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The lower bound theorem and minimal triangulations of sphere bundles over the circle

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Abstract

For integers $d \geq 2$ and $\varepsilon = 0$ or 1, let $S^{1,d-1}(\varepsilon)$ denote the sphere product $S^1 \times S^{d-1}$ if $\varepsilon = 0$ and the twisted sphere product $S^1 \mathbin{\underline{\times}} S^{d-1}$ if $\varepsilon = 1$. The main results of this paper are: (a) if $d \equiv \varepsilon \pmod 2$ then $S^{1,d-1}(\varepsilon)$ has a unique minimal triangulation using $2d+3$ vertices, and (b) if $d \equiv 1 - \varepsilon \pmod 2$ then $S^{1,d-1}(\varepsilon)$ has minimal triangulations (not unique) using $2d+4$ vertices. In this context, a minimal triangulation of a manifold is a triangulation using the least possible number of vertices. The second result confirms a recent conjecture of Lutz. The first result provides the first known infinite family of closed manifolds (other than spheres) for which the minimal triangulation is unique. Actually, we show that while $S^{1,d-1}(\varepsilon)$ has at most one $(2d+3)$-vertex triangulation (one if $d \equiv \varepsilon \pmod 2$, zero otherwise), in sharp contrast, the number of non-isomorphic $(2d+4)$-vertex triangulations of these d-manifolds grows exponentially with d for either choice of ε. The result in (a), as well as the minimality part in (b), is a consequence of the following result: (c) for $d \geq 3$, there is a unique $(2d+3)$-vertex simplicial complex which triangulates a non-simply connected closed pl manifold of dimension d. This amazing simplicial complex was first constructed by Kühnel in 1986. In 1987, Brehm and Kühnel proved that any non-simply connected closed pl d-manifold requires at least $2d+3$ vertices. The result (c) completely describes the case of equality in this theorem.

The proof of (c), presented in Section 1, depends crucially on Barnette's Lower Bound Theorem and the characterisation of the cases of equality in this theorem due to Kalai. In Section 2, we present a short and self-contained account of these results, mostly following ideas due to Gromov and Kalai.

1 Sphere bundles over the circle

1.1 Introduction

With a single exception in Subsection 1.4, all simplicial complexes considered here are finite. For a simplicial complex X, $V(X)$ will denote the set of all the vertices of X and $|X|$ will denote the geometric carrier of X. One says that X is a *triangulation* of the topological space $|X|$.

When a superscript (respectively, subscript) occurs in the name of a simplicial complex, it usually indicates the dimension (respectively, the number of vertices) of the complex. For instance, $S^d_{d+2}(V)$ (or simply S^d_{d+2}) stands for the $(d+2)$-vertex *standard d-sphere* whose faces are all the proper subsets of the vertex-set V. Likewise, $B^d_{d+1}(V)$ (or simply B^d_{d+1}) stands for the $(d+1)$-vertex *standard d-ball* whose faces are all subsets of the vertex-set

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V. The space $|S^d_{d+2}|$ (respectively, $|B^d_{d+1}|$) is a closed (respectively, compact) pl d-manifold with the induced piecewise linear (pl) structure from S^d_{d+2} (respectively, B^d_{d+1}). A simplicial complex X is called a *combinatorial d-sphere* (respectively, *combinatorial d-ball*) if $|X|$ (with the induced pl structure from X) is pl homeomorphic to $|S^d_{d+2}|$ (respectively, $|B^d_{d+1}|$). A simplicial complex X is said to be a *combinatorial d-manifold* if $|X|$ (with the induced pl structure) is a pl d-manifold. Equivalently, X is a combinatorial d-manifold if all its vertex links are combinatorial spheres or combinatorial balls. In this case, we also say that X is a *combinatorial triangulation* of $|X|$. (Recall that for any face α of a complex X, its *link* $\mathrm{lk}_X(\alpha)$ is the simplicial complex whose faces are the faces β of X such that $\alpha \cap \beta = \emptyset$ and $\alpha \cup \beta \in X$. Likewise, the *star* $\mathrm{st}_X(\alpha)$ of the face α has all the maximal faces $\gamma \supseteq \alpha$ of X as its maximal faces.) A simplicial complex X is a combinatorial manifold without boundary if all its vertex links are combinatorial spheres. If X is a combinatorial d-manifold with boundary then its *boundary* ∂X is the pure $(d-1)$-dimensional simplicial complex whose facets are the $(d-1)$-faces of X which are contained in a unique d-face of X. In the sequel, a combinatorial manifold will usually mean one without boundary. Thus we shall omit the qualification "without boundary".

In [5], Brehm and Kühnel showed that for $d \geq 3$, any combinatorial triangulation of a non-simply connected closed d-manifold requires at least $2d+3$ vertices. In [10], Kühnel proved that this bound is optimal by showing that there is a $(2d+3)$-vertex d-dimensional complex - herein denoted by K^d_{2d+3} - which triangulates $S^{1,d-1}(\varepsilon)$ with $d \equiv \varepsilon \pmod 2$ (in the notation of our abstract). In Subsection 1.5, we show that, up to isomorphism, K^d_{2d+3} is the unique $(2d+3)$-vertex triangulation of a non-simply connected closed pl d-manifold for all $d \geq 3$. We remark that for $d = 2$, K^d_{2d+3} is Möbius' seven-vertex torus, whose uniqueness is part of the folklore. However, the real projective plane admits a (unique) six-vertex triangulation.

It follows from our result that if $d \equiv 1 - \varepsilon \pmod 2$ then a triangulation of $S^{1,d-1}(\varepsilon)$ needs at least $2d+4$ vertices. In [12], Kühnel and Lassmann constructed a $(2d+4)$-vertex triangulation of $S^{1,d-1}(0)$ for all $d \geq 2$. In [13], Lutz has conjectured that $S^{1,d-1}(1)$ can be triangulated by $2d+4$ vertices for d even. In Subsection 1.4, we present a construction of $(2d+4)$-vertex triangulations of $S^{1,d-1}(\varepsilon)$ for $d \geq 2$, $\varepsilon = 0, 1$. Indeed, the non-isomorphic triangulations obtained are parametrized by the partitions of the number $d+1$; hence from well known results of Hardy and Ramanujan, their number grows exponentially with d. Subsection 1.2 outlines some constructions of new complexes from old: by handle addition and handle deletion. This technique originates with Walkup [18] and plays a crucial role in our proof of the uniqueness of K^d_{2d+3}, as well as in the proof of Kalai's theorem presented in Section 2. In view of its importance, we present a precise combinatorial description of the operation of handle deletion. This should be of some independent interest since such a precise description does not seem to be available in the existing literature.

1.2 Preliminaries

For $i = 1, 2$, the i-faces of a simplicial complex K are also called the *edges* and *triangles* of K, respectively. For a simplicial complex K, the graph whose vertices and edges are the vertices and edges of K is called the *edge graph* (or *1-skeleton*) of K. Recall that a *graph* is nothing but a simplicial complex of dimension ≤ 1. A set of vertices in a graph is called a *clique* if these vertices are mutually adjacent (i.e., any two of them form an edge). Note that any simplex in a simplicial complex is a clique in its edge graph.

For a simplex σ in a simplicial complex K, the number of vertices in $\mathrm{lk}_K(\sigma)$ is called the *degree* of σ in K and is denoted by $\deg_K(\sigma)$ (or by $\deg(\sigma)$). So, the degree of a vertex v in K is the same as the degree of v in the edge graph of K.

A simplicial complex K is called *pure* if all the maximal faces of K have the same dimension. A maximal face in a pure simplicial complex is also called a *facet*. For a pure d-dimensional simplicial complex K, let $\Lambda(K)$ be the graph whose vertices are the facets of K, two such vertices being adjacent in $\Lambda(K)$ if the corresponding facets intersect in a $(d-1)$-face.

Definition 1.1. For $d \geq 1$, a d-dimensional pure simplicial complex is said to be a *weak pseudomanifold* if each $(d-1)$-simplex is in exactly two facets. Clearly, any d-dimensional weak pseudomanifold has at least $d+2$ vertices, with equality only for S^d_{d+2}. A connected d-dimensional weak pseudomanifold is said to be a *normal pseudomanifold* if the links of all the simplices of dimension $\leq d-2$ are connected. Thus the 1-dimensional normal pseudomanifolds are cycles (circles) and the 2-dimensional normal pseudomanifolds are just the connected combinatorial 2-manifolds. But, normal pseudomanifolds of dimension d form a broader class than connected combinatorial d-manifolds for $d \geq 3$. In fact, any triangulation of a connected closed manifold is a normal pseudomanifold. Altshuler has constructed an 8-vertex normal pseudomanifold of dimension 3 in which each vertex link is an $\mathbb{R}P^2_7$ (cf. [6]). In [11], Kühnel has generalised it to a 2^d-vertex d-dimensional normal pseudomanifold in which each vertex link is an $\mathbb{R}P^{d-1}$.

Notice that all the links of positive dimensions (i.e., the links of simplices of dimension $\leq d-2$) in a d-dimensional normal pseudomanifold are normal pseudomanifolds. If X is a d-dimensional normal pseudomanifold then $\Lambda(X)$ is a connected $(d+1)$-regular graph. (If $\Lambda(X)$ is not connected then, since X is connected, $\Lambda(X)$ has two components G_1 and G_2 and two intersecting facets σ_1, σ_2 such that $\sigma_i \in G_i$, $i = 1, 2$. Choose σ_1, σ_2 among all such pairs such that $\dim(\sigma_1 \cap \sigma_2)$ is maximum. Then $\dim(\sigma_1 \cap \sigma_2) \leq d-2$ and $\mathrm{lk}_X(\sigma_1 \cap \sigma_2)$ is not connected, a contradiction.) This implies that a *d-dimensional normal pseudomanifold has no proper subcomplex which is also a d-dimensional normal pseudomanifold.* (Or else, the facets of such a subcomplex would provide a disconnection of $\Lambda(X)$.) In particular, a connected combinatorial d-manifold does not contain a combinatorial d-manifold as a proper subcomplex.

Let X, Y be two simplicial complexes with disjoint vertex sets. (Since we identify isomorphic complexes, this is no real restriction on X, Y.) Then their *join* $X * Y$ is the simplicial complex whose simplexes are those of X and of Y, and the (disjoint) unions of simplexes of X with simplexes of Y. It is easy to see that if X and Y are combinatorial spheres (respectively normal pseudomanifolds) then their join $X * Y$ is a combinatorial sphere (respectively normal pseudomanifold). If X consists of a single vertex x then the the join of X and Y is called the *cone* over Y and is denoted by $C(Y)$. One says that Y is the base and x is the cone-vertex of $C(Y)$.

By a *subdivision* of a simplicial complex K we mean a simplicial complex K' together with a homeomorphism from $|K'|$ onto $|K|$ which is facewise linear. Two complexes K, L have isomorphic subdivisions if and only if $|K|$ and $|L|$ are pl homeomorphic. Let X be a pure d-dimensional simplicial complex and σ be a facet of X, then take a symbol v outside $V(X)$ and consider the pure d-dimensional simplicial complex Y with vertex set $V(X) \cup \{v\}$ whose facets are facets of X other than σ and the $(d+1)$-sets $\tau \cup \{v\}$ where τ runs over the $(d-1)$-simplices in σ. Clearly, Y is a subdivision of X. The complex Y is called the subdivision obtained from X by *starring a new vertex v in the facet σ.*

If U is a non-empty subset of the vertex set $V(X)$ of a simplicial complex X then the simplices of X which are subsets of U form a simplicial complex. This simplicial complex is called the *induced subcomplex* of X on the vertex set U and is denoted by $X[U]$.

Definition 1.2. If Y is an induced subcomplex of a simplicial complex X then the *simplicial complement* $C(Y, X)$ of Y in X is the induced subcomplex of X with vertex set $V(X) \setminus V(Y)$. By abuse of notation, for any face σ of X, the induced subcomplex of X on the complement of σ will be denoted by $C(\sigma, X)$.

Definition 1.3. Let σ_1, σ_2 be two facets in a pure simplicial complex X. Let $\psi : \sigma_1 \to \sigma_2$ be a bijection. We shall say that ψ is *admissible* if (ψ is a bijection and) the distance between x and $\psi(x)$ in the edge graph of X is ≥ 3 for each $x \in \sigma_1$ (i.e., if every path in the edge graph joining x to $\psi(x)$ has length ≥ 3). Notice that if σ_1, σ_2 are from different connected components of X then any bijection between them is admissible. Also note that, in general, for the existence of an admissible map $\psi : \sigma_1 \to \sigma_2$, the facets σ_1 and σ_2 must be disjoint.

Definition 1.4. Let X be a weak pseudomanifold with disjoint facets σ_1, σ_2. Let $\psi\colon \sigma_1 \to \sigma_2$ be an admissible bijection. Let X^{ψ} denote the weak pseudomanifold obtained from $X \setminus \{\sigma_1, \sigma_2\}$ by identifying x with $\psi(x)$ for each $x \in \sigma_1$. Then X^{ψ} is said to be obtained from X by an *elementary handle addition.* If X_1, X_2 are two d-dimensional weak pseudomanifolds with disjoint vertex-sets, σ_i a facet of X_i $(i = 1, 2)$ and $\psi\colon \sigma_1 \to \sigma_2$ any bijection, then $(X_1 \sqcup X_2)^{\psi}$ is called an *elementary connected sum* of X_1 and X_2, and is denoted by $X_1 \#_{\psi} X_2$ (or simply by $X_1 \# X_2$). (Note that the combinatorial type of $X_1 \#_{\psi} X_2$ depends on the choice of the bijection ψ. However, when X_1, X_2 are connected combinatorial d-manifolds, $|X_1 \#_{\psi} X_2|$ is the topological connected sum of $|X_1|$ and $|X_2|$: independent of ψ.)

Lemma 1.1. *Let S be an induced subcomplex of a combinatorial d-manifold X. Suppose S is a combinatorial $(d-1)$-sphere and X triangulates S^d. Then the simplicial complement $C(S, X)$ has exactly two connected components.*

Proof. Note that $|X| \setminus |S|$ is the disjoint union of two open d-dimensional balls D_1, D_2. Since the two vertices in the link in X of each $(d-1)$-simplex of S must be in different components of $|X| \setminus |S|$, it follows that each D_i $(i = 1, 2)$ contains a vertex. For $i = 1, 2$, let V_i be the set of vertices of X in D_i and Y_i be the induced subcomplexes of X on the vertex-set V_i. Then $|Y_i|$ is a strong deformation retract of D_i and hence $|Y_i|$ is connected. Thus Y_1, Y_2 are the two connected components of $C(S, X)$. Hence the result. $\square$

Let N be an induced subcomplex of a simplicial complex M. One says that N is *two-sided* in M if $|N|$ has a (tubular) neighbourhood in $|M|$ homeomorphic to $|N| \times [-1, 1]$ such that the image of $|N|$ (under this homeomorphism) is $|N| \times \{0\}$.

Lemma 1.2. *Let M be a combinatorial d-manifold $(d \geq 2)$ and A be a set of vertices of M such that the induced subcomplex $M[A]$ of M on A is a connected combinatorial $(d-1)$-manifold. Let G be the graph whose vertices are the edges of M with exactly one end in A, two such vertices being adjacent in G if the union of the corresponding edges is a 2-simplex of M. Then G has at most two connected components. If, further, $M[A]$ is two-sided in M (which is automatically satisfied if either M is an orientable combinatorial 2-manifold or $M[A]$ is a $(d-1)$-sphere with $d \geq 3$) then G has exactly two connected components.*

Proof. Let $E = V(G)$ be the set of edges of M with exactly one end in A. For $x \in A$, set $E_x = \{e \in E : x \in e\}$, and let $G_x = G[E_x]$ be the induced subgraph of G on E_x. Note that G_x is isomorphic to the edge graph of $C(\mathrm{lk}_{M[A]}(x), \mathrm{lk}_M(x))$. Therefore, by Lemma 1.1, G_x has exactly two components for each $x \in A$. Also, for an edge xy in $M[A]$, there is a d-simplex σ of M such that xy is in σ. Since the induced complex $M[A]$ is $(d-1)$-dimensional, there is a vertex $u \in \sigma \setminus A$. Then $e_1 = xu \in E_x$ and $e_2 = yu \in E_y$ are adjacent in G. Thus, if x, y are adjacent vertices in $M[A]$ then there is an edge of G between E_x and E_y. Since $M[A]$ is connected and $V(G) = \cup_{x \in A} E_x$, it follows that G has at most two connected components.

Now suppose $S = M[A]$ is two-sided in M. Let U be a tubular neighbourhood of $|S|$ in $|M|$ such that $U \setminus |S|$ has two components, say U^+ and U^-. Since $|S|$ is compact, we can choose U sufficiently small so that U does not contain any vertex from $V(M) \setminus A$. Then, for $e \in E$, $|e|$ meets either U^+ or U^- but not both. Put $E^{\pm} = \{e \in E : |e| \cap U^{\pm} \neq \emptyset\}$. Then no element of E^+ is adjacent in G with any element of E^-. From the previous argument, one sees that each $x \in A$ is in an edge from E^+ and in an edge from E^-. Thus, both E^+ and E^- are non-empty. So, G is disconnected. □

Now, let X be a combinatorial d-manifold and S be an induced subcomplex of X isomorphic to S^{d-1}_{d+1}. Suppose S is two-sided in X. As above, let E be the set of all edges of X with exactly one end in S. Let E^+ and E^- be the connected components of the graph G (with vertex-set E) defined above. Notice that if a facet σ intersects $V(S)$ then σ contains edges from E, and the graph G induces a connected subgraph on the set $E_\sigma = \{e \in E : e \subseteq \sigma\}$. (Indeed, this subgraph is the line graph of a complete bipartite graph.) Consequently, either $E_\sigma \subseteq E^+$ or $E_\sigma \subseteq E^-$. Accordingly, we say that the facet σ is positive or negative (relative to S). If a facet σ of X does not intersect $V(S)$ then we shall say that σ is a neutral facet.

Let $V(S) = W$ and $V(X) \setminus V(S) = U$. Take two disjoint sets W^+ and W^-, both disjoint from U, together with two bijections $f_{\pm} \colon W \to W^{\pm}$. We define a pure simplicial complex $\widetilde{X}$ as follows. The vertex-set of $\widetilde{X}$ is $U \sqcup W^+ \sqcup W^-$. The facets of $\widetilde{X}$ are: (i) W^+, W^-, (ii) all the neutral facets of X, (iii) for each positive facet σ of X, the set $\widetilde{\sigma} := (\sigma \cap U) \sqcup f_+(\sigma \cap W)$, and (iv) for each negative facet τ of X, the set $\widetilde{\tau} := (\tau \cap U) \sqcup f_-(\tau \cap W)$.

Definition 1.5. If S is an induced two-sided S^{d-1}_{d+1} in a combinatorial d-manifold X, then the pure simplicial complex $\widetilde{X}$ constructed above is said to be obtained from X by an *elementary handle deletion* over S.

Lemma 1.3. *Let X be a combinatorial d-manifold with an induced two-sided standard $(d-1)$-sphere S. Let $\widetilde{X}$ be obtained from X by an elementary handle deletion over S. Then we have:*

(*a*) *X is obtained from $\widetilde{X}$ by elementary handle addition.*

(*b*) *The connected components of $\widetilde{X}$ are d-dimensional normal pseudomanifolds.*

(*c*) *If X is connected, then $\widetilde{X}$ has at most two connected components.*

(*d*) *If X is connected but $\widetilde{X}$ is not, then $X = Y_1 \# Y_2$, where Y_1, Y_2 are the connected components of $\widetilde{X}$.*

(*e*) *If $C(S, X)$ is connected then $\widetilde{X}$ is connected.*

Proof. With the notation as above, let $\psi = f_- \circ f_+^{-1} \colon W^+ \to W^-$. It is easy to see that ψ is admissible and $X = (\widetilde{X})^\psi$. This proves (a).

Clearly, each $(d-1)$-face in $\widetilde{X}$ is in two facets. Since the links of faces of dimension $\leq d-2$ in X are connected, it follows that the links of faces of dimension $\leq d-2$ in $\widetilde{X}$ are connected. This proves (b).

If X is connected, then choosing two vertices $f_\pm(x_0) \in W^\pm$ of $\widetilde{X}$, one sees that each vertex of $\widetilde{X}$ is joined by a path in the edge graph of $\widetilde{X}$ to either $f_+(x_0)$ or $f_-(x_0)$. Hence $\widetilde{X}$ has at most two components. This proves (c). This arguments also shows that when $\widetilde{X}$ is disconnected, W^+ and W^- are facets in different components of $\widetilde{X}$. Hence (d) follows.

Observe that $C(S, X) = C(W^+ \sqcup W^-, \widetilde{X})$. Assume that $C(S, X)$ is connected. Now, for any $(d-1)$-simplex $\tau \subseteq W^+$, there is a vertex x in $C(S, X)$ such that $\tau \cup \{x\}$ is a facet of $\widetilde{X}$. So, $C(S, X)$ and W^+ are in the same connected component of $\widetilde{X}$. Similarly, $C(S, X)$ and W^- are in the same connected component of $\widetilde{X}$. This proves (e). □

Remark 1.1. Under the hypothesis in Lemma 1.3, it is easy to see that $\widetilde{X}$ is a triangulated manifold.

Example 1.1. It is well known that the real projective plane has a unique 6-vertex triangulation, denoted by $\mathbb{R}P^2_6$. It is obtained from the boundary complex of the icosahedron by identifying antipodal vertices. The simplicial complement of any facet in $\mathbb{R}P^2_6$ is an S^1_3. But, it is not possible to obtain a combinatorial 2-manifold M by deleting the handle over this S^1_3. Such a 2-manifold would have face vector $(9, 18, 12)$ and hence Euler characteristic $\chi = 3$. But, arguing as in the proof of Lemma 1.3 (e), one can see that M must be connected - and any connected closed 2-manifold has Euler characteristic ≤ 2, a contradiction. Thus the hypothesis "two-sided" in Definition 1.5 is essential. Indeed, in this example, the graph G of Lemma 1.2 is connected: it is a 9-gon.

1.3 Stacked spheres

Let X be a pure d-dimensional simplicial complex and Y be obtained from X by starring a new vertex v in a facet σ. Clearly, Y is a normal pseudomanifold if and only if X is so. Since Y is a subdivision of X, it follows that X is a combinatorial manifold (respectively, combinatorial sphere) if and only if Y is a combinatorial manifold (respectively, combinatorial sphere). Notice that the new vertex v is of degree $d+1$ in Y, and when $d > 1$ the edge graph of X is the induced subgraph of the edge graph of Y on the vertex set $V(Y) \setminus \{v\}$.

Now, if Y is a d-dimensional normal pseudomanifold, then note that for any vertex u of Y, $\mathrm{lk}_Y(u)$ is a $(d-1)$-dimensional normal pseudomanifold, hence has at least $d+1$ vertices. Thus, each vertex of Y has degree $\geq d+1$. If u is a vertex of Y of (minimal) degree $d+1$ and the number of vertices in Y is $> d+2$, then consider the pure simplicial complex X with vertex set $V(Y) \setminus \{u\}$, whose facets are the facets of Y not passing through u, and the set of all $d+1$ neighbours of u. We say that X is obtained from Y by *collapsing* the vertex u. Clearly, this is the reverse of the operation of starring a vertex u in a facet of X.

Definition 1.6. A combinatorial d-sphere X is said to be a *stacked sphere* if there is a finite sequence $X_0, X_1, \dots, X_m$ of combinatorial d-spheres such that $X_0 = S^d_{d+2}$, the standard d-sphere, $X_m = X$ and X_i is obtained from X_{i-1} by starring a new vertex in a facet of X_{i-1} for $1 \leq i \leq m$. Thus an n-vertex stacked d-sphere is obtained from the standard d-sphere by $(n-d-2)$-fold starring. Since, for $d > 1$, each starring increases the number of edges by $d+1$, it follows that any n-vertex stacked d-sphere has exactly

$\binom{d+2}{2} + (n-d-2)(d+1) = n(d+1) - \binom{d+2}{2}$ edges. By the Lower Bound Theorem (Theorem 9 below), this is the smallest number of edges for an n-vertex normal pseudomanifold of dimension $d > 1$ and, for $d \geq 3$, this lower bound is attained only by the stacked spheres.

Lemma 1.4. *Let X be a normal pseudomanifold of dimension $d \geq 2$.*

(*a*) *If X is not the standard d-sphere then any two vertices of degree $d+1$ in X are non-adjacent.*

(*b*) *If X is a stacked sphere then X has at least two vertices of degree $d+1$.*

Proof. Let x_1, x_2 be two adjacent vertices of degree $d+1$ in X. Thus, $\mathrm{lk}(x_1) = S^{d-1}_{d+1}$, so that all the vertices in $V = V(\mathrm{st}(x_1))$ are adjacent. It follows that $V \setminus \{x_2\}$ is the set of neighbours of x_2. Hence all the facets through x_2 are contained in the $(d+2)$-set V. Since there must be a facet containing x_2 but not containing x_1, such a facet must be $V \setminus \{x_1\}$. Thus, X induces a standard d-sphere on V. Since X is a d-dimensional normal pseudomanifold, it follows that $X = S^d_{d+2}(V)$. This proves Part (a).

We prove (b) by induction on the number n of vertices of X. If $n = d+2$ then $X = S^d_{d+2}$ and the result is trivial. So assume $n > d+2$, and the result holds for all the smaller values of n. Since X is a stacked sphere, X is obtained from an $(n-1)$-vertex stacked sphere Y by starring a new vertex x in a facet σ of Y. Thus, x is a vertex of degree $d+1$ in X. If Y is the standard d-sphere then the unique vertex y in $V(Y) \setminus \sigma$ is also of degree $d+1$ in X. Otherwise, by induction hypothesis, Y has at least two vertices of degree $d+1$, and since any two of the vertices in σ are adjacent in Y - Part (a) implies that at least one of these degree $d+1$ vertices of Y is outside σ. Say $z \notin \sigma$ is of degree $d+1$ in Y. Then z (as well as x) is a vertex of degree $d+1$ in X. □

Lemma 1.5. *Let X, Y be d-dimensional normal pseudomanifolds. Suppose Y is obtained from X by starring a new vertex in a facet of X. Then Y is a stacked sphere if and only if X is a stacked sphere.*

Proof. The "if" part is immediate from the definition of stacked spheres. We prove the "only if" part by induction on the number $n \geq d+3$ of vertices of Y. The result is trivial for $n = d+3$. So, assume $n > d+3$. Let Y be obtained from X by starring a vertex x in a facet σ of X. Suppose Y is a stacked sphere. Then Y is obtained from some stacked sphere Z by starring a vertex y in a facet τ of Z. If $x = y$ then Z is obtained from Y by collapsing x, so that $X = Z$ is a stacked sphere, hence we are done. On the other hand, if $x \neq y$, then both x and y are of degree $d+1$ in Y, so that by Lemma 1.4, x and y are non-adjacent. Therefore, x is a vertex of degree $d+1$ in Z. Let W be obtained from Z by collapsing the vertex x. By induction hypothesis, W is a stacked sphere. But, X is obtained from W by starring the vertex y. Hence by the "if" part, X is a stacked sphere. □

Lemma 1.6. *The link of a vertex in a stacked sphere is a stacked sphere.*

Proof. Let X be a d-dimensional stacked sphere and v be a vertex of X. We prove the result by induction on the number n of vertices of X. The result is trivial for $n = d+2$. So, assume $n \geq d+3$ and the result is true for all stacked spheres on $\leq n-1$ vertices. Let X be obtained from an $(n-1)$-vertex stacked sphere Y by starring a vertex x in a facet σ of Y. If $v = x$ then $\mathrm{lk}_X(v)$ is a standard $(d-1)$-sphere and hence is a stacked sphere. So, assume that $v \neq x$. Since the number of vertices in Y is $n-1$, by induction hypothesis, $\mathrm{lk}_Y(v)$ is a stacked sphere. Clearly, either $\mathrm{lk}_X(v) = \mathrm{lk}_Y(v)$ or $\mathrm{lk}_X(v)$ is obtained from $\mathrm{lk}_Y(v)$ by starring x in a facet of $\mathrm{lk}_Y(v)$. In either case, $\mathrm{lk}_X(v)$ is a stacked sphere. □

Lemma 1.7. *Any stacked sphere is uniquely determined by its edge graph.*

Proof. Let X be an n-vertex d-dimensional stacked sphere with edge graph G. If $d = 1$ or $n = d+2$ then there is nothing to prove. So, assume $d > 1$, $n > d+2$, and we have the result for all smaller values of n. Let x be a vertex of degree $d+1$ in G. Let H be the induced subgraph of G on $V(G) \setminus \{x\}$, and let Y be obtained from X by collapsing the vertex x. By Lemma 1.5, Y is a stacked sphere; H is its edge graph. By induction hypothesis, H (and hence G) determines Y. Then all the facets of X not containing x are determined by G. Also, the facets of X through x are determined as the $(d+1)$-sets consisting of x together with d of its neighbours. □

Lemma 1.8. *Let X_1, X_2 be d-dimensional normal pseudomanifolds. Then* (*a*) *$X_1 \# X_2$ is a combinatorial* 2*-sphere if and only if both X_1 and X_2 are combinatorial* 2*-spheres; and* (*b*) *$X_1 \# X_2$ is a stacked d-sphere if and only if both X_1, X_2 are stacked d-spheres.*

Proof. Let $d = 2$. Then X_1, X_2 are connected combinatorial 2-manifolds and hence $X_1 \# X_2$ is a connected combinatorial 2-manifold. For $0 \leq i \leq 2$, $1 \leq j \leq 2$, let $f_i(X_j)$ denote the number of i-faces in X_j. Then, from the definition, $\chi(X_1 \# X_2) = (f_0(X_1) + f_0(X_2) - 3) - (f_1(X_1) + f_1(X_2) - 3) + (f_2(X_1) + f_2(X_2) - 2) = \chi(X_1) + \chi(X_2) - 2$. Part (*a*) now follows from the fact that the Euler characteristic of a connected closed 2-manifold M is ≤ 2 and equality holds if and only if M is a 2-sphere.

We prove Part (*b*) by induction on the number $n \geq d+3$ of vertices in $X_1 \# X_2$. If $n = d+3$ then both X_1, X_2 must be standard d-spheres (hence stacked spheres) and then $X_1 \# X_2 = S^0_2 * S^{d-1}_{d+1}$ is easily seen to be a stacked sphere. So, assume $n > d+3$, so that at least one of X_1, X_2 is not the standard d-sphere. Without loss of generality, say X_1 is not the standard d-sphere. Of course, $X = X_1 \# X_2$ is not a standard d-sphere. Let X be obtained from $X_1 \sqcup X_2 \setminus \{\sigma_1, \sigma_2\}$ by identifying a facet σ_1 of X_1 with a facet σ_2 of X_2 by some bijection. Then, $\sigma_1 = \sigma_2$ is a clique in the edge graph of X, though it is not a facet of X. Notice that a vertex $x \in V(X_1) \setminus \sigma_1$ is of degree $d+1$ in X_1 if and only if it is of degree $d+1$ in X. If either X_1 is a stacked sphere or X is a stacked sphere then, by Lemma 1.4, such a vertex x exists. Let $\widetilde{X}_1$ (respectively, $\widetilde{X}$) be obtained from X_1 (respectively, X) by collapsing this vertex x. Notice that $\widetilde{X} = \widetilde{X}_1 \# X_2$. Therefore, by induction hypothesis and Lemma 1.5, we have: X is a stacked sphere $\Longleftrightarrow$ $\widetilde{X}$ is a stacked sphere $\Longleftrightarrow$ both $\widetilde{X}_1$ and X_2 are stacked spheres $\Longleftrightarrow$ both X_1 and X_2 are stacked spheres. □

Definition 1.7. For $d \geq 2$, $\mathcal{K}(d)$ will denote the family of all combinatorial d-manifolds X such that the link of each vertex of X is a $(d-1)$-dimensional stacked sphere. Notice that, Lemma 1.6 says that all stacked d-spheres belong to the class $\mathcal{K}(d)$. Also, for $d \geq 2$, K^d_{2d+3} and all the simplicial complexes $K^d_{2d+4}(p)$ constructed in Subsection 1.4 are in the class $\mathcal{K}(d)$ (cf. Proof of Lemma 1.11).

Lemma 1.9 (Walkup [18]). *Let X be a normal pseudomanifold and $\psi \colon \sigma_1 \to \sigma_2$ be an admissible bijection, where σ_1, σ_2 are facets of X. Then* (*a*) *X^{ψ} is a combinatorial* 3*-manifold if and only if X is a combinatorial* 3*-manifold; and* (*b*) *$X^{\psi} \in \mathcal{K}(d)$ if and only if $X \in \mathcal{K}(d)$.*

Proof. For a vertex v of X, let $\bar{v}$ denote the corresponding vertex of X^{ψ}. Observe that $\mathrm{lk}_{X^{\psi}}(\bar{v})$ is isomorphic to $\mathrm{lk}_X(v)$ if $v \in V(X) \setminus (\sigma_1 \cup \sigma_2)$ and $\mathrm{lk}_{X^{\psi}}(\bar{v}) = \mathrm{lk}_X(v) \# \mathrm{lk}_X(\psi(v))$ if $v \in \sigma_1$. The results now follow from Lemma 1.8. □

Theorem 1. *For $d \geq 2$, there is a unique $(3d+4)$-vertex stacked d-sphere $\mathcal{S} = \mathcal{S}^d_{3d+4}$ which has a pair of facets with an admissible bijection between them. Further, this pair of facets and the admissible bijection between them is unique up to automorphisms of $\mathcal{S}$.*

Proof. Uniqueness: Let V^+ and V^- be two (disjoint) facets in a $(3d+4)$-vertex stacked d-sphere $\mathcal{S}$, and $\psi\colon V^+ \to V^-$ be an admissible bijection. Put $V(\mathcal{S}) = U \sqcup V^+ \sqcup V^-$. Thus, $\#(U) = d+2$. Since ψ is admissible, for each $x \in V^+$, none of the $3d+2$ vertices of $\mathcal{S}$ other than x and $\psi(x)$ is adjacent (in the edge graph of $\mathcal{S}$) with both x and $\psi(x)$. Further, x and $\psi(x)$ are non-adjacent. Therefore,

$$\deg(x) + \deg(\psi(x)) \leq 3d+2, \ x \in V^+. \tag{1}$$

Also, for $y \in U$, y is adjacent to at most one vertex in the pair $\{x, \psi(x)\}$ for each $x \in V^+$, and these $d+1$ pairs partition $V(\mathcal{S}) \setminus U$. So, each $y \in U$ has at most $d+1$ neighbours outside U. Since y can have at most $d+1 = \#(U \setminus \{y\})$ neighbours in U, it follows that

$$\deg(y) \leq 2d+2, \ y \in U. \tag{2}$$

From (1) and (2), we get by addition,

$$\sum_{x \in V^+} \deg(x) + \sum_{x \in V^+} \deg(\psi(x)) + \sum_{y \in U} \deg(y) \leq (d+1)(3d+2) + (d+2)(2d+2) = (d+1)(5d+6).$$

Now, the left hand side in this inequality is the sum of the degrees of all the vertices of $\mathcal{S}$, which equals twice the number of edges of $\mathcal{S}$. Thus $\mathcal{S}$ has at most $(d+1)(5d+6)/2$ edges. But, as $\mathcal{S}$ is a $(3d+4)$-vertex stacked d-sphere and $d \geq 2$, it has exactly $(3d+4)(d+1) - \binom{d+2}{2} = (d+1)(5d+6)/2$ edges. Hence we must have equality in (1) and (2). Thus we have equality throughout the arguments leading to (1) and (2). Therefore we have: (a) U is a $(d+2)$-clique in the edge graph G of $\mathcal{S}$, and (b) for each $y \in U$ and $x \in V^+$, y is adjacent to exactly one of the vertices x and $\psi(x)$. Notice that, since U, V^+ and V^- are cliques and there is no edge between V^+ and V^-, it follows that G is completely determined by its (bipartite) subgraph H whose edges are the edges of G between U and V^+.

Let $0 \leq m \leq d+1$.

Claim. There exist x_i^+, $1 \leq i \leq m$, in V^+ and y_i, $1 \leq i \leq m$, in U such that for each i $(1 \leq i \leq m)$, the i vertices $y_1, \dots, y_i$ are the only vertices from U adjacent to x_i^+. Further, there is a stacked d-sphere $X(m)$ with vertex-set $V(\mathcal{S}) \setminus \{x_i^+ : 1 \leq i \leq m\}$ whose edge graph is the induced subgraph G_m of G on this vertex set.

We prove the claim by finite induction on m. The claim is trivially correct for $m = 0$ (take $X(0) = \mathcal{S}$, $G_0 = G$). So, assume $1 \leq m \leq d+1$ and the claim is valid for all smaller values of m. By Lemma 1.4, $X(m-1)$ has at least two vertices of degree $d+1$ and they are non-adjacent in G_{m-1}. Since each vertex of U has degree $2d+2$ in G, it has degree $\geq 2d+2-(m-1) > d+1$ in G_{m-1}. Since V^- is a clique of G_{m-1}, at least one of the degree $d+1$ vertices of G_{m-1} is in $V^+ \setminus \{x_i^+ : 1 \leq i < m\}$. Let x_m^+ be a vertex of degree $d+1$ in G_{m-1} from $V^+ \setminus \{x_i^+ : 1 \leq i < m\}$. Notice that x_{m-1}^+ is a vertex of degree $d+1$ in $X(m-2)$; its set of neighbours in G_{m-2} is $\{y_j : 1 \leq j \leq m-1\} \sqcup (V^+ \setminus \{x_i^+ : 1 \leq i \leq m-1\})$. Since $\mathrm{lk}_{X(m-2)}(x_{m-1}^+)$ is an S_{d+1}^{d-1}, all the neighbours of x_{m-1}^+ are mutually adjacent (in G_{m-2} and hence) in G. Thus, the vertices y_j, $1 \leq j \leq m-1$, are adjacent in G with each vertex in $V^+ \setminus \{x_i^+ : 1 \leq i \leq m-1\}$. In particular, x_m^+ is adjacent (in G and hence) in G_{m-1} to the $m-1$ vertices y_j, $1 \leq j \leq m-1$ in U. It is also adjacent to the $d+1-m$ vertices in $V^+ \setminus \{x_i^+ : 1 \leq i \leq m\}$ and to no vertex in V^-. Since x_m^+ is of degree $d+1$ in G_{m-1}, it follows that there is a unique vertex $y_m \in U \setminus \{y_i : 1 \leq i \leq m-1\}$ which is adjacent to x_m^+ (in G_{m-1} and hence) in G. By construction, $y_1, \ldots, y_m$ are the only vertices in U adjacent to x_m^+. Now, let $X(m)$ be obtained from $X(m-1)$ by collapsing the vertex x_m^+ of degree $d+1$. By Lemma 1.5, $X(m)$ is a stacked sphere. Its edge graph is the induced subgraph G_m of G on the vertex-set $V(\mathcal{S}) \setminus \{x_i^+ : 1 \leq i \leq m\}$. This completes the induction step and hence proves the claim.

Now, by the final step $m = d+1$, we have named the vertices in V^+ as x_i^+, $1 \leq i \leq d+1$. We have also named $d+1$ of the vertices in U as y_i, $1 \leq i \leq d+1$. Let y_{d+2} be the unique vertex in $U \setminus \{y_i : 1 \leq i \leq d+1\}$. Also, put $x_i^- = \psi(x_i^+) \in V^-$, $1 \leq i \leq d+1$. Thus, x_i^- is adjacent to y_j if and only if x_i^+ is non-adjacent with y_j. This completes the description of the edge graph G of $\mathcal{S}$. The vertices of G are x_i^+, x_i^- $(1 \leq i \leq d+1)$ and y_j, $1 \leq j \leq d+2$. x_i^+ and x_j^+ (as well as x_i^- and x_j^-) are adjacent in G for $i \neq j$. y_i and y_j are adjacent in G for $i \neq j$. x_i^+ and x_j^- are non-adjacent in G for all i, j. x_i^+ and y_j are adjacent in G if and only if $j \leq i$. x_i^- and y_j are adjacent in G if and only if $j > i$.

Since the edge graph G is thus completely determined by the given datum, Lemma 1.7 implies that $\mathcal{S}$ is uniquely determined. Notice that the graph G has maximum vertex degree $2d+2$, and the set U is uniquely determined by G as the set of its vertices of maximum degree. Also, the facets V^+, V^- are determined by G as the connected components of the induced subgraph of G on the complement of U. Finally, the above argument shows that the admissible bijection $\psi \colon V^+ \to V^-$ is also determined by G since it must map the unique vertex of degree $d+i$ in V^+ to the unique vertex of degree $2d+2-i$ in V^- $(1 \leq i \leq d+1)$. Notice that there is an automorphism of order two which interchanges x_i^+ and x_{d+2-i}^- for each i and interchanges y_j and y_{d+3-j} for each j. This automorphism interchanges V^+ and V^- and replaces ψ by ψ^{-1}. This completes the uniqueness proof.

Existence of $\mathcal{S}_{3d+4}^d$: Consider the graph G explicitly described above. We sequence the $2d+2$ vertices in $V(G) \setminus U$ as follows:

$$v_i = \begin{cases} x_i^+, & 1 \leq i \leq d+1, \\ x_{2d+3-i}^-, & d+2 \leq i \leq 2d+2. \end{cases}$$

For $0 \leq m \leq 2d+2$, let G_m denote the induced subgraph of G on the vertex-set $V(G) \setminus \{v_i : 1 \leq i \leq m\}$. Notice that G_{2d+2} is the complete graph on the vertex set U, while $G_0 = G$. For $1 \leq m \leq 2d+2$, let σ_m denote the set of all $d+1$ neighbours of v_m in G_{m-1}. We use backward induction on m to construct, for $0 \leq m \leq 2d+2$, a simplicial complex X_m such that G_m is the edge graph of X_m and (when $m > 0$) σ_m is a facet of X_m. Let X_{2d+2} be the standard d-sphere on the vertex set $U = V(G_{2d+2})$. Clearly, G_{2d+2} is the edge graph, and σ_{2d+2} is a facet of X_{2d+2}. Now, for $0 \leq m < 2d+2$, having constructed X_{m+1}, we construct X_m from X_{m+1} by starring the vertex v_{m+1} in the facet σ_{m+1} of X_{m+1}. When $m > 0$, as $v_{m+1} \in \sigma_m \subseteq \sigma_{m+1} \cup \{v_{m+1}\}$, it follows that σ_m is a facet of X_m. Clearly, the edge graph of X_m is the induced subgraph of G_{m+1} on $V(X_m)$, namely G_m. By Lemma 1.5, each X_m is a stacked d-sphere. In particular, $\mathcal{S}_{3d+4}^d := X_0$ is a stacked sphere with edge graph G. $\Box$

Remark 1.2. (a) The proof of Theorem 1, in conjunction with the Lower Bound Theorem, actually shows the following. If X is an n-vertex d-dimensional normal pseudomanifold with an admissible bijection, then $n \geq 3d+4$, and equality holds only for $X = \mathcal{S}_{3d+4}^d$. (b) If ψ is the admissible bijection on $\mathcal{S}_{3d+4}^d$, then it is possible to verify directly that $(\mathcal{S}_{3d+4}^d)^\psi = K_{2d+3}^d$. This is also immediate from the proof of Theorem 3 below.

1.4 Some Examples

Recall that for any positive integer n, a *partition* of n is a finite weakly increasing sequence of positive integers adding to n. The terms of the sequence are called the *parts* of the partition. Let's say that a partition of n is *even* (respectively, odd) if it has an even (respectively, odd) number of even parts. Let $P(n)$ (respectively $P_0(n)$, respectively $P_1(n)$) denote the total number of partitions (respectively even partitions, respectively odd partitions) of n.

To appreciate the construction given below, it is important to understand the growth rate of these number theoretic functions P_ε, $\varepsilon = 0, 1$. Recall that if f, g are two real valued functions on the set of positive integers, then one says that f, g are *asymptotically equal* (in symbols, $f(n) \sim g(n)$) if $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 1$. A famous theorem of Hardy and Ramanujan (cf. [14]) says that

$$P(n) \sim \frac{c_1}{n} e^{c_2\sqrt{n}} \quad \text{as} \quad n \to \infty, \tag{3}$$

where the absolute constants c_1, c_2 are given by

$$c_1 = \frac{1}{4\sqrt{3}}, \quad c_2 = \pi\sqrt{\frac{2}{3}}.$$

We observe that:

Lemma 1.10. *$P_0(n) \sim \frac{c_1}{2n} e^{c_2\sqrt{n}}$, $P_1(n) \sim \frac{c_1}{2n} e^{c_2\sqrt{n}}$ as $n \to \infty$.*

Proof. In view of (3), it suffices to show that $P_0(n) \sim \frac{1}{2}P(n)$, $P_1(n) \sim \frac{1}{2}P(n)$ as $n \to \infty$. Now, $(p_1, \dots, p_k) \mapsto (1, p_1, \dots, p_k)$ is a one to one function from the set of even (respectively, odd) partitions of $n-1$ to the set of even (respectively, odd) partitions of n. Also, $(p_1, \dots, p_k) \mapsto (p_1, \dots, p_{k-1}, p_k+1)$ is a one to one function from the set of even (respectively, odd) partitions of $n-1$ to the set of odd (respectively, even) partitions of n. Therefore, $\min(P_0(n), P_1(n)) \geq \max(P_0(n-1), P_1(n-1))$. Since $P_0(n-1) + P_1(n-1) = P(n-1)$, it follows that

$$P_0(n) \geq \frac{1}{2}P(n-1) \quad \text{and} \quad P_1(n) \geq \frac{1}{2}P(n-1).$$

But, from (3) it follows that $P(n-1) \sim P(n)$. Therefore, $\liminf\limits_{n\to\infty} \frac{P_0(n)}{P(n)} \geq \frac{1}{2}$, $\liminf\limits_{n\to\infty} \frac{P_1(n)}{P(n)} \geq \frac{1}{2}$. But, $P_0(n) + P_1(n) = P(n)$. Therefore, $\lim\limits_{n\to\infty} \frac{P_0(n)}{P(n)} = \frac{1}{2} = \lim\limits_{n\to\infty} \frac{P_1(n)}{P(n)}$. $\square$

The Construction: For $d \geq 2$, let N^{d+1} denote the pure $(d+1)$-dimensional simplicial complex with vertex-set $\mathbb{Z}$ (the set of all integers) such that the facets of N^{d+1} are the sets of $d+2$ consecutive integers. Then N^{d+1} is a combinatorial $(d+1)$-manifold with boundary $M^d = \partial N^{d+1}$. Now, M^d is a combinatorial d-manifold ($\in \mathcal{K}(d)$) and triangulates $\mathbb{R} \times S^{d-1}$ (cf. [10]). Clearly, the facets of M^d are of the form $\sigma_{n,i} := \{n, n+1, \dots, n+d+1\} \setminus \{n+i\}$, $1 \leq i \leq d$, $n \in \mathbb{Z}$ (intervals of length $d+2$ minus an interior point).

For $m \geq 1$, let N^{d+1}_{m+d+1} (respectively, M^d_{m+d+1}) denote the induced subcomplex of N^{d+1} (respectively, M^d) on $m+d+1$ consecutive vertices (without loss of generality we may take $V(N^{d+1}_{m+d+1}) = V(M^d_{m+d+1}) = \{1, 2, \dots, m+d+1\}$). Clearly, M^d_{m+d+1} triangulates $[0,1] \times S^{d-1}$ and $\partial M^d_{m+d+1} = S^{d-1}_{d+1}(A_m) \sqcup S^{d-1}_{d+1}(B_m)$, where $A_m = \{1, \dots, d+1\}$ and $B_m = \{m+1, \dots, m+d+1\}$.

Lemma 1.11. *(a) ∂N^{d+1}_{m+d+1} is a stacked d-sphere and A_m, B_m are two of its facets. (b) If $\psi\colon B_m \to A_m$ is an admissible bijection then $X^d_m(\psi) := (\partial N^{d+1}_{m+d+1})^\psi$ is a combinatorial d-manifold and triangulates $S^{1,d-1}(\varepsilon)$, where $\varepsilon = 0$ if $X^d_m(\psi)$ is orientable and $\varepsilon = 1$ otherwise.*

Proof. Observe that ∂N^{d+1}_{d+2} is the standard d-sphere and for $m \geq 2$, ∂N^{d+1}_{m+d+1} is obtained from ∂N^{d+1}_{m+d} by starring the new vertex $m+d+1$ in the facet $B_{m-1} = \{m, \dots, m+d\}$ of ∂N^{d+1}_{m+d}. Thus, ∂N^{d+1}_{m+d+1} is a stacked d-sphere. A_m is a facet of ∂N^{d+1}_{i+d+1} for all $i \geq 1$ and from construction, B_m is a facet of ∂N^{d+1}_{m+d+1}. This proves (a).

Thus, by Lemma 1.6, ∂N^{d+1}_{m+d+1} is in $\mathcal{K}(d)$. Then, by Lemma 1.9 (b), $X^d_m(\psi)$ is in the class $\mathcal{K}(d)$. In consequence, $X^d_m(\psi)$ is a combinatorial d-manifold. Since M^d_{m+d+1} triangulates $[0,1] \times S^{d-1}$ and $M^d_{m+d+1} = \partial N^{d+1}_{m+d+1} \setminus \{A_m, B_m\}$, it follows that $X^d_m(\psi)$ $(= (\partial N^{d+1}_{m+d+1})^{\psi})$ triangulates an S^{d-1}-bundle over S^1. But, there are only two such bundles: $S^{1,d-1}(\varepsilon)$, $\varepsilon = 0, 1$ (cf. [16, pages 134–135]). This is orientable for $\varepsilon = 0$ and non-orientable for $\varepsilon = 1$. Hence the result. $\square$

Notice that $x \in B_m$ is at a distance ≥ 3 from $y \in A_m$ (in the edge graph of ∂N^{d+1}_{m+d+1}) if and only if $x - y \geq 2d+3$. Therefore, if $m \leq 2d+2$, it is easy to see that there is no admissible bijection $\psi \colon B_m \to A_m$. For $m \geq 2d+3$ the map $\psi_0 \colon B_m \to A_m$ given by $\psi_0(m+i) = i$ is admissible. When $m = 2d+3$, it is the only admissible map and the resulting combinatorial manifold $X^d_{2d+3}(\psi_0)$ is Kühnel's K^d_{2d+3}, triangulating $S^{1,d-1}(\varepsilon)$, $d \equiv \varepsilon \pmod 2$, whose uniqueness we prove in Subsection 1.5 below. For $m \geq 2d+3$, Kühnel and Lassmann constructed $X^d_m(\psi_0)$ and proved that for m odd $X^d_m(\psi_0)$ is orientable if and only if d is even (cf. [12]). Here we have:

Lemma 1.12. *Let $m \geq 2d+3$. If md is even then for any admissible $\psi \colon B_m \to A_m$, the combinatorial d-manifold $X^d_m(\psi)$ is orientable if and only if $\psi \circ \psi_0^{-1}$ is an even permutation. In other words, if $\psi \circ \psi_0^{-1}$ is an even (respectively, odd) permutation then $X^d_m(\psi)$ is a combinatorial triangulation of $S^{1,d-1}(0)$ (respectively, $S^{1,d-1}(1)$).*

Proof. For $1 \leq k \leq m$, $1 \leq i \leq d$, let $\sigma_{k,i}$ denote the facet $\{k, k+1, \dots, k+d+1\} \setminus \{k+i\}$ and for $0 \leq i < j \leq d+1$, $(i,j) \neq (0, d+1)$, let $\sigma_{k,i,j}$ denote the $(d-1)$-simplex $\{k, k+1, \dots, k+d+1\} \setminus \{k+i, k+j\}$ of M^d_{m+d+1}. Consider the orientation on M^d_{m+d+1} given by:

$$\begin{aligned} +\sigma_{k,i,j} &= (-1)^{kd+i+j} \langle k, \dots, k+i-1, k+i+1, \dots, k+j-1, k+j+1, \dots, k+d+1 \rangle, \\ +\sigma_{k,i} &= (-1)^{kd+i} \langle k, k+1, \dots, k+i-1, k+i+1, \dots, k+d+1 \rangle. \end{aligned} \tag{4}$$

By an easy computation one sees that the incidence numbers satisfy the following: $[\sigma_{k,i}, \sigma_{k,i,j}] = -1$, $[\sigma_{k,j}, \sigma_{k,i,j}] = 1$ for $1 \leq i < j \leq d$, $1 \leq k \leq m$ and $[\sigma_{k,i}, \sigma_{k,0,i}] = 1$, $[\sigma_{k+1,i-1}, \sigma_{k,0,i}] = [\sigma_{k+1,i-1}, \sigma_{k+1,i-1,d+1}] = (-1)^{2d-1} = -1$ for $1 \leq i \leq d$, $1 \leq k < m$. Thus, (4) gives an orientation on M^d_{m+d+1}.

Let $\bar{\sigma}_{k,i}$ and $\bar{\sigma}_{k,i,j}$ denote the corresponding simplices in $X^d_m(\psi_0)$. Observe that $\bar{\sigma}_{k,0,j} = \bar{\sigma}_{k+1,j-1,d+1}$ for $1 \leq k < m$ and $\bar{\sigma}_{m,0,j} = \bar{\sigma}_{1,j-1,d+1}$. (The vertex-set of $X^d_m(\psi_0)$ is the set of integers modulo m.) Then the above orientation induces an orientation on $X^d_m(\psi_0)$. (This is well defined since $+\sigma_{m,0,j} = (-1)^{md+j} \langle m+1, \dots, m+j-1, m+j+1, \dots, m+d+1 \rangle = (-1)^j \langle 1, \dots, j-1, j+1, \dots, d+1 \rangle = (-1)^{d+(j-1)+(d+1)} \langle 1, \dots, j-1, j+1, \dots, d+1 \rangle = +\sigma_{1,j-1,d+1}$.) Now, $[\bar{\sigma}_{m,j}, \bar{\sigma}_{m,0,j}] = 1$, $[\bar{\sigma}_{1,j-1}, \bar{\sigma}_{m,0,j}] = [\bar{\sigma}_{1,j-1}, \bar{\sigma}_{1,j-1,d+1}] = -1$. Thus, $[\bar{\sigma}_{m,j}, \bar{\sigma}_{m,0,j}] = -[\bar{\sigma}_{1,j-1}, \bar{\sigma}_{m,0,j}]$. Therefore, the induced orientation on $X^d_m(\psi_0)$ is coherent. So, $X^d_m(\psi_0)$ is orientable. This implies that $X^d_m(\psi_0)$ triangulates $S^1 \times S^{d-1} = S^{1,d-1}(0)$.

Since $|M^d_{m+d+1}|$ is homeomorphic to $|S^{d-1}_{d+1}(B_m)| \times [0,1]$, we can choose an orientation on $|S^{d-1}_{d+1}(B_m)|$ so that the orientation on $|M^d_{m+d+1}|$ as the product $|S^{d-1}_{d+1}(B_m)| \times [0,1]$ is the same as the orientation given in (4). This also induces an orientation on $|S^{d-1}_{d+1}(A_m)|$. Let S_B (respectively, S_A) denote the oriented sphere $|S^{d-1}_{d+1}(B_m)|$ (respectively $|S^{d-1}_{d+1}(A_m)|$) with this orientation. Then, as the boundary of an oriented manifold, $\partial(|M^d_{d+m+1}|) = S_A \cup (-S_B)$. [In fact, it is not difficult to see that the orientation defined in (4) on $S^{d-1}_{d+1}(A_m)$ (respectively $S^{d-1}_{d+1}(B_m)$) is the same as the orientation in S_A (respectively S_B).]

Let $|\psi_0|: S_B \to S_A$ be the homeomorphism induced by ψ_0. Since $|X^d_m(\psi_0)|$ is orientable, it follows that $|\psi_0|: S_B \to S_A$ is orientation preserving (cf. [16, pages 134–135]).

Therefore, $\psi \circ \psi_0^{-1}$ is an even (respectively odd) permutation $\Longrightarrow |\psi \circ \psi_0^{-1}|: S_A \to S_A$ is orientation preserving (respectively reversing) $\Longrightarrow |\psi| = |\psi \circ \psi_0^{-1}| \circ |\psi_0|: S_B \to S_A$ is orientation preserving (respectively reversing) $\Longrightarrow |X^d_m(\psi)|$ is orientable (respectively non-orientable). Hence, the result follows from Lemma 1.11. □

Now take $m = 2d+4$. A bijection $\psi: \{2d+5, \ldots, 3d+5\} \to \{1, \ldots, d+1\}$ is admissible for ∂N^{d+1}_{3d+5} if and only if $x - \psi(x) \geq 2d+3$ for $2d+5 \leq x \leq 3d+5$. It turns out that there are 2^d distinct admissible choices for ψ. But it seems difficult to decide when two admissible choices for ψ yield isomorphic complexes $X^d_{2d+4}(\psi)$. So, we specialize as follows:

Let $p = (p_1, p_2, \ldots, p_k)$ be a partition of $d+1$. Put $s_0 = 0$ and $s_j = \sum_{i=1}^{j} p_i$ for $1 \leq j \leq k$. (Thus, in particular, $s_1 = p_1$ and $s_k = d+1$.) Let π_p be the permutation of $\{1, 2, \ldots, d+1\}$ which is the product of k disjoint cycles $(s_{j-1}+1, s_{j-1}+2, \ldots, s_j)$, $1 \leq j \leq k$. Notice that π_p is an even (respectively, odd) permutation if p is an even (respectively, odd) partition of $d+1$. Now, define the bijection $\psi_p: \{2d+5, 2d+6, \ldots, 3d+5\} \to \{1, 2, \ldots, d+1\}$ by $\psi_p(2d+4+i) = \pi_p(i)$, $1 \leq i \leq d+1$. Since $\pi_p(i) \leq i+1$ for $1 \leq i \leq d+1$, it follows that ψ_p is an admissible bijection. Clearly, the corresponding complex $X^d_{2d+4}(\psi_p)$ depends only on the partition p of $d+1$. We denote it by $K^d_{2d+4}(p)$. Note that $\pi_p = \psi_p \circ \psi_0^{-1}$. Therefore, by Lemma 1.12, $K^d_{2d+4}(p)$ triangulates $S^{1,d-1}(0)$ (respectively, $S^{1,d-1}(1)$) if p is an even (respectively odd) partition of $d+1$.

Let G_p denote the non-edge graph of $K^d_{2d+4}(p)$. Its vertex-set is $V(K^d_{2d+4}(p))$, and two distinct vertices x, y are adjacent in G_p if xy is not an edge of $K^d_{2d+4}(p)$. It turns out that G_p has a clear description in terms of the partition p. For $b \geq 1$, let $K_{1,b}$ denote the unique graph with one vertex of degree b and b vertices of degree one. Also, let $p = (p_1, p_2, \ldots, p_k)$, and put $p_0 = 1$. Then a computation shows that G_p is the disjoint union of K_{1,p_i}, $0 \leq i \leq k$. Thus, if p and q are distinct partitions of $d+1$ then G_p and G_q are non-isomorphic (this is where our assumption that p, q are weakly increasing sequences comes into play!) and hence $K^d_{2d+4}(p)$ and $K^d_{2d+4}(q)$ are non-isomorphic complexes. Thus we have proved:

Theorem 2. *For any partition p of $d+1 \geq 3$, let $\varepsilon = \varepsilon(p) = 0$ if p is even and $=$ 1 if p is odd. Then $K^d_{2d+4}(p)$ is a $(2d+4)$-vertex triangulation of $S^{1,d-1}(\varepsilon)$. Further, distinct partitions p of $d+1$ correspond to non-isomorphic triangulations of $S^{1,d-1}(\varepsilon)$. In consequence, for $\varepsilon = 0, 1$, there are $(2d+4)$-vertex combinatorial triangulations of $S^{1,d-1}(\varepsilon)$ and the number of non-isomorphic triangulations is at least $P_\varepsilon(d+1) \sim \frac{c_1}{2d} e^{c_2\sqrt{d}}$.*

1.5 Uniqueness of K^d_{2d+3}

Recall from Subsection 1.4 that for $d \geq 2$, K^d_{2d+3} is the $(2d+3)$-vertex combinatorial d-manifold constructed by Kühnel in [10]. It triangulates $S^{1,d-1}(\varepsilon)$, where $\varepsilon \in \{0,1\}$ is given by $\varepsilon \equiv d \pmod 2$. One description of K^d_{2d+3} is implicit in Subsection 1.4. An equivalent (and somewhat simpler) description is as follows. It is the boundary complex of the combinatorial $(d+1)$-manifold with boundary whose vertices are the vertices of a cycle S^1_{2d+3} of length $2d+3$, and facets are the sets of $d+2$ vertices spanning a path in the cycle. From this picture, it is clear that the dihedral group of order $4d+6$ $(= \mathrm{Aut}(S^1_{2d+3}))$ is the full automorphism group of K^d_{2d+3}. Here we prove that for $d \geq 3$, up to simplicial isomorphism, K^d_{2d+3} is the unique $(2d+3)$-vertex non-simply connected combinatorial d-manifold.

Lemma 1.13. *Let X be a non-simply connected n-vertex combinatorial manifold of dimension $d \geq 3$. Then $n \geq 2d+3$. If further, $n = 2d+3$, then for any facet σ of X and any vertex x outside σ, either the induced subcomplex of X on $V(X) \setminus (\sigma \cup \{x\})$ is an S^{d-1}_{d+1} or the induced subcomplex $\mathrm{lk}_X(x)[\sigma]$ of $\mathrm{lk}_X(x)$ on the vertex set σ is disconnected.*

Proof. Let σ be a facet and $C = C(\sigma, X)$ be its simplicial complement. Choose a small (simply connected) neighbourhood U of $|\sigma|$ in $|X|$ such that $U \cap (|X| \setminus |\sigma|)$ is homeomorphic to $S^{d-1} \times (0,1)$. Now, $|X|$ is non-simply connected, $|X| = U \cup (|X| \setminus |\sigma|)$ and $d \geq 3$. So, by Van Kampen's theorem, $|X| \setminus |\sigma|$ is non-simply connected. But $|C|$ is a strong deformation retract of $|X| \setminus |\sigma|$. Therefore, C is non-simply connected.

Now, let's order the vertices of X arbitrarily as $x_1, x_2, \ldots, x_n$. For $1 \leq i \leq n$, let L_i (respectively R_i) be the induced subcomplex of X on the vertex-set $\{x_1, \ldots, x_i\}$ (respectively $\{x_{i+1}, \ldots, x_n\}$). We claim that for $1 \leq j \leq d$, $1 \leq i < n$,

$$H_j(R_i, R_{i+1}) \cong H_{d-j}(L_{i+1}, L_i), \tag{5}$$

where the homologies are taken with coefficients in $\mathbb{Z}_2$.

To prove (5), let $f\colon |X| \to \mathbb{R}$ be the simplexwise linear function given by $f(x_i) = i$, $1 \leq i \leq n$. Define $\mathcal{L}_i = \{x \in |X| : f(x) \leq i + 1/2\}$ and $\mathcal{R}_i = \{x \in |X| : f(x) > i + 1/2\}$. Then observe that $(|L_{i+1}|, |L_i|)$ (respectively $(|R_i|, |R_{i+1}|)$) is a strong deformation retract of $(\mathcal{L}_{i+1}, \mathcal{L}_i)$ (respectively $(\mathcal{R}_i, \mathcal{R}_{i+1})$). Hence we have

$$\begin{aligned} H_{d-j}(L_{i+1}, L_i) &\cong H_{d-j}(|L_{i+1}|, |L_i|) \cong H_{d-j}(\mathcal{L}_{i+1}, \mathcal{L}_i) \cong H^{d-j}(\mathcal{L}_{i+1}, \mathcal{L}_i) \\ &\cong H_j(\mathcal{R}_i, \mathcal{R}_{i+1}) \cong H_j(|R_i|, |R_{i+1}|) \cong H_j(R_i, R_{i+1}). \end{aligned}$$

This proves (5). (Here, the fourth isomorphism is because of Alexander duality (cf. [15, Theorem 17, Page 296]). The usual statement of this duality refers to Alexander cohomology, but this agrees with singular cohomology for polyhedral pairs (cf. [15, Corollary 11, Page 291]). Also, Alexander duality applies to orientable closed manifolds, but any closed pl manifold (such as $|X|$ in our application) is orientable over $\mathbb{Z}_2$. The third isomorphism holds since over a field, homology and cohomology are isomorphic.)

Now fix a facet σ of X. Choose the ordering of $V(X)$ so that $\sigma = \{x_1, \ldots, x_{d+1}\}$. Notice that, since $L_1 = \{x_1\}$ is simply connected but $L_n = X$ is not, it follows that there is a (smallest) index i such that L_i is simply connected but L_{i+1} is not. Note that $i \geq d+1$. Choose this i. Since $L_{i+1} = L_i \cup \mathrm{st}_{L_{i+1}}(x_{i+1})$ and $L_i \cap \mathrm{st}_{L_{i+1}}(x_{i+1}) = \mathrm{lk}_{L_{i+1}}(x_{i+1})$, Van Kampen's theorem implies that $\mathrm{lk}_{L_{i+1}}(x_{i+1})$ is not connected. Hence $H_1(L_{i+1}, L_i) \cong H_1(\mathrm{st}_{L_{i+1}}(x_{i+1}), \mathrm{lk}_{L_{i+1}}(x_{i+1})) \cong H_0^{\#}(\mathrm{lk}_{L_{i+1}}(x_{i+1})) \neq \{0\}$. Thus, there is an index $i \geq d+1$ such that $H_1(L_{i+1}, L_i) \neq \{0\}$. Hence, from (5), it follows that

$$H_{d-2}(\mathrm{lk}_{R_i}(x_{i+1})) \cong H_{d-1}(R_i, R_{i+1}) \neq \{0\} \text{ for some } i \geq d+1. \tag{6}$$

Notice that we have $R_{i+1} \subset R_i \subseteq C = C(\sigma, X)$. Since $H_{d-1}(R_i, R_{i+1}) \neq \{0\}$, R_i contains at least two $(d-1)$-faces. Hence the number of vertices in R_i is $\geq d+1$.

First suppose R_i has exactly $d+1$ vertices. Since $H_{d-2}(\mathrm{lk}_{R_i}(x_{i+1})) \neq \{0\}$ and $\mathrm{lk}_{R_i}(x_{i+1})$ has at most d vertices, it follows that $\mathrm{lk}_{R_i}(x_{i+1}) = S^{d-2}_d$. Since $d \geq 3$, it follows that R_i is simply connected. As C is not simply connected, we have $R_i \subset C$ (proper inclusion). Thus

$n \geq (d+1)+1+(d+1) = 2d+3$. Also, if the number $n-i$ of vertices in R_i is $\geq d+2$. Then $n \geq i+d+2 \geq 2d+3$. This proves the inequality.

Now assume that $n = 2d+3$. Let $x \notin \sigma$ be a vertex such that $\mathrm{lk}_X(x) \cap L_{d+1}$ $(= \mathrm{st}_X(x) \cap L_{d+1})$ is connected. Choosing the vertex order so that $x_{d+2} = x$, we get that L_{d+2} is simply connected (by Van Kampen theorem). Therefore $i \geq d+2$. Hence R_i has $\leq n-d-2 = d+1$ vertices. But, $H_{d-1}(R_i, R_{i+1}) \neq \{0\}$, so that R_i has $\geq d+1$ vertices. Therefore R_i has exactly $d+1$ vertices and hence $i = d+2$. Thus, $H_{d-2}(\mathrm{lk}_{R_{d+2}}(x_{d+3})) \cong H_{d-1}(R_{d+2}, R_{d+3}) \neq \{0\}$. Since $\mathrm{lk}_{R_{d+2}}(x_{d+3})$ has at most d vertices, it follows that $\mathrm{lk}_{R_{d+2}}(x_{d+3}) = S^{d-2}_{d}$. Since any vertex of R_{d+2} may be chosen to be x_{d+3} in this argument, we get that all the vertex links of R_{d+2} are isomorphic to S^{d-2}_{d}. Hence the induced subcomplex R_{d+2} of C on the vertex set $V(X) \setminus (\sigma \cup \{x\})$ is an S^{d-1}_{d+1}. This proves the lemma. □

Remark 1.3. The inequality in Lemma 1.13 is a theorem due to Brehm and Kühnel [5].

Lemma 1.14. *Let X be a $(2d+3)$-vertex non-simply connected combinatorial manifold of dimension $d \geq 3$. Then, there is a facet σ of X such that its simplicial complement $C(\sigma, X)$ contains an induced S^{d-1}_{d+1}.*

Proof. Suppose the contrary. Then, by Lemma 1.13, for each facet σ of X and each vertex $x \notin \sigma$, the induced subcomplex $\mathrm{lk}_X(x)[\sigma]$ of $\mathrm{lk}_X(x)$ on σ is disconnected. If τ were a $(d-2)$-face of X of degree 3, say with $\mathrm{lk}_X(\tau) = S^1_3(\{x_1, x_2, x_3\})$, then the induced subcomplex of $\mathrm{lk}_X(x_3)$ on the facet $\tau \cup \{x_1, x_2\}$ would be connected - a contradiction. So, X has no $(d-2)$-face of degree 3. Now, no face γ of X of dimension $e \leq d-2$ can have (minimal) degree $d-e+1$. (In other words, the link of γ can not be a standard sphere.) Or else, any $(d-2)$-face $\tau \supseteq \gamma$ of X would have degree 3. So, no standard sphere of positive dimension occurs as a link in X.

Now fix a facet σ of X. For each $x \in \sigma$, there is a unique vertex $x' \notin \sigma$ such that $(\sigma \setminus \{x\}) \cup \{x'\}$ is a facet. This defines a map $x \mapsto x'$ from σ to its complement. This map is injective: if we had $x'_1 = y = x'_2$ for $x_1 \neq x_2$ then the induced subcomplex of $\mathrm{lk}_X(y)$ on σ would be connected. Also, since $\mathrm{lk}_X(x')[\sigma]$ is disconnected, it follows that x must be an isolated vertex in $\mathrm{lk}_X(x')[\sigma]$. This implies that xx' is an edge of X, and $V(\mathrm{lk}_X(xx')) \subseteq V(X) \setminus (\sigma \cup \{x'\})$. Hence xx' is an edge of degree $\leq d+1$. Therefore, by the observation in the previous paragraph (with $e = 1$), $\deg_X(xx') = d+1$. In consequence, $\mathrm{lk}_X(xx')$ is a $(d+1)$-vertex $(d-2)$-sphere. But all such spheres are known: we must have $\mathrm{lk}_X(xx') = S^m_{m+2} * S^n_{n+2}$ for some $m, n \geq 0$ with $m+n = d-3$ (cf. [2]). If $m > 0$ or $n > 0$ then S^1_3 occurs as a link (of some $(d-4)$-simplex) in this sphere and hence it occurs as the link of a $(d-2)$-simplex (containing xx') in X. Hence, we must have $m = n = 0$. Thus $d = 3$ and each of the four edges xx' $(x \in \sigma)$ is of degree 4.

Then $\mathrm{lk}_X(xx')$ is an $S^1_4 = S^0_2 * S^0_2$ with vertex set $V(X) \setminus (\sigma \cup \{x'\})$. In consequence, putting $C = C(\sigma, X)$, one sees that C is a 5-vertex non-simply connected simplicial complex (by the proof of Lemma 1.13) such that for at least four of the vertices x' in C, $\mathrm{lk}_C(x') \supseteq S^1_4$. In consequence, all $\binom{5}{2} = 10$ edges occur in C. Since C is non-simply connected, it follows that C has at least one missing triangle (induced S^1_3), say with vertices y_1, y_2, y_3. At least two of these vertices (say y_1, y_2) have S^1_4 in their links. It follows that $\mathrm{lk}_C(y_1) \supseteq S^0_2(\{y_2, y_3\}) * S^0_2(\{y_4, y_5\})$ and $\mathrm{lk}_C(y_2) \supseteq S^0_2(\{y_1, y_3\}) * S^0_2(\{y_4, y_5\})$ where y_4, y_5 are the two other vertices of C. Hence $C \supseteq C_0 = (S^1_3(\{y_1, y_2, y_3\}) * S^0_2(\{y_4, y_5\})) \cup \{y_4y_5\}$. But all 5-vertex simplicial complexes properly containing C_0 and not containing the 2-simplex $y_1y_2y_3$ are simply connected. So, $C = C_0$. But, then two of the vertices of C (viz. y_4, y_5) have no S^1_4 in their links, a contradiction. This completes the proof. □

Theorem 3. *For $d \geq 3$, Kühnel's complex K^d_{2d+3} is the only non-simply connected $(2d+3)$-vertex combinatorial manifold of dimension d.*

Proof. Let X be a non-simply connected $(2d+3)$-vertex combinatorial manifold of dimension $d \geq 3$. By Lemma 1.14, X must have a facet σ such that $C(\sigma, X)$ contains an induced subcomplex S which is an S^{d-1}_{d+1}. Let x be the unique vertex in $C(\sigma, X) \setminus S$. If xy is a non-edge for each $y \in \sigma$ then the $(d-1)$-sphere $\mathrm{lk}_X(x)$ is a subcomplex of the $(d-1)$-sphere S and hence $\mathrm{lk}_X(x) = S$. This implies that $C(\sigma, X)$ is the combinatorial d-ball $\{x\} * S$. This is not possible since $C(\sigma, X)$ is non-simply connected. Thus, x forms an edge with a vertex in σ. This implies that $C(S, X)$ is connected.

Thus, S is an induced S^{d-1}_{d+1} in X, and $C(S, X)$ is connected. Since $d \geq 3$, S is two-sided in X. By Lemma 1.3, we may delete the handle over S to get a $(3d+4)$-vertex d-dimensional normal pseudomanifold $\widetilde{X}$. Since X has at most $\binom{2d+3}{2}$ edges, it follows that $\widetilde{X}$ has at most $\binom{2d+3}{2} + \binom{d+1}{2}$ edges. But $\binom{2d+3}{2} + \binom{d+1}{2} = (3d+4)(d+1) - \binom{d+2}{2}$ is the lower bound on the number of edges of a $(3d+4)$-vertex d-dimensional normal pseudomanifold given by the Lower Bound Theorem (cf Theorem 9 below). Therefore, $\widetilde{X}$ attains the lower bound, and hence, by Theorem 9, $\widetilde{X}$ is a stacked sphere. Since $\widetilde{X}$ was obtained from X by deleting a handle over an S^{d-1}_{d+1}, Lemma 1.3 implies that $X = \widetilde{X}^{\psi}$ where $\psi \colon \sigma_1 \to \sigma_2$ is an admissible bijection between two facets of $\widetilde{X}$. Thus, $\widetilde{X}$ is a $(3d+4)$-vertex stacked d-sphere with an admissible bijection ψ. Therefore, by Theorem 1, $\widetilde{X} = \mathcal{S}^d_{3d+4}$ and ψ are uniquely determined, hence so is $X = \widetilde{X}^{\psi}$. Since K^d_{2d+3} satisfies the hypothesis, it follows that $X = K^d_{2d+3}$. □

Corollary 4. *If $d \geq 2$, $\varepsilon \equiv d \ (mod\ 2)$ then $S^{1,d-1}(\varepsilon)$ has a unique $(2d+3)$-vertex combinatorial triangulation, namely K^d_{2d+3}.*

Proof. Since $S^{1,d-1}(\varepsilon)$ (with $\varepsilon \equiv d \pmod 2$) is non-simply connected and is the geometric carrier of K^d_{2d+3}, the result is immediate from Theorem 3 for $d \geq 3$. For $d = 2$, this result is classical. □

Corollary 5. *If $d \geq 2$, $\varepsilon \not\equiv d \ (mod\ 2)$ then any combinatorial triangulation of $S^{1,d-1}(\varepsilon)$ requires at least $2d+4$ vertices. Thus, for this manifold, the $(2d+4)$-vertex combinatorial triangulations in Subsection 1.4 are vertex minimal.*

Proof. Since $S^{1,d-1}(\varepsilon)$ (with $\varepsilon \not\equiv d \pmod 2$) is non-simply connected and K^d_{2d+3} does not triangulate this space, the result is immediate from Theorem 3 for $d \geq 3$. For $d = 2$, this result is classical. □

Corollary 6 (Walkup [18], Altshuler and Steinberg [1]). *K^3_9 is the unique 9-vertex combinatorial 3-manifold which is not a combinatorial sphere. In consequence, every closed 3-manifold other than S^3 and $S^{1,2}(1) = S^1 \times S^2$ requires at least 10 vertices for a combinatorial triangulation.*

Proof. Clearly, any triangulation of a 3-manifold is a combinatorial triangulation. The result is immediate from Theorem 3, since by the Poincaré-Perelman theorem, the 3-sphere is the only simply connected closed 3-manifold. However, it is not necessary to invoke such a powerful result. Since a simply connected 3-manifold is clearly a homology 3-sphere, and by a result of [3] any homology 3-sphere (other that S^3) requires at least 12 vertices, the corollary follows from Theorem 3. □

2 The Lower Bound Theorem

2.1 Introduction

Barnette's Lower Bound Theorem ([4]) says that any connected combinatorial manifold has at least as many edges as a stacked sphere of the same dimension with the same number of vertices. In [9], Kalai reproved this result using ideas from the theory of rigidity of frameworks and also showed that equality holds in this theorem only in the case of stacked spheres, provided the dimension is ≥ 3. (Notice that, in the case $d = 1$ these results are trivial: all connected combinatorial 1-manifolds are stacked spheres. Also, for a connected combinatorial 2-manifold on n vertices, the number of edges is $3(n - \chi) \geq 3(n - 2)$, where χ is the Euler characteristic; with equality if and only if $\chi = 2$, i.e., precisely in the case of 2-spheres. Thus, Kalai's theorem is false for dimension $d = 2$.)

Actually, Kalai showed that for $d \geq 3$, the edge graph of any connected combinatorial d-manifold is "generically $(d + 1)$-rigid" in the sense of rigidity of frameworks. From this theory, one knows that any n-vertex generically q-rigid graph has at least $qn - \binom{q+1}{2}$ edges, and this bound is optimal in the strong sense that any generically q-rigid graph on n vertices contains a generically q-rigid spanning subgraph with exactly $qn - \binom{q+1}{2}$ edges. Thus the Lower Bound Theorem is an immediate consequence of Kalai's rigidity theorem. Kalai also succeeds in using these ideas to characterize the case of equality.

The beautiful application of these results reported in Section 1 led us to take a close look at Kalai's proof. The first author observed that there appeared to be a minor loophole in Kalai's proof. The second author quickly confirmed this suspicion by means of explicit counter examples, and also showed how to fill up this loophole. Explicitly, in the second sentence in the proof of his Lemma 6.2, Kalai claimed that if C is a strongly connected (i.e., $\Lambda(C)$ is connected) d-dimensional simplicial complex which is not 2-neighbourly (i.e., there are missing edges in C) then it has non-adjacent vertices u, v and facets S, T such that $u \in S$, $v \in T$ and S and T meet maximally. This is false: indeed there are combinatorial d-manifolds C which are counterexamples (cf. [6]). However, the following weaker statement is true: C has non-adjacent vertices u, v and a facet T such that $v \in T$ and u is adjacent to all vertices in $T \setminus \{v\}$. This suffices to complete the proof of Kalai's Lemma 6.2 as indicated by him.

More importantly, we found it difficult to follow Kalai's proof in its totality because of our lack of familiarity with the rigidity theory of frameworks (which in turn is heavily dependent on analytic considerations that seem foreign to the questions at hand). We suspect that many experts in Combinatorial Topology share our discomfort, so that it should be helpful to have a self-contained combinatorial proof of the Lower Bound Theorem including a treatment of the cases of equality. A pointer in this direction is given in Gromov's book "Partial differential relations" [8, pages 211–212], where he presents a combinatorial definition of q-rigidity. It is a trivial consequence of his definition that if an n-vertex combinatorial d-manifold is $(d+1)$-rigid according to Gromov then it has at least $(d+1)n - \binom{d+2}{2}$ edges (cf. Lemma 2.1 below). Therefore, to prove the Lower Bound Theorem, it is sufficient to show that any connected combinatorial d-manifold (with $d \geq 3$) is $(d+1)$-rigid in the sense of Gromov. It is also fairly easy to see (cf. Lemma 2.5 below) that if all the vertex links of a connected combinatorial d-manifold X are q-rigid in this sense, then X is $(q + 1)$-rigid. This sets up an inductive proof provided we have a starting point. In [8], Gromov sketches an argument which purports to prove that all connected combinatorial 2-manifolds are 3-rigid in his sense. It was later observed that Gromov's proof has some gaps. Connelly and Whiteley filled this gap (cf. [19]). In [17], Tay presented another proof.

Here, we present a simple proof of this result. The notion of generalised bistellar move (cf. Definition 2.3 below) plays a crucial role in our proof. This suffices to start the induction.

Gromov himself did not investigate the case of equality in the Lower Bound Theorem. To do so, we closely follow Kalai's arguments, but using Gromov's definition. Even here, we are able to achieve considerable simplification. In particular, we find no need to introduce the notion of chordal graphs. Instead, our proof of Kalai's theorem hinges on a close examination of the cases of equality in Gromov's argument, aided by the notion of handle addition and handle deletion (cf. Definitions 1.4 and 1.5 above).

We should note that the notion of generic rigidity pertains primarily to graphs and Kalai calls a simplicial complex generically q-rigid if its edge graph is generically q-rigid. On the other hand, Gromov's definition pertains to simplicial complexes. For this reason, it is not possible to compare these two notions in general. However, such a comparison is possible when the dimension d of the simplicial complex is $\geq q-1$ (and we are interested in the case $d = q-1$). In these cases, Gromov's notion of rigidity is weaker than the notion of generic rigidity. From the theory of rigidity of frameworks, it is known that if an n-vertex graph G is minimally generically q-rigid (i.e., G is generically q-rigid but no proper spanning subgraph of G is generically q-rigid) then either G is a complete graph on $\leq q+1$ vertices, or else G has $n \geq q+1$ vertices and has exactly $nq - \binom{q+1}{2}$ edges, and any induced subgraph of G (say, with $p \geq q$ vertices) has at most $pq - \binom{q+1}{2}$ edges (cf. [7]. By a theorem of Laman, this fact characterizes minimally generically q-rigid graphs for $q \leq 2$). Using this result, it is easy to deduce that generic q-rigidity (of the edge graph) implies Gromov's q-rigidity for any simplicial complex of dimension $\geq q-1$.

On the other hand, it is easy to see that $S^{d-2}_d * (S^1_3 \sqcup S^1_3)$ is $(d+1)$-rigid according to Gromov, but it is not generically $(d+1)$-rigid. Thus, Gromov's rigidity is strictly weaker than generic rigidity. In consequence, the rigidity theorem proved here is a weaker result than the corresponding theorem of Kalai. Yet, it suffices to derive the Lower Bound Theorem together with a characterization of the equality case - and has the advantage that Gromov's definition is purely combinatorial (even if the motivation behind this definition is rather mysterious!). In contrast, no combinatorial characterization of generic q-rigidity is known for $q > 2$.

Finally, we mention that we prove the main results in the larger category of "normal pseudomanifolds" (cf. Definition 1.1), since all the proofs go through naturally in this class. It may be noted that Kalai himself proved his theorem for a smaller class (namely, the class of normal pseudomanifolds whose 2-dimensional links are 2-spheres), and it was extended to the class of all normal pseudomanifolds by Tay in [17]. However, Tay's proof depends on Kalai's theorem, while the proof of Theorem 9 given below is self-contained.

2.2 Gromov's q-rigidity and Lower Bound Theorem

Throughout this concluding subsection, we use the following definition due to Gromov (except that Gromov does not include connectedness as a requirement for rigidity; but it seems anathema to call a disconnected object rigid!). Thus q-rigidity hitherto refers to Gromov's q-rigidity, without further mention.

Definition 2.1. Let X be a d-dimensional simplicial complex and q a positive integer. We shall say that X is *q-rigid* if X is connected and, for any set $A \subseteq V(X)$ which is disjoint from at least one d-simplex of X, the number of edges of X intersecting A is $\geq mq$, where $m = \#(A)$.

Lemma 2.1. *Let X be an n-vertex d-dimensional simplicial complex. If X is q-rigid then the number of edges of X is $\geq (n-d-1)q + \binom{d+1}{2}$.*

Proof. Let e be the number of edges of X. Fix a d-simplex σ of X and put $A = V(X) \setminus \sigma$. Then $\#(A) = n - d - 1$ and exactly $e - \binom{d+1}{2}$ edges intersect A. □

Definition 2.2. Let X be an n-vertex d-dimensional simplicial complex and q a positive integer. We shall say that X is *minimally q-rigid* if X is q-rigid and has exactly $(n-d-1)q + \binom{d+1}{2}$ edges (i.e., if the lower bound in Lemma 2.1 is attained by X).

Lemma 2.2. *A connected simplicial complex is q-rigid if and only if the cone over it is $(q+1)$-rigid. It is minimally q-rigid if and only if the cone over it is minimally $(q+1)$-rigid.*

Proof. Let X be an n-vertex d-dimensional simplicial complex and $C(X)$ be the cone over X with cone-vertex x. Note that all the $(d+1)$-simplices of $C(X)$ pass through x, so that $A \subseteq V(C(X))$ is disjoint from a $(d+1)$-simplex if and only if $A \subseteq V(X)$ and A is disjoint from a d-simplex of X. Also $C(X)$ has exactly $m = \#(A)$ more edges than X which intersect A (viz., the edges joining x with the vertices of A). In consequence, the number of edges of X intersecting A is $\geq mq$ if and only if the number of edges of $C(X)$ intersecting A is $\geq m(q+1)$. This proves the first part. The second part follows since $C(X)$ has one more vertex and n more edges than X. □

Lemma 2.3. *Let X_1, X_2 be subcomplexes of a simplicial complex X such that $X = X_1 \cup X_2$ and $\dim(X_1 \cap X_2) = \dim(X)$. If X_1, X_2 are both q-rigid then X is q-rigid. If, further, X is minimally q rigid then both X_1, X_2 are minimally q-rigid.*

Proof. Since X_1, X_2 are both connected, our assumption implies that X is connected. Let $\dim(X) = d$. Since $\dim(X_1 \cap X_2) = \dim(X)$, it follows that $\dim(X_1) = \dim(X_2) = \dim(X_1 \cap X_2) = d$. Let $A \subseteq V(X)$ be disjoint from some d-simplex $\sigma \in X = X_1 \cup X_2$. Without loss of generality, $\sigma \in X_1$. Write $A_1 = A \cap V(X_1)$ and $A_2 = A \setminus V(X_1)$. Say $m = \#(A)$, $m_i = \#(A_i)$, $i = 1, 2$. Thus, $m = m_1 + m_2$. Note that $A_1 \subseteq V(X_1)$ is disjoint from the d-simplex σ of X_1. Also, if τ is a d-simplex of $X_1 \cap X_2$, then τ is a d-simplex of X_2 disjoint from A_2 (since $\tau \subseteq V(X_1)$ and A_2 is disjoint from $V(X_1)$). Since, X_1, X_2 are q-rigid, we have at least m_1q edges of X_1 meeting A_1 and at least m_2q edges of X_2 meeting A_2. Also, as $V(X_1)$ and A_2 are disjoint, no edge of X_1 meets A_2. Therefore, we have at least $m_1q + m_2q = mq$ distinct edges of X meeting A. This proves that X is q-rigid.

Now, if X is minimally q-rigid, then taking A to be the complement in $V(X)$ of a d-simplex of X_1, one gets exactly mq edges of X meeting A. Since we have equality in the above argument, it follows that exactly m_1q edges of X_1 intersect $A_1 = A \cap V(X_1)$. Since A_1 is the complement in $V(X_1)$ of a d-simplex of X_1, this shows that X_1 is then minimally q-rigid. Since the assumptions are symmetric in X_1 and X_2, in this case X_2 is also minimally q-rigid. □

Lemma 2.4. *Let $\{X_\alpha : \alpha \in I\}$ be a finite family of q-rigid subcomplexes of a simplicial complex X. Suppose there is a connected graph H with vertex set I such that whenever $\alpha, \beta \in I$ are adjacent in H, we have $\dim(X_\alpha \cap X_\beta) = \dim(X)$. Also suppose $\cup_{\alpha \in I} X_\alpha = X$. Then X is q-rigid. If, further, X is minimally q-rigid, then each X_α is minimally q-rigid.*

Proof. Induction on $\#(I)$. If $\#(I) = 1$ then the result is trivial. For $\#(I) = 2$, the result is just Lemma 2.3. So suppose $\#(I) > 2$ and we have the result for smaller values of $\#(I)$.

Since H is a connected graph, there is $\alpha_0 \in I$ such that the induced subgraph of H on the vertex set $I \setminus \{\alpha_0\}$ is connected (for instance, one may take α_0 to be an end vertex of a spanning tree in H). Applying the induction hypothesis to the family $\{X_\alpha : \alpha \neq \alpha_0\}$, one gets that $Y_1 = \cup_{\alpha \neq \alpha_0} X_\alpha$ is q-rigid. Since $Y_2 = X_{\alpha_0}$ is also q-rigid, $X = Y_1 \cup Y_2$, and $\dim(Y_1 \cap Y_2) = \dim(X)$ (if α_0 is adjacent to α_1 in H then $\dim(X) \geq \dim(Y_1 \cap Y_2) \geq \dim(X_{\alpha_1} \cap Y_2) = \dim(X)$), induction hypothesis (or Lemma 2.3) implies that X is q-rigid. Now, if X is minimally q-rigid then, by Lemma 2.3, so are Y_1 and Y_2. Since Y_1 is minimally q-rigid, induction hypothesis then implies that X_α is minimally q-rigid for $\alpha \neq \alpha_0$ (and also for $\alpha = \alpha_0$ since $X_{\alpha_0} = Y_2$). □

Lemma 2.5. *Let X be a connected pure d-dimensional simplicial complex. (a) If each vertex link of X is q-rigid then X is $(q+1)$-rigid. (b) If, further, X is minimally $(q+1)$-rigid then all the vertex links of X are minimally q-rigid.*

Proof. Let $I = V(X)$ and H be the edge graph of X. Since X is connected, so is H. For $\alpha \in I$, $\mathrm{st}(\alpha)$ is a cone over the q-rigid complex $\mathrm{lk}(\alpha)$, and hence by Lemma 2.2, $\mathrm{st}(\alpha)$ is $(q+1)$-rigid for each $\alpha \in I$. Since X is pure, the family $\{\mathrm{st}(\alpha) : \alpha \in I\}$ satisfies the hypothesis of Lemma 2.4. Hence X is $(q+1)$-rigid. If it is minimally $(q+1)$-rigid, then by Lemma 2.4, each $\mathrm{st}(\alpha)$ is minimally $(q+1)$-rigid, and hence, by Lemma 2.2, $\mathrm{lk}(\alpha)$ is minimally q-rigid for all $\alpha \in I$. □

Definition 2.3. Let X be a d-dimensional weak pseudomanifold. Let B_1, B_2 be two combinatorial d-balls such that B_1 is a subcomplex of X and $\partial B_1 = \partial B_2 = B_2 \cap X$. Then the pure d-dimensional simplicial complex $\widetilde{X} = (X \setminus B_1) \cup B_2$ is said to be obtained from X by a *generalised bistellar move* (with respect to the pair (B_1, B_2)). Observe that $\widetilde{X}$ is also a d-dimensional weak pseudomanifold. [Let τ be a $(d-1)$-simplex of $\widetilde{X}$. If $\tau \in B_2 \setminus \partial B_2$ then τ is in two facets in B_2. If $\tau \in \widetilde{X} \setminus B_2$ then τ is in two facets in $X \setminus B_1 = \widetilde{X} \setminus B_2$. If $\tau \in \partial B_1 = \partial B_2$ then τ is in one facet in $X \setminus B_1 = \widetilde{X} \setminus B_2$ and in one facet in B_2.] Notice that we then have $\partial B_2 = \partial B_1 = B_1 \cap \widetilde{X}$, and X is obtained from $\widetilde{X}$ by the (reverse) generalised bistellar move with respect to the pair (B_2, B_1). In case both B_1 and B_2 are d-balls with at most $d+2$ vertices (and hence at least one has $d+2$ vertices) then this construction reduces to the usual bistellar move. Clearly, if $\widetilde{X}$ is obtained from X by a generalised bistellar move then $|\widetilde{X}|$ is homeomorphic to $|X|$ and if the dimension of X is at most 3 then $|\widetilde{X}|$ is pl homeomorphic to $|X|$.

Lemma 2.6. *If $\widetilde{X}$ is obtained from X by a generalised bistellar move, then $\widetilde{X}$ is a normal pseudomanifold if and only if X is a normal pseudomanifold.*

Proof. Let X be a normal pseudomanifold. We prove that $\widetilde{X}$ is a normal pseudomanifold by induction on the dimension d of X. If $d = 1$ then the result is trivial. Assume that the result is true for all normal pseudomanifolds of dimension $< d$ and X is a normal pseudomanifold of dimension $d \geq 2$. Let $\widetilde{X}$ be obtained from X by a generalised bistellar move with respect to the pair (B_1, B_2). Since X is connected, it follows that $\widetilde{X}$ is connected. We have observed that $\widetilde{X}$ is a weak pseudomanifold. Let α be a face of dimension $\leq d-2$. If $\alpha \in B_2 \setminus \partial B_2$ then $\mathrm{lk}_{\widetilde{X}}(\alpha) = \mathrm{lk}_{B_2}(\alpha)$ is connected. If $\alpha \in \widetilde{X} \setminus B_2$ then $\mathrm{lk}_{\widetilde{X}}(\alpha) = \mathrm{lk}_X(\alpha)$ is connected. If $\alpha \in \partial B_1 = \partial B_2$ then $\mathrm{lk}_{\widetilde{X}}(\alpha)$ is obtained from $\mathrm{lk}_X(\alpha)$ by the generalised bistellar move with respect the pair $(\mathrm{lk}_{B_1}(\alpha), \mathrm{lk}_{B_2}(\alpha))$. Since $\mathrm{lk}_X(\alpha)$ is a normal pseudomanifold of dimension $< d$, by induction hypothesis, $\mathrm{lk}_{\widetilde{X}}(\alpha)$ is a normal pseudomanifold. In particular, $\mathrm{lk}_{\widetilde{X}}(\alpha)$ is connected. This implies that $\widetilde{X}$ is a normal pseudomanifold. Since X is obtained from $\widetilde{X}$ by the reverse generalised bistellar move, the converse follows. □

Lemma 2.7. *Let X be a combinatorial 2-sphere with more than 4 vertices. For any vertex u in X, there is a combinatorial 2-ball B with $V(B) = V(\mathrm{lk}(u))$ such that $B \cap X = \partial B = \mathrm{lk}(u)$.*

Proof. Let $\deg(u) = k$. The proof is by induction on $k \geq 3$. The result is obvious for $k = 3$ (B must be the standard 2-ball on three vertices). So, assume $k > 3$ and we have the result for smaller values of k. By the easy part of Kuratowski's theorem, the $k + 1$ (≥ 5) vertices in $\mathrm{st}(u)$ can't be mutually adjacent in the edge graph of X, so that there are two vertices v, w in $\mathrm{lk}(u)$ such that vw is not an edge of X. Then $\mathrm{lk}(u) \cup \{vw\}$ is the union of two cycles C_1, C_2 with vw as their only common edge. Let X_1, X_2 be the cones over C_1, C_2 with cone vertices u_1 and u_2 respectively ($u_1, u_2 \notin V(X)$). Let $D_1 = \mathrm{st}_X(u)$, $D_2 = X_1 \cup X_2$ and $\widetilde{X} = (X \setminus D_1) \cup D_2$. Then $\widetilde{X}$ is obtained from X by a generalised bistellar move. This implies that $\widetilde{X}$ is a combinatorial 2-sphere with vertices u_1, u_2 such that $\deg_{\widetilde{X}}(u_1) < k$, $\deg_{\widetilde{X}}(u_2) < k$. By, induction hypothesis, there exist two 2-balls B_1, B_2, with vertex sets $V(\mathrm{lk}_{\widetilde{X}}(u_1))$, $V(\mathrm{lk}_{\widetilde{X}}(u_2))$ respectively, satisfying the requirement. Then $B = B_1 \cup B_2$ is the 2-ball as required. □

Lemma 2.8. *All combinatorial 2-spheres are (minimally) 3-rigid.*

Proof. Let X be a combinatorial 2-sphere, say with n vertices. We prove the 3-rigidity of X by induction on n. If $n = 4$, then X is the standard 2-sphere, and the result is obvious. So, assume $n > 4$, and we have the result for smaller values of n. Take any set $A \subseteq V(X)$ which is disjoint from at least one 2-simplex σ of X. Say $\#(A) = m$. Fix a vertex $x \in A$, say of degree k. Take a 2-ball B with vertex set $V(B) = V(\mathrm{lk}(x))$ as in Lemma 2.7. Note that B is a k-vertex 2-ball with k edges in the boundary (viz., the edges of $\mathrm{lk}(x)$), hence it has $k - 3$ edges in the interior: these are not edges of X. Define $\widetilde{X} = (X \setminus \mathrm{st}(x)) \sqcup B$. So, $\widetilde{X}$ is obtained from X by a generalized bistellar move. Therefore, $\widetilde{X}$ is an $(n-1)$-vertex combinatorial 2-sphere, and $\widetilde{A} := A \setminus \{x\}$ is a subset of $V(\widetilde{X}) = V(X) \setminus \{x\}$, which is disjoint from the 2-simplex σ of $\widetilde{X}$. By induction hypothesis, $\widetilde{X}$ is 3-rigid, so that at least $3(m-1)$ edges of $\widetilde{X}$ intersect $\widetilde{A}$, and hence also A. Of these edges, at most $k-3$ edges are not in X. Thus at least $3(m-1) - (k-3)$ edges of X (not passing through x) meet A. Also, all the k edges of X through x meet A. Thus we have a total of at least $3(m-1) - (k-3) + k = 3m$ edges of X meeting A. Hence X is 3-rigid. Since $\chi(X) = 2$, it has $3(n-2)$ edges. Hence X is minimally 3-rigid. □

Lemma 2.9. *Let X_1, X_2 be combinatorial d-manifolds. If X_1, X_2 are $(d+1)$-rigid then their elementary connected sum $X_1 \# X_2$ is $(d+1)$-rigid. If, further, $X_1 \# X_2$ is minimally $(d+1)$-rigid then both X_1 and X_2 are minimally $(d+1)$-rigid.*

Proof. Since X_1, X_2 are both connected, so is $X_1 \# X_2$. Let σ_i be a facet of X_i ($i = 1, 2$) and $f \colon \sigma_1 \to \sigma_2$ be a bijection, such that $X = X_1 \# X_2$ is obtained from $X_1 \sqcup X_2 \setminus \{\sigma_1, \sigma_2\}$ via an identification through f. We view $V(X_i)$ as a subset of $V(X)$ in the obvious fashion. Put $\widetilde{X} = (X_1 \# X_2) \cup \{\sigma_1 = \sigma_2\}$. Then X_1, X_2 are subcomplexes of $\widetilde{X}$ satisfying the hypothesis of Lemma 2.3 with $q = d + 1$. Hence, by Lemma 2.3, $\widetilde{X}$ is $(d+1)$-rigid. Since $X_1 \# X_2$ is a subcomplex of $\widetilde{X}$ of the same dimension with the same set of edges, it follows that $X_1 \# X_2$ is $(d+1)$-rigid.

If $X_1 \# X_2$ is minimally $(d+1)$-rigid, then so is $\widetilde{X}$ and hence, by Lemma 2.3, so are X_1, X_2. □

Lemma 2.10. *Let Y be a connected combinatorial d-manifold which is obtained from a combinatorial d-manifold X by an elementary handle addition. If X is $(d+1)$-rigid then Y is $(d+1)$-rigid.*

Proof. Let $Y = X^{\psi}$, where $\psi: \sigma_1 \to \sigma_2$ is an admissible bijection between two disjoint facets σ_1, σ_2 of X. Thus Y is obtained from $X \setminus \{\sigma_1, \sigma_2\}$ by identifying x with $\psi(x)$ for each $x \in \sigma_1$ (cf. Definition 1.4). Let's identify $V(Y)$ with $V(X) \setminus \sigma_2$ via the quotient map $V(X) \to V(Y)$. Let $A \subseteq V(Y)$ be an m-set disjoint from a facet σ of Y. Then, under this identification $A \subseteq V(X)$ is disjoint from σ and it follows from the definition of X^{ψ} that σ is a facet of X. This implies, by $(d+1)$-rigidity of X, that at least $m(d+1)$ edges of X meet A. Since $A \cap \sigma_2 = \emptyset$, these edges corresponds to distinct edges of Y under our identification. Hence Y is $(d+1)$-rigid. □

Lemma 2.11. *Let X be an n-vertex 2-dimensional normal pseudomanifold. Then X is 3-rigid. X is minimally 3-rigid if and only if X is a combinatorial 2-sphere.*

Proof. Since X is 2-dimensional, it follows that X is a connected combinatorial 2-manifold.

First suppose X is orientable, say of genus $g \geq 0$. We prove by induction on g that X is 3-rigid. The $g = 0$ case is Lemma 2.8. So, assume $g > 0$ and we have the result for lesser genus. Now, for the fixed genus g, we do an induction on n. Fix any m-set $A \subseteq V(X)$, such that A is disjoint from a 2-simplex σ of X. Take $x \in A$ and look at $\mathrm{lk}_X(x)$.

If none of the diagonals of the cycle $\mathrm{lk}_X(x)$ are edges of X then take a combinatorial 2-ball D such that $V(D) = V(\mathrm{lk}_X(x))$ and $\partial D = \mathrm{lk}_X(x)$ and put $\widetilde{X} = (X \setminus \mathrm{st}_X(x)) \cup D$. Then $\widetilde{X}$ is obtained from X by a generalised bistellar move, so that $\widetilde{X}$ is an $(n-1)$-vertex combinatorial 2-manifold ($|\widetilde{X}|$ is pl homeomorphic to $|X|$ and hence) with the same genus g. (In particular, this case can't arise if n is the smallest possible number of vertices for a triangulation of $|X|$.) Therefore, by induction hypothesis, $\widetilde{X}$ is 3-rigid and hence at least $3(m-1)$ edges of $\widetilde{X}$ meet $A \setminus \{x\}$. Arguing exactly as in the proof of Lemma 2.8, one sees that at least $3m$ edges of X meet A, so that X is 3-rigid.

Next suppose there is a diagonal yz of $\mathrm{lk}_X(x)$ which is an edge of X. Then $U = \{x, y, z\}$ induces an S^1_3 in X. Since X is orientable, this S^1_3 is two-sided in X. Let Y be the combinatorial 2-manifold obtained from X by deleting the handle over $S^1_3(U)$. If Y is disconnected, say with components Y_1 and Y_2, then, by Lemma 1.3, $X = Y_1 \# Y_2$ and Y_1, Y_2 are combinatorial 2-manifolds. A little computation shows that the genus g_1, g_2 of Y_1, Y_2 are related by $g_1 + g_2 = g$. If, say, $g_2 = 0$, then Y_1 is a triangulation of $|X|$ using fewer vertices and Y_2 is a combinatorial 2-sphere, so that by induction hypothesis (on number of vertices) and by Lemma 2.8, both Y_1 and Y_2 are 3-rigid. Hence, by Lemma 2.9, X is 3-rigid in this case. (Note that this case does not arise if X is a minimal triangulation of $|X|$.) Otherwise, $g_1 > 0$, $g_2 > 0$ and hence $g_1 < g$, $g_2 < g$. Therefore, by our original induction hypothesis on genus, both Y_1, Y_2 are 3-rigid, so that we are done as before. (This case may arise even if X is minimal triangulation of $|X|$.) Now suppose Y is connected. Then Y is a connected combinatorial 2-manifold of genus $g-1$ and hence by our original induction hypothesis, Y is 3-rigid. Since X is obtained from Y by handle addition (cf. Lemma 1.3), it follows from Lemma 2.10 that X is 3-rigid. This completes the induction.

Now suppose X is non-orientable. Let $\widehat{X}$ be the orientable double cover of X. By the above, $\widehat{X}$ is 3-rigid. Since the covering map $V(\widehat{X}) \to V(X)$ is a two-to-one simplicial map, it is immediate that X is 3-rigid. This proves the first part.

The last part follows from the following: X is minimally 3-rigid $\Longleftrightarrow$ number of edges in X is $3(n-2) \Longleftrightarrow \chi(X) = 2$. □

Remark 2.1. The proof of Lemma 2.11 shows, in particular, that any minimal triangulation of a connected and orientable combinatorial 2-manifold of positive genus must arise as the connected sum of two triangulations of smaller genus or from handle addition over a combinatorial 2-manifold of smaller genus. This fact should be useful in the explicit classification of minimal triangulations of orientable 2-manifolds of small genus. Lemma 2.7 shows that any combinatorial 2-sphere on n (> 4) vertices arises from an $(n-1)$-vertex combinatorial 2-sphere by a generalised bistellar move. This should help in simplifying the existing classifications and obtaining new classifications of combinatorial 2-spheres with few vertices.

Theorem 7. *Let X be a d-dimensional normal pseudomanifold. If $d \geq 2$ then X is $(d+1)$-rigid. If, further, $d \geq 3$ and X is minimally $(d+1)$-rigid, then all the vertex links of X are minimally d-rigid.*

Proof. The proof is by induction on d. For $d = 2$ this is Lemma 2.11. For $d \geq 3$, all the vertex links of X are $(d-1)$-dimensional normal pseudomanifolds and hence, by the induction hypothesis, all vertex links of X are d-rigid. So the result follows from Lemma 2.5. □

Lemma 2.12. *Let X be a minimally $(d+1)$-rigid normal pseudomanifold of dimension $d \geq 3$. Then every clique of size $\leq d$ in the edge graph of X is a simplex of X.*

Proof. Let $I = V(X)$ and let H be the edge graph of X. For $\alpha \in I$, let H_α be the induced subgraph of H on the vertex-set $V(\mathrm{lk}(\alpha))$ and put $X_\alpha = \mathrm{st}(\alpha) \cup H_\alpha$. By Lemma 2.2 and Theorem 7, $\mathrm{st}(\alpha)$ is $(d+1)$-rigid and hence so is X_α. Thus $\{X_\alpha : \alpha \in I\}$ satisfies the hypothesis of Lemma 2.4. Since X is minimally $(d+1)$-rigid, it follows that X_α is minimally $(d+1)$-rigid for each $\alpha \in I$. But $X_\alpha \supseteq \mathrm{st}(\alpha)$, $V(X_\alpha) = V(\mathrm{st}(\alpha))$ and $\mathrm{st}(\alpha)$ is $(d+1)$-rigid. Therefore, X_α and $\mathrm{st}(\alpha)$ have the same edge graph. That is, $H_\alpha \subseteq \mathrm{st}(\alpha)$. Thus, each clique of size ≤ 3 through α is a simplex of X. Since this holds for each $\alpha \in I$, it follows that each clique of size ≤ 3 in H is a simplex of X.

Now, by an induction on k, one sees that for $k \leq d$, any k-clique of H is a face of X: if C is a k-clique (and $k \geq 4$ and hence $d \geq 4$), then for any $x \in C$, $C \setminus \{x\}$ is a $(k-1)$-clique of $\mathrm{lk}(x)$ and $\dim(\mathrm{lk}(x)) = d - 1 \geq 3$. Therefore, $C \setminus \{x\}$ is a simplex of $\mathrm{lk}(x)$ and hence C is a simplex of X. □

Lemma 2.13. *Let X be a minimally $(d+1)$-rigid normal pseudomanifold of dimension $d \geq 3$. Then the edge graph of X has a clique of size $d + 2$.*

Proof. If we have the result for $d = 3$ then the result follows for all $d \geq 3$ by a trivial induction on dimension (using the second statement in Theorem 7). So, we may assume $d = 3$.

Let $n \geq 5$ be the number of vertices of X. Since X is minimally 4-rigid, it has $4n - 10$ edges and hence the average degree of the vertices is $\frac{2(4n-10)}{n} < 8$. Therefore, X has a vertex x of degree ≤ 7. Then, by Lemmas 2.5 and 2.11, $\mathrm{lk}(x)$ is a combinatorial 2-sphere on ≤ 7 vertices. If possible, suppose $\mathrm{lk}(x)$ has no vertex of degree 3. It is easy to see that up to isomorphism there are only two such S^2, namely $S^0_2 * S^1_m$ with $m = 4$ or 5. Thus $\mathrm{lk}(x)$ is one of these two spheres, say $\mathrm{lk}(x) = S^0_2(\{y, z\}) * S^1_m(A)$. Since xyz is not a 2-simplex, by Lemma 2.12, yz is not an edge of X. Put $B_1 = \mathrm{st}_X(x)$, $B_2 = B^1_2(\{x, y\}) * S^1_m(A)$. Set $\widetilde{X} = (X \setminus B_1) \cup B_2$. Then $\widetilde{X}$ is obtained from X by a generalised bistellar move. Hence $\widetilde{X}$ is a 3-dimensional normal pseudomanifold with $n - 1$ vertices and $4n - 10 - (m + 2) + 1 = 4n - 11 - m < 4(n - 1) - 10$ edges (as $m \geq 4$). This is impossible since $\widetilde{X}$ is 4-rigid by Theorem 7. This proves that $\mathrm{lk}(x)$ has a vertex y of degree 3. Then the vertex-set of $\mathrm{st}(xy)$ is a 5-clique. This completes the proof. □

Lemma 2.14. *Let X be an n-vertex minimally $(d+1)$-rigid d-dimensional normal pseudomanifold. If $d \geq 3$ and $n > d+2$ then X contains a standard $(d-1)$-sphere S as an induced subcomplex.*

Proof. By Lemma 2.13, there is a $(d+2)$-set $C \subseteq V(X)$ which is a clique of the edge graph of X. If all the $(d+1)$-subsets of C were facets of X then the induced subcomplex of X on the vertex-set C would be a proper subcomplex which is a (standard) d-sphere. This is not possible since X is a d-dimensional normal pseudomanifold. So, there is a $(d+1)$-set $C_0 \subseteq C$ such that C_0 is not a facet of X. But C_0 is a $(d+1)$-clique of the edge graph of X, so by Lemma 2.12, all proper non-empty subsets of C_0 are faces of X. Thus the induced subcomplex S of X on the vertex-set C_0 is a standard $(d-1)$-sphere. □

Lemma 2.15. *If X is minimally 4-rigid 3-dimensional normal pseudomanifold then X is a stacked 3-sphere.*

Proof. By Theorem 7, all the vertex links are minimally 3-rigid. Therefore, by Lemma 2.11, X is a combinatorial 3-manifold. Let the number of vertices in X be n. We wish to prove by induction on n that X must be a stacked 3-sphere. This is trivial for $n = 5$, so that we may assume that $n > 5$ and we have the result for smaller values of n.

By Lemma 2.14, X contains a standard 2-sphere S as an induced subcomplex. Since S is a 2-sphere, S is two-sided in X. Let Y be the simplicial complex obtained from X by deleting the "handle" over S. Since X is a combinatorial 3-manifold, by Lemma 1.9 (a), Y is a combinatorial 3-manifold. Also, Y has $n+4$ vertices and $4n - 10 + \binom{4}{2} < 4(n+4) - \binom{5}{2}$ edges. Therefore Y is not 4-rigid and hence, by Theorem 7, Y must be disconnected. Since X is connected, Lemma 1.3 implies that $X = Y_1 \# Y_2$, where Y_1, Y_2 are 3-dimensional normal pseudomanifolds. Since X is minimally 4-rigid, Lemma 2.9 implies that Y_1, Y_2 are both minimally 4-rigid. Let Y_i have n_i vertices $(i = 1, 2)$. Since $n_1 + n_2 = n + 4$, $n_1 > 4$, $n_2 > 4$, it follows that $n_1 < n$, $n_2 < n$. Therefore, by induction hypothesis, Y_1, Y_2 are stacked 3-spheres. Since X is an elementary connected sum of Y_1 and Y_2, Lemma 1.8 (b) implies that X is a stacked 3-sphere. □

Theorem 8. *For $d \geq 3$, the stacked d-spheres are the only minimally $(d+1)$-rigid d-dimensional normal pseudomanifolds.*

Proof. If X is an n-vertex stacked d-sphere then (cf. Definition 1.6) the number of edges of X is $(d+1)n - \binom{d+2}{2}$, so that X is minimally $(d+1)$-rigid by Theorem 7.

For the converse, let X be a minimally $(d+1)$-rigid d-dimensional normal pseudomanifold, with $d \geq 3$. We prove by induction on d that X is a stacked d-sphere. The $d = 3$ case is Lemma 2.15. So, assume $d > 3$ and we have the result for smaller values of d. By Theorem 7 and induction hypothesis, all the vertex links of X are stacked $(d-1)$-spheres. That is, X is in the class $\mathcal{K}(d)$ (cf. Definition 1.7). In particular, X is a combinatorial d-manifold.

Let the number of vertices in X be n. We wish to prove by induction on n that X must be a stacked d-sphere. This is trivial for $n = d+2$, so that we may assume that $n > d+2$ and we have the result for smaller values of n.

By Lemma 2.14, X contains a standard $(d-1)$-sphere S as an induced subcomplex. Since $d > 3$, S is two-sided in X. Let Y be the simplicial complex obtained from X by deleting the "handle" over S. Since X is in the class $\mathcal{K}(d)$, by Lemma 1.9 (b), Y is in the class $\mathcal{K}(d)$. In particular, Y is a combinatorial d-manifold. Also, Y has $n+d+1$ vertices and $((d+1)n - \binom{d+2}{2}) + \binom{d+1}{2} = (n+d+1)(d+1) - (d+1)(d+2) < (n+d+1)(d+1) - \binom{d+2}{2}$ edges. Therefore Y is not $(d+1)$-rigid and hence, by Theorem 7, Y must be disconnected. Since X is connected, Lemma 1.3 implies that $X = Y_1 \# Y_2$, where Y_1, Y_2 are d-dimensional normal pseudomanifolds. Since X is minimally $(d+1)$-rigid, Lemma 2.9 implies that Y_1, Y_2 are both minimally $(d+1)$-rigid. Let Y_i have n_i vertices $(i = 1, 2)$. Since $n_1 + n_2 = n+d+1$, $n_1 > d+1$, $n_2 > d+1$, it follows that $n_1 < n$, $n_2 < n$. Therefore, by induction hypothesis, Y_1, Y_2 are stacked d-spheres. Since X is an elementary connected sum of Y_1 and Y_2, Lemma 1.8 (b) implies that X is a stacked d-sphere. □

As an immediate consequence, we have:

Theorem 9 (The Lower Bound Theorem). *For $d \geq 2$, any n-vertex d-dimensional normal pseudomanifold has $\geq n(d+1) - \binom{d+2}{2}$ edges. For $d \geq 3$, equality holds only for stacked spheres.*

Proof. The inequality is obvious from Lemma 2.1 and Theorem 7. The case of equality follows from Theorem 8 and Definition 2.2. □

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