguments used to obtain (11), we get

$$P\{g^{l}(\mathbf{u}_{i}) \neq g^{l}(\mathbf{u}_{j}) \text{ for all } l \ge 1 \text{ and for } 1 \le i \ne j \le k\} > 0.$$

$$(13)$$

Finally, by translation invariance of the underlying measure, we have that

 $P(\mathcal{G} \text{ contains at least } k \text{ trees}) > 0,$ 

which shows that  $\mathcal{G}$  contains infinitely many trees almost surely.

**Remark 3.2** It is easy to see that the set  $W = \{g^l(\mathbf{u}_i) \neq g^l(\mathbf{u}_j) \text{ for all } l \geq 1 \text{ and for } 1 \leq i \neq j \leq k\}$  is an invariant set, but since the measure  $\mu$  is non-ergodic, hence proving P(W) > 0 is not sufficient to prove that P(W) = 1.

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