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Limit Laws for the Local Times of Fractional Brownian and Stable Motions

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Abstract. Consider a linear fractional stable motion $\Lambda(t)$, indexed by the Hurst parameter $0 < H < 1$ and the index of stability $0 < \alpha \leq 2$. (When $\alpha = 2$, the process $\Lambda(t)$ becomes the fractional Brownian motion.) Let L_t^x be the local time of $\Lambda(t)$ at x upto the time t. When H is restricted to $\frac{1}{3} < H < 1$, we obtain the convergence in distribution of: the process $(t, x, u) \longmapsto \varepsilon^{-\frac{1-H}{2H}} \left(L_t^{\varepsilon x+u} - L_t^u\right)$ as $\varepsilon \to 0$, and the process $(t, u) \longmapsto \kappa^{-\frac{1-H}{2}} \int_0^{\kappa t} f\left(\Lambda(s) - \kappa^H u\right) ds$ as $\kappa \to \infty$, where the Borel function $f(x)$ is such that $\int_{-\infty}^{\infty} f(x)\, dx = 0$. The restriction $\frac{1}{3} < H < 1$ cannot be relaxed. These results generalize the known results for the Brownian motion (the case $\alpha = 2$ and $H = \frac{1}{2}$) and the symmetric stable process (the case $H = \frac{1}{\alpha}$, $1 < \alpha < 2$).

1 Introduction and the main results

Let $\{Z_\alpha(t), t \in \mathbb{R}\}$, $0 < \alpha \leq 2$, be an *α-stable Lévy motion.* By this we mean that $Z_\alpha(t)$ has stationary independent increments having a strictly α-stable distribution, that is, for $s < t$,

$$E\left[e^{iu(Z_\alpha(t)-Z_\alpha(s))}\right] = \begin{cases} e^{-(t-s)\sigma^\alpha |u|^\alpha \left(1-i\beta\, \text{sign}(u)\, \tan\left(\frac{\pi\alpha}{2}\right)\right)} & \text{if } \alpha \neq 1 \\ e^{-\sigma(t-s)|u|} & \text{if } \alpha = 1, \end{cases} \tag{1}$$

where $|\beta| \leq 1$ and $\sigma > 0$ is the scale parameter. For convenience we henceforth take $\sigma = 1$. (Note that this definition of strict α-stability for the case $\alpha = 1$ differs from the usual one in that we take the shift parameter to be 0.) When $\alpha = 2$, $Z_\alpha(t)$ becomes the *Brownian motion* with variance 2.

A process $\{\Lambda_{\alpha,H}(t), t \geq 0\}$ is called a *linear fractional stable motion* (LFSM) with Hurst parameter H, $0 < H < 1$, if it is given by

$$\Lambda_{\alpha,H}(t) = \int_{-\infty}^0 \left\{(t-u)^{H-\frac{1}{\alpha}} - (-u)^{H-\frac{1}{\alpha}}\right\} Z_\alpha(du) + \int_0^t (t-u)^{H-\frac{1}{\alpha}}\, Z_\alpha(du)$$

if $H \neq \frac{1}{\alpha}$, and if $H = \frac{1}{\alpha}$, $\Lambda_{\alpha,H}(t)$ is taken to be $Z_\alpha(t)$, that is,

$$\Lambda_{\alpha,\frac{1}{\alpha}}(t) = \int_0^t Z_\alpha(du) = Z_\alpha(t),$$

where $Z_\alpha(t)$ is an α-stable Levy motion as above. Note that in the case $H = \frac{1}{\alpha}$, the restriction $0 < H < 1$ reduces to $1 < \alpha \leq 2$. When $\alpha = 2$, the LFSM reduces to the *Fractional Brownian Motion* (FBM). See Samorodnitsky and Taqqu (1994) and Maejima (1989) for the details of LFSM and FBM.

Let L_t^x be the *local time* of the LFSM $\Lambda_{\alpha,H}(t)$ at x upto the time t. See Jeganathan (2004) for the existence and other details of the local time of the LFSM. In particular we have the representation

$$L_t^x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^t e^{iu(\Lambda_{\alpha,H}(s)-x)} ds du. \tag{2}$$

Note that the local time for $Z_\alpha(t)$ does not exist when $0 < \alpha \leq 1$, but our interest is restricted to $\Lambda_{\alpha,\frac{1}{\alpha}}(t) = Z_\alpha(t)$, $1 < \alpha \leq 2$.

For convenience, we shall

henceforth denote $\Lambda_{\alpha,H}(t)$ by $\Lambda(t)$ and $Z_\alpha(t)$ by $Z(t)$.

We next state the main results of this paper

Theorem 1. *Assume that $\frac{1}{3} < H < 1$. Let the Borel function $f(x)$ be such that*

$$\int_{-\infty}^{\infty} \left(|f(x)| + |xf(x)|\right) dx < \infty \tag{3}$$

and

$$\int_{-\infty}^{\infty} f(x)\, dx = 0. \tag{4}$$

Then for every $M > 0$, and for any distinct reals u_1,...,u_q,

$$\begin{aligned} t \quad &\longmapsto \quad \left(\kappa^{-\frac{1-H}{2}} \int_0^{\kappa t} f\left(\Lambda(s) - \kappa^H u_i\right) ds, i = 1, ..., q\right) \\ &\Longrightarrow \quad \left(\sqrt{b} W^{(u_i)}\left(L_t^{u_i}\right), i = 1, ..., q\right) \quad \text{in } \mathbb{C}\left([0, M] \longmapsto \mathbb{R}^q\right) \end{aligned}$$

as $\kappa \to \infty$, where $W^{(u_i)}(t), i = 1, ..., q$, are independent standard Brownian motions independent of the process $\Lambda(t)$, and

$$b = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty \left|\widehat{f}(\mu)\right|^2 E\left[e^{-i\mu\Lambda(t)}\right] d\mu dt.$$

Here $\mathbb{C}([0, M] \longmapsto \mathbb{R}^q)$ stands for the space of all continuous functions defined on the interval $[0, M]$ and taking values in $\mathbb{R}^q$. See Billingsly (1968, Chapter 2) for the weak convergence in $\mathbb{C}([0, M] \longmapsto \mathbb{R}^q)$. It will become clear later that, for any Borel function $h(y)$ such that $\int_{-\infty}^\infty |h(y)|\, dy < \infty$, the convergence in Theorem 1 can be extended to the form

$$\begin{aligned} &\left(\kappa^{-H}\Lambda(\kappa^H t), \kappa^{-(1-H)} \int_0^{\kappa t} h\left(\Lambda(s) - \kappa^H u_i\right) ds,\right. \\ &\left.\kappa^{-\frac{1-H}{2}} \int_0^{\kappa t} f\left(\Lambda(s) - \kappa^H u_i\right) ds, i = 1, ..., q\right) \\ \Longrightarrow \quad &\left(\Lambda(t), L_t^{u_i} \int_{-\infty}^\infty h(y)\, dy, \sqrt{b} W^{(u_i)}\left(L_t^{u_i}\right), i = 1, ..., q\right) \end{aligned} \tag{5}$$

in $\mathbb{C}\left([0, M] \longmapsto \mathbb{R}^{1+2q}\right)$.

It can be seen using Plancherel's theorem that the constant b in Theorem 1 has an alternative form

$$b = 2 \int_0^\infty \int_{-\infty}^\infty E\left[f(x) f(x + \Lambda(t))\right] dx dt.$$

Now let $\phi(y)$ be the probability density function of $\Lambda(1)$. Then the constant b has also the form

$$b = -2 \int\int f(x) \left\{c_1 \mathbb{I}_{\{y>x\}} (y - x)^{\frac{1}{H}-1} + c_2 \mathbb{I}_{\{y<x\}} (x - y)^{\frac{1}{H}-1}\right\} f(y)\, dy dx,$$

where $\mathbb{I}_{\{y>x\}}$ stands for the indicator function of the set $\{y > x\}$, and

$$c_1 = \int_0^\infty \left(\frac{1}{t^H}\phi(0) - \frac{1}{t^H}\phi\left(\frac{1}{t^H}\right)\right) dt, \qquad c_2 = \int_0^\infty \left(\frac{1}{t^H}\phi(0) - \frac{1}{t^H}\phi\left(-\frac{1}{t^H}\right)\right) dt.$$

This is because

$$
\begin{aligned}
& \int_0^\infty \int_{-\infty}^\infty E\left[f(x) f(x+\Lambda(t))\right] dxdt \\
= & \int_0^\infty \int_{-\infty}^\infty f(x) E\left[f\left(x+t^H \Lambda(1)\right)\right] dxdt \\
= & \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f(x) f\left(x+t^H y\right)(\phi(y)-\phi(0))\, dydxdt \qquad \text{using (4)} \\
= & \int_0^\infty \frac{1}{t^H} \int_{-\infty}^\infty \int_{-\infty}^\infty f(x) f(y)\left(\phi\left(\frac{y-x}{t^H}\right)-\phi(0)\right) dydxdt,
\end{aligned}
$$

which gives the stated form by making, for each x and y, the transformations $\frac{(y-x)^{\frac{1}{H}}}{t} \longmapsto \frac{1}{t}$ and $\frac{(x-y)^{\frac{1}{H}}}{t} \longmapsto \frac{1}{t}$ respectively when $y > x$ and $y < x$. Note that when $\phi(y)$ symmetric around 0, $c_1 = c_2 = c$, say, then

$$b = -2c \int \int f(x) |x-y|^{\frac{1}{H}-1} f(y)\, dydx.$$

Note further that c is finite only when $\frac{1}{3} < H < 1$. It will be indicated below that the restriction $\frac{1}{3} < H < 1$ in Theorem 1 cannot be relaxed.

Theorem 1 for the special cases mentioned next is usually stated for $q = 1$ and for $u_i = 0$ and the following remarks pertain to this case. In the case $\Lambda(t)$ is a Brownian motion (that is, the case $\alpha = 2$ and $H = \frac{1}{2}$), Theorem 1 for the one-dimensional convergence for $t = 1$, goes back to Skorokhod and Slobodenjuk (1966), and in the above stated general form of convergence in $\mathbb{C}([0, M] \longmapsto \mathbb{R})$, it is due, independently of Skorokhod and Slobodenjuk (1966), to Papanicolaou, Strook and Varadhan (1977). The statement for this case is also presented in Ikeda and Watanabe (1989, Chapter III, Section 4.4) and Revuz and Yor (1991, Chapter XIII, Section 2). (It may be noted that the statements in Skorokhod and Slobodenjuk (1966) are more general with respect to the forms of the functions $f(x)$ involved, in the sense that they allow $f(x)$ to be regularly varying at infinity also, for which the limiting forms in Theorem 1 will be different.)

In the case $\Lambda(t)$ is a symmetric stable process (that is, the case $1 < \alpha \leq 2$ and $H = \frac{1}{\alpha}$), Theorem 1 is due to Rosen (1991, Theorem 1.1). In this case note that Theorem 1 above does not assume the symmetry requirement. However the statement for this case without the symmetry assumption is implicit in Borodin and Ibragimov (1995) because it is a continuous analogue of the discrete versions established there and it was stated there (Chapter III, Section 6.3, page 129) that appropriate continuous analogues of the discrete versions will hold.

Theorem 1 itself for the one-dimensional convergence for $t = 1$ was stated (without proof) in Jeganathan (2006, Remark 3) as a continuous time analogue of the discrete time version established there.

To state the next statement, let, for $0 < \gamma < 2$,

$$\Gamma_\gamma(x, y) = \frac{1}{2}\left(|x|^\gamma + |y|^\gamma - |x-y|^\gamma\right).$$

Let

$$(t, x) \longmapsto B_\gamma(t, x)$$

be a Gaussian process with covariance

$$E\left[B_\gamma(s, x) B_\gamma(t, y)\right] = (s \wedge t) \Gamma_\gamma(x, y).$$

Theorem 2. *Assume that $1/3 < H < 1$. Then for every $M > 0$ and $A > 0$, and for any distinct reals $u_1, ..., u_q$,*

$$\begin{aligned}(t,x) &\longmapsto \left(\varepsilon^{-\frac{1-H}{2H}}\left(L_t^{\varepsilon x+u_i} - L_t^{u_i}\right), i = 1,...,q\right) \\ &\Longrightarrow \left(\sqrt{b}B^{(u_i)}_{\frac{1}{H}-1}\left(L_t^{u_i}, x\right), i = 1,...,q\right) \quad \text{in } \mathbb{C}\left([0,M]\times[-A,A]\longmapsto \mathbb{R}^q\right)\end{aligned}$$

as $\varepsilon \to 0$, where $B^{(u_i)}_{\frac{1}{H}-1}(t,x)$, $i = 1,...,q$, are independent processes, each having the same distribution as that of $B_{\frac{1}{H}-1}(t,x)$ introduced above, and independent of the process $\Lambda(t)$, and

$$b = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \left(4\sin^2\left(\frac{\mu}{2}\right)\right) E\left[e^{-i\mu\Lambda(t)}\right] d\mu dt.$$

Note that the requirement $1/3 < H < 1$ is the same as $0 < \frac{1}{H} - 1 < 2$.

Similar to the extension (5) of Theorem 1, Theorem 2 has the extension:

$$\begin{aligned}(t,x) &\longmapsto \left(\Lambda(t), L_t^{u_i}, \varepsilon^{-\frac{1-H}{2H}}\left(L_t^{\varepsilon x+u_i} - L_t^{u_i}\right), i = 1,...,q\right) \\ &\Longrightarrow \left(\Lambda(t), L_t^{u_i}, \sqrt{b}B^{(u_i)}_{\frac{1}{H}-1}\left(L_t^{u_i}, x\right), i = 1,...,q\right) \qquad (6)\end{aligned}$$

in $\mathbb{C}\left([0,M]\times[-A,A]\longmapsto \mathbb{R}^{1+2q}\right)$.

In the case $\Lambda(t)$ is a Brownian motion, Theorem 2 is due to Yor (1983). In the case $\Lambda(t)$ is a symmetric stable process, it is due to Rosen (1991, Theorem 1.2) for $q = 1$ and to Eisenbaum (1996) for the general q.

Regarding these results, it may be noted that the restriction $1/3 < H < 1$ cannot be relaxed, as can be seen from the known regularity properties of L_1^x with respect to the space variable x when L_1^x is the local time of the fractional Brownian motion (the case $\alpha = 2$), see Geman and Horowitz (1980, Table 2).

We shall also obtain the following bounds, which will in particular give the required tightness for convergencies in Theorems 1 and 2 above.

Proposition 3. *Assume that $1/3 < H < 1$, and let $f(x)$ be as in Theorem 1 above, satisfying (3) and (4). Then for any integer $l \geq 1$, there is a constant C depending only on l and H, such that*

$$\sup_{\kappa>0,u} E\left[\left(\kappa^{-\frac{1-H}{2}}\int_{\kappa s}^{\kappa t} f\left(\Lambda(v) + \kappa^H u\right) dv\right)^{2l}\right] \leq C\left|t-s\right|^{l(1-H)}.$$

Proposition 4. *Assume that $1/3 < H < 1$. Then for any integer $l \geq 1$, there is a constant C depending only on l and H, such that*

$$\begin{aligned}&\sup_{\varepsilon>0,u} E\left[\left(\varepsilon^{-\frac{1-H}{2H}}\left(L_t^{\varepsilon x+u} - L_t^u - \left(L_s^{\varepsilon y+u} - L_s^u\right)\right)\right)^{2l}\right] \\ \leq\ & C\left|t\right|^{l(1-H)}\left|x-y\right|^{\frac{l(1-H)}{H}} + C\left|y\right|^{\frac{l(1-H)}{H}}\left|t-s\right|^{l(1-H)}.\end{aligned}$$

The methodology we employ is a continuous time analogue of that employed in Jeganathan (2006). This will allow us to treat both the Theorems 1 and 2 in a unified way. Before indicating this we briefly recall the methods employed in the existing works mentioned earlier.

First, in obtaining Theorem 1, both Skorokhod and Slobodenjuk (1966) and Papanicolaou, Strook and Varadhan (1977) initially invoke Itô's formula, thereby reducing the convergence to that of a stochastic integral, but the former work uses the method of moments to obtain the required convergence whereas the later work uses a much simpler approach of employing an appropriate continuous time martingale CLT. The method of Yor (1983), in obtaining Theorem 2, is similar to that of Papanicolaou, Strook and Varadhan (1977).

In the case $\Lambda(t)$ is a symmetric stable process, in which the Itô's formula is not available, Rosen (1991) employs the method of moments directly in obtaining Theorem 2, using the representation (2). On the other hand Borodin and Ibragimov (1995) when dealing with the discrete analogue of Theorem 1, employ Fourier analytic method and reduce the situation in the limit to essentially a continuous time martingale CLT, and hence their method is much simpler than that of Rosen (1991). As noted earlier though only the discrete situation is dealt with in detail in Borodin and Ibragimov (1995), it was mentioned, independently of Rosen (1991), that the method is applicable to the continuous time situations such as Theorem 1 also. In the context of Theorem 2, see Eisenbaum (1996) for yet another approach for this case under symmetry.

The present method is adopted from Jeganathan (2006) and is Fourier analytic similar to that in Borodin and Ibragimov (1995) but we reduce the situation directly to a discrete time martingale CLT, see Section 2 below. Note however that in all the situations mentioned above, the processes involved have stationary independent increments, and the above mentioned methods themselves are tied to such structures, but unfortunately such structures are not available in the present case.

We next indicate that both the Theorems 1 and 2 can be treated in a unified way.

Unification of Theorems 1 and 2 and of Propositions 3 and 4. Let

$$\widehat{f}(\mu) = \int_{-\infty}^{\infty} e^{i\mu x} f(x)\, dx.$$

Note that the restrictions (3) and (4) in Theorem 1 entail that

$$\left|\widehat{f}(\mu)\right| \leq C \min(|\mu|, 1). \tag{7}$$

We have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu x}\widehat{f}(\mu)\, d\mu.$$

In particular

$$f\left(\Lambda(t) - \kappa^H u\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu\left(\Lambda(t)-\kappa^H u\right)}\widehat{f}(\mu)\, d\mu,$$

and hence

$$\int_0^{\kappa t} f\left(\Lambda(s) - \kappa^H u\right) ds = \frac{1}{2\pi}\int_0^{\kappa t}\int_{-\infty}^{\infty} e^{-i\mu\left(\Lambda(s)-\kappa^H u\right)}\widehat{f}(\mu)\, d\mu ds, \tag{8}$$

which is involved in Theorem 1. It will become clear that in dealing with this, we shall invoke only the requirement (7).

Regarding Theorem 2, using the representation (2), we have

$$\begin{aligned}
L_t^{\varepsilon x+u} - L_t^u &= \frac{1}{2\pi}\int_0^t \int_{-\infty}^{\infty} e^{i\mu(\Lambda(s)-u)}\left(e^{-i\mu\varepsilon x} - 1\right) d\mu ds \\
&=^D \frac{1}{\kappa 2\pi}\int_0^{\kappa t}\int_{-\infty}^{\infty} e^{i\mu\left(\kappa^{-H}\Lambda(s)-u\right)}\left(e^{-i\mu\varepsilon x} - 1\right) d\mu ds \\
&= \frac{1}{\kappa^{1-H}2\pi}\int_0^{\kappa t}\int_{-\infty}^{\infty} e^{i\mu\left(\Lambda(s)-\kappa^H u\right)}\left(e^{-i\mu\kappa^H\varepsilon x} - 1\right) d\mu ds, \qquad (9)
\end{aligned}$$

where $=^D$ stands for the equivalence of all finite dimensional distributions of the random processes involved. Thus, if we take κ such that

$$\kappa^H \varepsilon = 1,$$

and let

$$F(\mu) = \left(e^{i\mu x} - 1\right),$$

then

$$\kappa^{1-H}\left(L_t^{\varepsilon x+u} - L_t^u\right) = \frac{1}{2\pi}\int_0^{\kappa t}\int_{-\infty}^{\infty} e^{-i\mu\left(\Lambda(s)-\kappa^H u\right)}F(\mu)\,d\mu ds. \tag{10}$$

This is exactly of the form (8), in addition to having the requirement (7) for $F(\mu)$, that is, $|F(\mu)| \leq C\min(|\mu|, 1)$, where the constant C may depend on x.

In the same way, Propositions 3 and 4 are unified. More specifically, in place Proposition 3, we establish the following

Proposition 5. *Assume that $1/3 < H < 1$, and let $F(\mu)$ be such that*

$$|F(\mu)| \leq C\min(|\mu|, 1)$$

for a constant C.

Then for any integer $l \geq 1$, there is a constant D depending only on l, H and the constant C, such that

$$\sup_{\kappa>0,u} E\left[\left(\kappa^{-\frac{1-H}{2}}\int_{\kappa s}^{\kappa t}\int_{-\infty}^{\infty} e^{-i\mu\left(\Lambda(v)-\kappa^H u\right)}F(\mu)\,d\mu dv\right)^{2l}\right] \leq D\,|t-s|^{l(1-H)}.$$

It is clear that Proposition 3 follows from this, in view of (7) and (8). The same is the case for Proposition 4. To see this, fix x and y with $x > y$, and take κ such that

$$\kappa^H \varepsilon (x-y) = 1.$$

Then we have, as in (9) and (10),

$$\begin{aligned}\kappa^{1-H}\left(L_t^{\varepsilon x+u} - L_t^{\varepsilon y+u}\right) &=^D \frac{1}{2\pi}\int_0^{\kappa t}\int_{-\infty}^{\infty} e^{-i\mu\left(\Lambda(s)-\kappa^H u\right)}e^{i\mu\frac{y}{x-y}}\left(e^{i\mu}-1\right)d\mu ds \\ &= \frac{1}{2\pi}\int_0^{\kappa t}\int_{-\infty}^{\infty} e^{-i\mu\left(\Lambda(s)-\kappa^H u\right)}F(\mu)\,d\mu ds \end{aligned} \tag{11}$$

where now

$$F(\mu) = e^{i\mu\frac{y}{x-y}}\left(e^{i\mu}-1\right), \tag{12}$$

which clearly satisfies $|F(\mu)| \leq C\min(|\mu|, 1)$, where the constant C does not depend on x and y. Hence we have by applying Proposition 5,

$$\begin{aligned} E\left[\left(\kappa^{\frac{(1-H)}{2}}\left(L_t^{\varepsilon x+u} - L_t^{\varepsilon y+u}\right)\right)^{2l}\right] &= E\left[\left((\varepsilon(x-y))^{-\frac{(1-H)}{2H}}\left(L_t^{\varepsilon x} - L_t^{\varepsilon y}\right)\right)^{2l}\right] \\ &\leq D\,|t|^{l(1-H)}. \end{aligned}$$

This is the same as

$$E\left[\left(\varepsilon^{-\frac{(1-H)}{2H}}\left(L_t^{\varepsilon x+u} - L_t^{\varepsilon y+u}\right)\right)^{2l}\right] \leq D\,|t|^{l(1-H)}\,|x-y|^{\frac{l(1-H)}{H}}.$$

Also, using (10) we have

$$
\begin{aligned}
& 2\pi\,(\varepsilon y)^{-\frac{1-H}{2H}}\left(L_t^{\varepsilon y+u}-L_t^u-\left(L_s^{\varepsilon y+u}-L_s^u\right)\right)\\
=^D\quad & \kappa^{-\frac{1-H}{2}}\int_{\kappa s}^{\kappa t}\int_{-\infty}^{\infty} e^{-i\mu\left(\Lambda(v)-\kappa^H u\right)}F(\mu)\,d\mu dv
\end{aligned}
$$

with $\kappa^H\varepsilon y=1$ and $F(\mu)=\left(e^{i\mu}-1\right)$. Thus, by Proposition 5 again

$$
E\left[\left(\varepsilon^{-\frac{1-H}{2H}}\left(L_t^{\varepsilon y+u}-L_t^u-\left(L_s^{\varepsilon y+u}-L_s^u\right)\right)\right)^{2l}\right]\le D\,|y|^{\frac{l(1-H)}{H}}\,|t-s|^{l(1-H)}.
$$

The preceding two inequalities give Proposition 4, in view of

$$
\begin{aligned}
& E\left[\left(\varepsilon^{-\frac{1-H}{2H}}\left(L_t^{\varepsilon x+u}-L_t^u-\left(L_s^{\varepsilon y+u}-L_s^u\right)\right)\right)^{2l}\right]\\
\le\quad & CE\left[\left(\varepsilon^{-\frac{(1-H)}{2H}}\left(L_t^{\varepsilon x+u}-L_t^{\varepsilon y+u}\right)\right)^{2l}\right]\\
& +CE\left[\left(\varepsilon^{-\frac{1-H}{2H}}\left(L_t^{\varepsilon y+u}-L_t^u-\left(L_s^{\varepsilon y+u}-L_s^u\right)\right)\right)^{2l}\right].
\end{aligned}
$$

■

The plan of the rest of the paper is as follows. In Section 2, we shall obtain the convergence of finite dimensional distributions in Theorems 1 and 2. As in Jeganathan (2006), this is done by reducing this problem to a discrete time martingale CLT, so that most of Section 2 is devoted to the verification of the requirements of this CLT. The verification of the Lindeberg condition of this CLT and the proof of Proposition 5 are presented in Section 3 because they involve the similar ideas as well as similar computations. (It may noted that in the special case $H=\frac{1}{\alpha}$, in which $\Lambda(t)=\int_0^t Z_\alpha(du)=Z_\alpha(t)$, many of the computations of this paper will become unnecessary and the method itself will take a much simpler form.)

Notations. In addition to the notation $=^D$ used above for the equality of the finite dimensional distributions of two random processes, $\overset{fdd}{\Longrightarrow}$ stands for the convergence in distribution of all finite dimensional distributions of a sequence of random processes.

E_t will stand for the conditional expectation given the σ-field $\sigma\left(Z(s);\ s\le t\right)$.

Throughout below we let

$$
\Lambda_{(s_1,s_2]}(t)=\int_{s_1}^{s_2}(t-u)^{H-1/\alpha}Z(du),\quad 0\le s_1<s_2\le t,
$$

and more generally, for $-\infty<s_1<s_2\le t$,

$$
\Lambda_{(s_1,s_2]}(t)=\int_{s_1}^{s_1\vee 0}\left\{(t-u)^{H-1/\alpha}-(-u)^{H-1/\alpha}\right\}Z(du)+\int_{s_1\vee 0}^{s_2}(t-u)^{H-1/\alpha}Z(du).
$$

Note that

$$
\Lambda_{(s_1,s_2]}(t)\text{ and }\Lambda_{(t_1,t_2]}(t)\text{ are independent if }(s_1,s_2]\text{ and }(t_1,t_2]\text{ are disjoint.}
$$

We also have the self-similarity property (already used in some form)

$$
\Lambda_{(s_1,s_2]}(t)=^D a^{-H}\Lambda_{(as_1,as_2]}(at)\quad\text{for any constant }a>0.
$$

In particular

$$
\left|E\left[e^{i\mu\Lambda(t)}\right]\right|\le E\left[e^{i\mu\Lambda_{(0,t]}(t)}\right]=E\left[e^{i\mu t^H\Lambda_{(0,1]}(1)}\right]\le e^{-\left|\mu t^H\right|^\alpha\int_0^1\left|u^{H-\frac{1}{\alpha}}\right|^\alpha du},
$$

where in obtaining the last inequality, we have used the fact that $\Lambda_{(0,1]}(1)$ has the stable distribution with scale parameter σ such that $\sigma^\alpha = \int_0^1 \left| u^{H-\frac{1}{\alpha}} \right|^\alpha du$, see (1). The preceding inequality and its analogues will be used extensively below.

2 Convergence of finite dimensional distributions

It follows from the discussions of Section 1, and as will be indicated at the end of this section, that the convergence of finite dimensional distributions in both the theorems 1 and 2 will follows from the following result.

Proposition 6. *Assume that* $1/3 < H < 1$*, and let* $F(\mu)$ *be such that*

$$|F(\mu)| \leq C \min(|\mu|, 1). \tag{13}$$

Then, for any distinct reals $u_1, ..., u_q$*,*

$$\begin{aligned} t \quad &\longmapsto \quad \left(\kappa^{-\frac{1-H}{2}} \int_0^{\kappa t} \int_{-\infty}^{\infty} e^{-i\mu\left(\Lambda(s)-\kappa^H u_i\right)} F(\mu)\, d\mu ds, i = 1, ..., q \right) \\ &\overset{fdd}{\Longrightarrow} \left(\sqrt{b^*} W^{(u_i)}\left(L_t^{u_i}\right), i = 1, ..., q \right) \end{aligned}$$

where $W^{(u_i)}(t), i = 1, ..., q$*, are independent standard Brownian motions, independent of the process* $\Lambda(t)$*, and*

$$b^* = 2 \int_0^\infty \int_{-\infty}^\infty |F(\mu)|^2 E\left[e^{-i\mu\Lambda(t)} \right] d\mu dt.$$

The proof of this result given below is directly based on Jeganathan (2006), see Proposition 7 below, and therefore, one actually has the following more general statement, where $F_1(\mu)$ is such that $|F_1(\mu)| \leq C$.

$$\begin{aligned} &\left(\kappa^{-H} \Lambda(\kappa^H t), \kappa^{-(1-H)} \int_0^{\kappa t} \int_{-\infty}^{\infty} e^{-i\mu\left(\Lambda(s)-\kappa^H u_i\right)} F_1(\mu)\, d\mu ds, \right. \\ &\left. \kappa^{-\frac{1-H}{2}} \int_0^{\kappa t} \int_{-\infty}^{\infty} e^{-i\mu\left(\Lambda(s)-\kappa^H u_i\right)} F(\mu)\, d\mu ds, i = 1, ..., q \right) \\ &\overset{fdd}{\Longrightarrow} \left(\Lambda(t), F_1(0) L_t^{u_i}, \sqrt{b^*} W^{(u_i)}\left(L_t^{u_i}\right), i = 1, ..., q \right). \end{aligned} \tag{14}$$

This will give the generalizations (5) and (6) of Theorems 1 and 2 stated earlier.

It is convenient to present the detailed proof for the case $u_i = 0, i = 1, ..., q$, and then indicate the required modifications for the general case. For this particular case we need to show that

$$t \longmapsto \kappa^{-\frac{1-H}{2}} \int_0^{\kappa t} \int_{-\infty}^{\infty} e^{-i\mu\Lambda(s)} F(\mu)\, d\mu ds \overset{fdd}{\Longrightarrow} \sqrt{b} W\left(L_t^0\right)$$

where $W(t)$ is a standard Brownian motion independent of the process $\Lambda(t)$. More specifically, for each finite $0 < t_1 < ... < t_r < \infty$ and for each reals $v_1, ..., v_r$, we first show that

$$\kappa^{-\frac{1-H}{2}} \sum_{i=0}^r v_i \int_{\kappa t_{i-1}}^{\kappa t_i} \int_{-\infty}^{\infty} e^{-i\mu\Lambda(s)} F(\mu)\, d\mu ds \Longrightarrow \sqrt{b^*} \sum_{i=0}^r v_i \left(W\left(L_{t_i}^0\right) - W\left(L_{t_{i-1}}^0\right) \right). \tag{15}$$

As in Jeganathan (2006), we shall reduce this problem to the application of a martingale CLT. For this purpose, let, for integer $m > 1$,

$$\zeta_{\kappa m l} = \kappa^{-\frac{1-H}{2}} \int_{\kappa \frac{l-1}{m}}^{\kappa \frac{l}{m}} \int_{-\infty}^{\infty} e^{-i\mu\Lambda(s)} F(\mu)\, d\mu ds, \qquad l = 1, 2, ...$$

Let $\ell_{1m}, ..., \ell_{rm}$ be integers such that such that

$$\frac{\ell_{im}}{m} \leq t_i < \frac{\ell_{im}+1}{m}. \tag{16}$$

Then it is clear from Proposition 5 of Section 1 above that, in order to obtain (15), we need to show that

$$\sum_{i=0}^{r} v_i \sum_{l=\ell_{i-1,m}+1}^{\ell_{i,m}} \zeta_{\kappa ml} \Longrightarrow \sqrt{b^*} \sum_{i=0}^{r} v_i \left(W\left(L_{t_i}^0\right) - W\left(L_{t_{i-1}}^0\right)\right) \tag{17}$$

as $\kappa \to \infty$ first and then $m \to \infty$.

For this purpose we shall show that the following requirements hold (recall that E_t stands for the conditional expectation given $\sigma\left(Z(s);\ s \leq t\right)$).

(R1) There is a nonrandom $\Delta\left(\kappa, m, L\right)$ such that, for each m and $L > 0$,

$$\sum_{l=1}^{[mL]} \left| E_{\kappa \frac{l-1}{m}} \left[\zeta_{\kappa ml}\right]\right| \leq \Delta\left(\kappa, m, L\right) \to 0 \quad \text{as } \kappa \to \infty.$$

(R2) For the integers ℓ_{im} as in (16),

$$\sum_{i=0}^{r} v_i^2 \sum_{l=\ell_{i-1,m}+1}^{\ell_{i,m}} E_{\kappa \frac{l-1}{m}} \left[\zeta_{\kappa ml}^2\right] \Longrightarrow b^* \sum_{i=0}^{r} v_i^2 \left(L_{t_i}^0 - L_{t_{i-1}}^0\right)$$

as $\kappa \to \infty$ first and then $m \to \infty$, where the constant b^* is as specified above in Proposition 6.

Here recall that the convergence in distribution of a sequence of distribution functions is metrizable, for example by the Lévy distance (see Loève (1963, page 215)). Then the preceding convergence means that the distribution of the left hand side converges in such a metric to that of the right hand side as $\kappa \to \infty$ first and then $m \to \infty$.

(R3) For every $L > 0$,

$$\lim_{m\to\infty} \limsup_{\kappa\to\infty} \sum_{l=1}^{[mL]} E\left[\zeta_{\kappa ml}^4\right] = 0.$$

The next condition (R4) pertains only to the case $\alpha = 2$, in which case $Z\left(t\right)$ reduces to a Brownian motion. To state it define

$$\chi_{\kappa ml} = \frac{1}{\sqrt{\kappa}} \left(Z\left(\kappa \frac{l}{m}\right) - Z\left(\kappa \frac{l-1}{m}\right)\right). \tag{18}$$

(R4) *When* $\alpha = 2$

$$\limsup_{\kappa\to\infty} P\left[\sum_{l=1}^{[mL]} \left| E_{\kappa \frac{l-1}{m}} \left[\zeta_{\kappa ml}\chi_{\kappa ml}\right]\right| > \varepsilon \right] = 0 \text{ for each } m,\ L > 0 \text{ and } \varepsilon > 0.$$

Proposition 7. *Suppose that (R1) - (R4) above are satisfied. Then the convergence (17), and hence that in (15) holds. (Actually the more general convergence (14) holds for* $u_i = 0, i = 1, ..., q$*.)*

Proof. This statement (including the generalization (14)), as well as its detailed proof, is essentially contained in Jeganathan (2006, Section 2) because the above requirements (R1) - (R4) are the same as the (R1) - (R4) stated there. ■

We next consider the verification of (R1) - (R4). In the rest of the paper,

we shall assume without further mentioning that (13) holds.

Verification of (R3). (R3) follows from Proposition 5, by choosing the integer l such that $2l > 4$ and $l(1-H) > 1$ and taking $s = \kappa\frac{l-1}{m}$, $t = \kappa\frac{l}{m}$. However, this will require the computation of the moments of order greater than 4. We shall indicate later (see the Remark at the end of Section 3) that the computation of the fourth moment is sufficient. ■

The next lemma verifies (R1).

Lemma 8. *One has*

$$\left|E_{\kappa\frac{l-1}{m}}\left[\zeta_{\kappa ml}\right]\right| \leq C\kappa^{-\frac{1-H}{2}} + Cm^{2H-1}\kappa^{\frac{1-3H}{2}},$$

and hence, because $\frac{1}{3} < H < 1$, (R1) holds.

Proof. We have

$$E_{\kappa\frac{l-1}{m}}\left[\zeta_{\kappa ml}\right] = \kappa^{-\frac{1-H}{2}} \int_{\kappa\frac{l-1}{m}}^{\kappa\frac{l}{m}} \int_{-\infty}^{\infty} E_{\kappa\frac{l-1}{m}}\left[e^{-i\mu\Lambda(t)}\right] F(\mu)\, d\mu dt.$$

Recall that $\Lambda(t) = \Lambda_{\left(-\infty,\kappa\frac{l-1}{m}\right]}(t) + \Lambda_{\left(\kappa\frac{l-1}{m},t\right]}(t)$ where $\Lambda_{\left(-\infty,\kappa\frac{l-1}{m}\right]}(t)$ and $\Lambda_{\left(\kappa\frac{l-1}{m},t\right]}(t)$ are independent. Hence

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left|E_{\kappa\frac{l-1}{m}}\left[e^{-i\mu\Lambda(t)}\right]\right| |F(\mu)|\, d\mu \\
\leq\ & \int_{-\infty}^{\infty} \left|E\left[e^{-i\mu\Lambda_{\left[\kappa\frac{l-1}{m},t\right]}(t)}\right]\right| |F(\mu)|\, d\mu = \int_{-\infty}^{\infty} \left|E\left[e^{-i\mu\left(t-\kappa\frac{l-1}{m}\right)^H\Lambda_{(0,1]}(1)}\right]\right| |F(\mu)|\, d\mu \\
=\ & \left(t-\kappa\frac{l-1}{m}\right)^{-H} \int_{-\infty}^{\infty} \left|E\left[e^{-i\mu\Lambda_{(0,1]}(1)}\right]\right| \left|F\left(\left(t-\kappa\frac{l-1}{m}\right)^{-H}\mu\right)\right| d\mu \\
\leq\ & \left(t-\kappa\frac{l-1}{m}\right)^{-H} \int_{-\infty}^{\infty} e^{-c|\mu|^\alpha} \left|F\left(\left(t-\kappa\frac{l-1}{m}\right)^{-H}\mu\right)\right| d\mu. \qquad (19)
\end{aligned}$$

Suppose that $\kappa\frac{l}{m} - \kappa\frac{l-1}{m} = \frac{\kappa}{m} \leq 1$. Then, using $|F(\mu)| \leq C$, the preceding bound is bounded by $C\left(t-\kappa\frac{l-1}{m}\right)^{-H}$, and hence

$$\begin{aligned}
\left|E_{\kappa\frac{l-1}{m}}\left[\zeta_{\kappa ml}\right]\right| &\leq C\kappa^{-\frac{1-H}{2}} \int_{\kappa\frac{l-1}{m}}^{\kappa\frac{l}{m}} \left(t-\kappa\frac{l-1}{m}\right)^{-H} dt \\
&\leq C\kappa^{-\frac{1-H}{2}} \int_{\kappa\frac{l-1}{m}}^{\kappa\frac{l-1}{m}+1} \left(t-\kappa\frac{l-1}{m}\right)^{-H} dt \leq C\kappa^{-\frac{1-H}{2}}.
\end{aligned}$$

If $\kappa\frac{l}{m} - \kappa\frac{l-1}{m} = \frac{\kappa}{m} > 1$, then

$$\begin{aligned}
\left|E_{\kappa\frac{l-1}{m}}\left[\zeta_{\kappa ml}\right]\right| &\leq C\kappa^{-\frac{1-H}{2}} \int_{\kappa\frac{l-1}{m}}^{\kappa\frac{l-1}{m}+1} \left(t-\kappa\frac{l-1}{m}\right)^{-H} dt \\
&\quad + C\kappa^{-\frac{1-H}{2}} \left(\int_{-\infty}^{\infty} |\mu|\, e^{-c|\mu|^\alpha} d\mu\right) \int_{\kappa\frac{l-1}{m}+1}^{\kappa\frac{l}{m}} \left(t-\kappa\frac{l-1}{m}\right)^{-2H} dt,
\end{aligned}$$

where the first factor of the sum on the right hand side is obtained from (19) using $|F(\mu)| \leq C$ as in previous inequality, and the second factor is obtained from (19) using $|F(\mu)| \leq C|\mu|$. Now

$$\begin{aligned}
\kappa^{-\frac{1-H}{2}} \int_{\kappa\frac{l-1}{m}+1}^{\kappa\frac{l}{m}} \left(t-\kappa\frac{l-1}{m}\right)^{-2H} dt &\leq C\kappa^{-\frac{1-H}{2}}\left(1+m^{2H-1}\kappa^{1-2H}\right) \\
&= C\kappa^{-\frac{1-H}{2}} + Cm^{2H-1}\kappa^{\frac{1-3H}{2}}.
\end{aligned}$$

This completes the proof of the lemma. ■

The next lemma verifies (R4), where recall that (R4) pertains only to the case $\alpha = 2$.

Lemma 9. *Assume $\alpha = 2$ and let $\chi_{\kappa ml}$ be as in (18). Then*

$$\left|E_{\kappa\frac{l-1}{m}}\left[\zeta_{\kappa ml}\chi_{\kappa ml}\right]\right| \le C\kappa^{-\frac{1-H}{2}-\frac{1}{2}} + Cm^{2H-\frac{3}{2}}\kappa^{\frac{1-3H}{2}},$$

from which (R4) follows, because $\frac{1}{3} < H < 1$.

Proof. First note that because $\alpha = 2$, $Z(t)$ is a Brownian motion, and also recall that $\Lambda(t)$ is an integral with respect to $Z(t)$. We have

$$\begin{aligned}
&\kappa^{\frac{1-H}{2}+\frac{1}{2}}E_{\kappa\frac{l-1}{m}}\left[\zeta_{\kappa ml}\chi_{\kappa ml}\right] \\
=& \int_{\kappa\frac{l-1}{m}}^{\kappa\frac{l}{m}}\int_{-\infty}^{\infty}E_{\kappa\frac{l-1}{m}}\left[\left(Z\left(\kappa\frac{l}{m}\right)-Z\left(\kappa\frac{l-1}{m}\right)\right)e^{-i\mu\Lambda(s)}\right]F(\mu)\,d\mu ds.
\end{aligned}$$

Here

$$\begin{aligned}
&E_{\kappa\frac{l-1}{m}}\left[\left(Z\left(\kappa\frac{l}{m}\right)-Z\left(\kappa\frac{l-1}{m}\right)\right)e^{-i\mu\Lambda(s)}\right] \\
=& e^{-i\mu\Lambda_{\left(-\infty,\kappa\frac{l-1}{m}\right]}(s)}E\left[\left(Z(s)-Z\left(\kappa\frac{l-1}{m}\right)\right)e^{-i\mu\Lambda_{\left(\kappa\frac{l-1}{m},s\right]}(s)}\right] \\
=& e^{-i\mu\Lambda_{\left(-\infty,\kappa\frac{l-1}{m}\right]}(s)}\left(s-\kappa\frac{l-1}{m}\right)^{\frac{1}{2}}E\left[Z(1)\,e^{-i\mu\left(s-\kappa\frac{l-1}{m}\right)^{H}\Lambda_{(0,1]}(1)}\right]
\end{aligned}$$

where in obtaining the first equality we use $E_{\kappa\frac{l-1}{m}}\left[\left(Z\left(\kappa\frac{l}{m}\right)-Z(s)\right)e^{-i\mu\Lambda(s)}\right] = 0$, which is a consequence of $E\left[Z\left(\kappa\frac{l}{m}\right)-Z(s)\right] = 0$ and the independence of $Z\left(\kappa\frac{l}{m}\right)-Z(s)$ with respect to $\sigma\left(Z(t)\,;t\le s\right)$. Therefore

$$\begin{aligned}
&\kappa^{\frac{1-H}{2}+\frac{1}{2}}\left|E_{\kappa\frac{l-1}{m}}\left[\zeta_{\kappa ml}\chi_{\kappa ml}\right]\right| \\
\le& \int_{\kappa\frac{l-1}{m}}^{\kappa\frac{l}{m}}\int_{-\infty}^{\infty}\left(s-\kappa\frac{l-1}{m}\right)^{\frac{1}{2}-H}\left|E\left[Z_2(1)\,e^{-i\mu\Lambda_{(0,1]}(1)}\right]\right| \\
&\times\left|F\left(\left(s-\kappa\frac{l-1}{m}\right)^{-H}\mu\right)\right|d\mu ds.
\end{aligned}$$

Suppose that $\kappa\frac{l}{m}-\kappa\frac{l-1}{m} = \frac{\kappa}{m} \le 1$. Then, using $|F(\mu)| \le C$, the preceding bound is bounded by

$$\int_{\kappa\frac{l-1}{m}}^{\kappa\frac{l-1}{m}+1}\left(s-\kappa\frac{l-1}{m}\right)^{\frac{1}{2}-H}ds \le C.$$

Here we have used the fact that $\int_{-\infty}^{\infty}\left|E\left[Z(1)\,e^{-i\mu\Lambda_{(0,1]}(1)}\right]\right|d\mu < \infty$, which is a consequence of the fact that $\left(Z(1)\,,\Lambda_{(0,1]}(1)\right)$ has a non-degenerate bivariate normal distribution. In the case $\kappa\frac{l}{m}-\kappa\frac{l-1}{m} = \frac{\kappa}{m} > 1$, using in addition $|F(\mu)| \le C|\mu|$ and the fact $\int_{-\infty}^{\infty}|\mu|\left|E\left[Z(1)\,e^{-i\mu\Lambda_{(0,1]}(1)}\right]\right|d\mu < \infty$, we have the bound

$$\int_{\kappa\frac{l-1}{m}}^{\kappa\frac{l-1}{m}+1}\left(s-\kappa\frac{l-1}{m}\right)^{\frac{1}{2}-H}ds + \int_{\kappa\frac{l-1}{m}+1}^{\kappa\frac{l}{m}}\left(s-\kappa\frac{l-1}{m}\right)^{\frac{1}{2}-2H}ds \le C + C\left(\frac{\kappa}{m}\right)^{\frac{3}{2}-2H}.$$

Thus

$$\left|E_{\kappa\frac{l-1}{m}}\left[\zeta_{\kappa ml}\chi_{\kappa ml}\right]\right| \le C\kappa^{-\frac{1-H}{2}-\frac{1}{2}}\left(1+\left(\frac{\kappa}{m}\right)^{\frac{3}{2}-2H}\right),$$

which is the same as the inequality in the statement of the lemma. ■

We next verify (R2). We begin with some preliminaries. We have

$$\begin{aligned}
&\kappa^{1-H}E_{\kappa\frac{l-1}{m}}\left[\zeta_{\kappa ml}^{2}\right]\\
=\;& E_{\kappa\frac{l-1}{m}}\left[\left(\int_{\kappa\frac{l-1}{m}}^{\kappa\frac{l}{m}}\int_{-\infty}^{\infty}e^{-i\mu\Lambda(s)}F\left(\mu\right)d\mu ds\right)^{2}\right]\\
=\;& 2\int_{\kappa\frac{l-1}{m}}^{\kappa\frac{l}{m}}\int_{s_1}^{\kappa\frac{l}{m}}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E_{\kappa\frac{l-1}{m}}\left[e^{-i\mu_1\Lambda(s_1)-i\mu_2\Lambda(s_2)}\right]F\left(\mu_1\right)F\left(\mu_2\right)d\mu_1 d\mu_2\right\}ds_2 ds_1.
\end{aligned}$$

We write $(\Lambda(s_1),\Lambda(s_2))$, $\kappa\frac{l-1}{m}<s_1<s_2<\kappa\frac{l}{m}$, in the form

$$\left(\Lambda\left(\kappa\frac{l-1}{m}+t_1\right),\Lambda\left(\kappa\frac{l-1}{m}+t_1+t_2\right)\right),\qquad 0<t_1<\frac{\kappa}{m},\;0<t_2<\frac{\kappa}{m}-t_1.$$

Here recall that $\Lambda\left(\kappa\frac{l-1}{m}+t_1\right)=\Lambda_{\left(-\infty,\kappa\frac{l-1}{m}\right]}\left(\kappa\frac{l-1}{m}+t_1\right)+\Lambda_{\left(\kappa\frac{l-1}{m},\kappa\frac{l-1}{m}+t_1\right]}\left(\kappa\frac{l-1}{m}+t_1\right)$ and similarly for $\Lambda\left(\kappa\frac{l-1}{m}+t_1+t_2\right)$. We observe that

$\left(\Lambda_{\left(\kappa\frac{l-1}{m},\kappa\frac{l-1}{m}+t_1\right]}\left(\kappa\frac{l-1}{m}+t_1\right),\Lambda_{\left(\kappa\frac{l-1}{m},\kappa\frac{l-1}{m}+t_1+t_2\right]}\left(\kappa\frac{l-1}{m}+t_1+t_2\right)\right)$ is independent of $\sigma\left(Z(t);t\leq\kappa\frac{l-1}{m}\right)$, and in addition has the same distribution as that of

$$\begin{aligned}
&\left(\Lambda_{(0,t_1]}(t_1),\Lambda_{(0,t_1+t_2]}(t_1+t_2)\right)\\
&=^{D}\left(\int_0^{t_1}u^{H-1/\alpha}Z(du),\int_0^{t_1}\left(u+t_2\right)^{H-\frac{1}{\alpha}}Z(du)+\int_0^{t_2}u^{H-\frac{1}{\alpha}}Z^*(du)\right),
\end{aligned}\tag{20}$$

where $Z^*(du)$ is an independent copy of Z.

Hence one can write

$$\begin{aligned}
&E_{\kappa\frac{l-1}{m}}\left[e^{-i\mu_1\Lambda\left(\kappa\frac{l-1}{m}+t_1\right)-\mu_2\Lambda\left(\kappa\frac{l-1}{m}+t_1+t_2\right)}\right]\\
=\;&E\left[e^{-i\mu_1\left(y_1+\Lambda_{(0,t_1]}(t_1)\right)-\mu_2\left(y_2+\Lambda_{(0,t_1+t_2]}(t_1+t_2)\right)}\right]
\end{aligned}$$

with

$$(y_1,y_2)=\left(\Lambda_{\left(-\infty,\kappa\frac{l-1}{m}\right]}\left(\kappa\frac{l-1}{m}+t_1\right),\Lambda_{\left(-\infty,\kappa\frac{l-1}{m}\right]}\left(\kappa\frac{l-1}{m}+t_1+t_2\right)\right).\tag{21}$$

Thus

$$E_{\kappa\frac{l-1}{m}}\left[\zeta_{\kappa ml}^{2}\right]=2\kappa^{-(1-H)}\int_0^{\frac{\kappa}{m}}\int_0^{\frac{\kappa}{m}-t_1}I\left(y_1,y_2;t_1,t_2\right)dt_2dt_1,\tag{22}$$

where we set

$$\begin{aligned}
&I\left(y_1,y_2;t_1,t_2\right)\\
=\;&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i\mu_1y_1-\mu_2y_2}E\left[e^{-i\mu_1\Lambda_{(0,t_1]}(t_1)-\mu_2\Lambda_{(0,t_1+t_2]}(t_1+t_2)}\right]F\left(\mu_1\right)F\left(\mu_2\right)d\mu_1d\mu_2\\
=\;&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i\mu_1y_1-\mu_2(y_2-y_1)}\\
&\times E\left[e^{-i\mu_1\Lambda_{(0,t_1]}(t_1)-\mu_2\left(\Lambda_{(0,t_1+t_2]}(t_1+t_2)-\Lambda_{(0,t_1]}(t_1)\right)}\right]F\left(\mu_1-\mu_2\right)F\left(\mu_2\right)d\mu_1d\mu_2.
\end{aligned}\tag{23}$$

Note that

$$\begin{aligned}
&\left|I\left(y_1,y_2;t_1,t_2\right)\right|\\
\leq\;&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|E\left[e^{-i\mu_1\Lambda_{(0,t_1]}(t_1)-\mu_2\left(\Lambda_{(0,t_1+t_2]}(t_1+t_2)-\Lambda_{(0,t_1]}(t_1)\right)}\right]\right|\left|F\left(\mu_1-\mu_2\right)F\left(\mu_2\right)\right|d\mu_1d\mu_2\\
=\;&I_*\left(t_1,t_2\right),\text{ say.}
\end{aligned}\tag{24}$$

The requirement (R2) is then a consequence of the following four statements, stated in the form of Lemmas 10 - 13.

Lemma 10. *For each m, with $I_*(t_1,t_2)$ as in (24) above,*

$$\lim_{q\to\infty}\limsup_{\kappa\to\infty}\kappa^{-(1-H)}\int_0^{\frac{\kappa}{m}}\int_q^{\frac{\kappa}{m}} I_*(t_1,t_2)\,dt_2dt_1=0.$$

Lemma 11.

$$\lim_{q\to\infty}\int_q^{\infty}\int_{-\infty}^{\infty}\left|E\left[e^{-i\mu\Lambda(t)}\right]\right|\left|F(\mu)\right|^2 d\mu dt=0.$$

Lemma 12. *For each m and $q>0$, the difference between*

$$\kappa^{-(1-H)}\int_0^{\frac{\kappa}{m}}\int_0^{q} I(y_1,y_2;t_1,t_2)\,dt_2dt_1,\quad \text{with } (y_1,y_2) \text{ as in (21)},$$

and

$$\frac{1}{m^{1-H}}\int_0^1\int_{-\infty}^{\infty} e^{-i\mu_1 m^H\kappa^{-H}\Lambda_{\left(-\infty,\kappa\frac{l-1}{m}\right]}\left(\kappa\frac{l-1}{m}+\kappa\frac{t_1}{m}\right)}E\left[e^{-i\mu_1\Lambda_{(0,t_1]}(t_1)}\right]d\mu_1dt_1$$
$$\times\left(\int_0^q\int_{-\infty}^{\infty}E\left[e^{-i\mu_2\Lambda(t_2)}\right]\left|F(\mu_2)\right|^2 d\mu_2dt_2\right) \tag{25}$$

converges to 0 in probability.

In the preceding lemma 12, note that

$$\kappa^{-H}\Lambda_{\left(-\infty,\kappa\frac{l-1}{m}\right]}\left(\kappa\frac{l-1}{m}+\kappa\frac{t_1}{m}\right)=^D\Lambda_{\left(-\infty,\frac{l-1}{m}\right]}\left(\frac{l-1}{m}+\frac{t_1}{m}\right).$$

Lemma 13. With the integers $\ell_{i,m}$ as in (16),

$$\sum_{i=0}^{r}v_i^2\frac{1}{m^{1-H}}\sum_{l=\ell_{i-1,m}+1}^{\ell_{i,m}}\int_0^1\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\mu m^H\Lambda_{\left(-\infty,\frac{l-1}{m}\right]}\left(\frac{l-1}{m}+\frac{t}{m}\right)}E\left[e^{-i\mu\Lambda_{(0,t]}(t)}\right]d\mu\right]dt$$
$$\Longrightarrow\sum_{i=0}^{r}v_i^2\left(L^0_{t_i}-L^0_{t_{i-1}}\right)\qquad\text{as } m\to\infty.$$

Proof of Lemma 13. Taking (16) into account, lemma 13 is essentially the Lemma 18 in Jeganathan (2006), and therefore we shall not give the details of its proof. ■

Proof of Lemma 11. Recall that $\left|E\left[e^{-i\mu\Lambda(t)}\right]\right|\leq\left|E\left[e^{-i\mu_{(0,t]}\Lambda(t)}\right]\right|=\left|E\left[e^{-i\mu t^H\Lambda_{(0,1]}(1)}\right]\right|\leq e^{-c\left|\mu t^H\right|^{\alpha}}$, so that, using $|F(\mu)|\leq C|\mu|$ and making the transformation $\mu t^H\longmapsto\mu$,

$$\begin{aligned}\int_q^{\infty}\int_{-\infty}^{\infty}|F(\mu)|^2\left|E\left[e^{-i\mu\Lambda(t)}\right]\right|d\mu dt &\leq \int_q^{\infty}\frac{1}{t^{3H}}\int_{-\infty}^{\infty}|\mu|^2\left|\left|E\left[e^{-i\mu\Lambda_{(0,1]}(1)}\right]\right|\right|d\mu dt\\ &\leq C\int_q^{\infty}\frac{1}{t^{3H}}dt\to 0,\end{aligned}$$

as $q\to\infty$ because $3H>1$. ■

The proof of Lemma 10 will be given in the next Section 3 because the ideas involved are similar to the verification of (R3) or the proof of Proposition 5. We next concentrate on the proof of Lemma 12.

Note that, using $|F(\mu)|\leq C$, for $I_*(t_1,t_2)$ as defined in (24),

$$\begin{aligned}I_*(t_1,t_2) &\leq C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|E\left[e^{-i\mu_1\Lambda_{(0,t_1]}(t_1)-\mu_2\left(\Lambda_{(0,t_1+t_2]}(t_1+t_2)-\Lambda_{(0,t_1]}(t_1)\right)}\right]\right|d\mu_1d\mu_2\\ &= I_{**}(t_1,t_2)\,,\text{ say.}\end{aligned}$$

We next show that

$$I_{**}(t_1,t_2) \le C\frac{1}{t_1^H t_2^H}. \tag{26}$$

Using (20), one has

$$\begin{aligned} & \left| E\left[e^{-i\mu_1 \Lambda_{(0,t_1]}(t_1) - \mu_2\left(\Lambda_{(0,t_1+t_2]}(t_1+t_2) - \Lambda_{(0,t_1]}(t_1)\right)} \right] \right| \\ \le \quad & e^{-\int_0^{t_1} \left| \mu_1 u^{H-\frac{1}{\alpha}} + \mu_2\left((u+t_2)^{H-\frac{1}{\alpha}} - u^{H-\frac{1}{\alpha}} \right) \right|^\alpha du - \int_0^{t_2} \left| \mu_2 u^{H-\frac{1}{\alpha}} \right|^\alpha du}. \end{aligned} \tag{27}$$

Hence, making the transformations $\mu_1 \longmapsto \frac{\mu_1}{t_1^H}$, $\mu_2 \longmapsto \frac{\mu_2}{t_2^H}$,

$$\begin{aligned} & I_{**}(t_1,t_2) \\ \le \quad & \frac{C}{t_1^H t_2^H} \int\int e^{-\int_0^{t_1} \left| \frac{\mu_1}{t_1^H} u^{H-\frac{1}{\alpha}} + \frac{\mu_2}{t_2^H}\left((u+t_2)^{H-\frac{1}{\alpha}} - u^{H-\frac{1}{\alpha}} \right) \right|^\alpha du - \int_0^{t_2} \left| \frac{\mu_2}{t_2^H} u^{H-\frac{1}{\alpha}} \right|^\alpha du} d\mu_1 d\mu_2. \end{aligned}$$

We have

$$\begin{aligned} \int_0^{t_2} \left| \frac{\mu_2}{t_2^H} u^{H-\frac{1}{\alpha}} \right|^\alpha du &= \int_0^1 \left| \mu_2 u^{H-\frac{1}{\alpha}} \right|^\alpha du \\ &\ge \int_{\frac{1}{2}}^1 \left| \mu_2 u^{H-\frac{1}{\alpha}} \right|^\alpha du = \frac{1}{2} \left| \mu_2 u_2^{H-\frac{1}{\alpha}} \right|^\alpha, \end{aligned}$$

for $\frac{1}{2} \le u_2 \le 1$, where we have used the mean value theorem for integrals. Similarly

$$\begin{aligned} & \int_0^{t_1} \left| \frac{\mu_1}{t_1^H} u^{H-\frac{1}{\alpha}} + \frac{\mu_2}{t_2^H} \left((u+t_2)^{H-\frac{1}{\alpha}} - u^{H-\frac{1}{\alpha}} \right) \right|^\alpha du \\ = \quad & \int_0^{t_1} \left| \frac{\mu_1}{t_1^{H-\frac{1}{\alpha}}} u^{H-\frac{1}{\alpha}} + t_1^H \frac{\mu_2}{t_2^H} \frac{1}{t_1^{H-\frac{1}{\alpha}}} \left((u+t_2)^{H-\frac{1}{\alpha}} - u^{H-\frac{1}{\alpha}} \right) \right|^\alpha \frac{1}{t_1} du \\ = \quad & \int_0^1 \left| \mu_1 u^{H-\frac{1}{\alpha}} + t_1^H \frac{\mu_2}{t_2^H} \left(\left(u + \frac{t_2}{t_1} \right)^{H-\frac{1}{\alpha}} - u^{H-\frac{1}{\alpha}} \right) \right|^\alpha du \\ \ge \quad & \frac{1}{2} \left| \mu_1 u_1^{H-\frac{1}{\alpha}} + t_1^H \frac{\mu_2}{t_2^H} \left(\left(u_1 + \frac{t_2}{t_1} \right)^{H-\frac{1}{\alpha}} - u_1^{H-\frac{1}{\alpha}} \right) \right|^\alpha \end{aligned}$$

for $\frac{1}{2} \le u_1 \le 1$. Thus

$$\begin{aligned} & t_1^H t_2^H I_{**}(t_1,t_2) \\ \le \quad & C \int\int e^{-\frac{1}{2}\left| \mu_1 u_1^{H-\frac{1}{\alpha}} + t_1^H \frac{\mu_2}{t_2^H} \left(\left(u_1 + \frac{t_2}{t_1} \right)^{H-\frac{1}{\alpha}} - u_1^{H-\frac{1}{\alpha}} \right) \right|^\alpha - \frac{1}{2}\left| \mu_2 u_2^{H-\frac{1}{\alpha}} \right|^\alpha} d\mu_1 d\mu_2. \end{aligned} \tag{28}$$

Here we have

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left| \mu_1 u_1^{H-\frac{1}{\alpha}} + t_1^H \frac{\mu_2}{t_2^H} \left(\left(u_1 + \frac{t_2}{t_1} \right)^{H-\frac{1}{\alpha}} - u_1^{H-\frac{1}{\alpha}} \right) \right|^\alpha} d\mu_1 \\ = \quad & \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left| \mu_1 u_1^{H-\frac{1}{\alpha}} \right|^\alpha} d\mu_1 = C u_1^{-\left(H-\frac{1}{\alpha}\right)} \le C \end{aligned} \tag{29}$$

where we have used the fact that $\frac{1}{2} \le u_1 \le 1$. Thus

$$I_{**}(t_1,t_2) \le C\frac{1}{t_1^H t_2^H} \int e^{-\frac{1}{2}\left| \mu_2 u_2^{H-\frac{1}{\alpha}} \right|^\alpha} d\mu_2 \le C\frac{1}{t_1^H t_2^H},$$

obtaining (26).

The following consequence is immediate.

Lemma 14

$$\sup_{y_1,y_2} \kappa^{-(1-H)} \int_0^{\kappa\delta} \int_0^{q} |I(y_1,y_2;t_1,t_2)|\, dt_2 dt_1 \le C q^{1-H} \delta^{1-H}.$$

Proof. Follows from $|I(y_1,y_2;t_1,t_2)| \le I_{**}(t_1,t_2)$ and (26). ■

The next step consists of finding a suitable approximation to

$$\kappa^{-(1-H)} \int_{\kappa\delta}^{\kappa} \int_0^{q} I(y_1,y_2;t_1,t_2)\, dt_2 dt_1$$

for each $q>0$ and $\delta>0$, that will lead to the proof of Lemma 12. For this purpose note that, making the transformations $\mu_1 \longmapsto \frac{\mu_1}{t_1^H}$, $\mu_2 \longmapsto \mu_2$ in (23), we have

$$\begin{aligned} & I(y_1,y_2;t_1,t_2) \\ = & \frac{1}{t_1^H} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\frac{\mu_1}{t_1^H} y_1 - \mu_2(y_2-y_1)} \\ & \times E\left[e^{-i\frac{\mu_1}{t_1^H}\Lambda_{(0,t_1]}(t_1) - \mu_2\left(\Lambda_{(0,t_1+t_2]}(t_1+t_2) - \Lambda_{(0,t_1]}(t_1)\right)} \right] F\left(\frac{\mu_1}{t_1^H} - \mu_2\right) F(\mu_2)\, d\mu_1 d\mu_2. \end{aligned}$$

Define

$$\begin{aligned} & I_a(y_1,y_2;t_1,t_2) \\ = & \frac{1}{t_1^H} \int\int_{\{|\mu_1|\le a, |\mu_2|\le a\}} e^{-i\frac{\mu_1}{t_1^H} y_1 - \mu_2(y_2-y_1)} \\ & \times E\left[e^{-i\frac{\mu_1}{t_1^H}\Lambda_{(0,t_1]}(t_1) - \mu_2\left(\Lambda_{(0,t_1+t_2]}(t_1+t_2) - \Lambda_{(0,t_1]}(t_1)\right)} \right] F\left(\frac{\mu_1}{t_1^H} - \mu_2\right) F(\mu_2)\, d\mu_1 d\mu_2. \end{aligned} \tag{30}$$

Let us first obtain

Lemma 15.

$$\sup_{y_1,y_2} \kappa^{-(1-H)} \int_{\kappa\delta}^{\kappa} \int_0^{q} |I(y_1,y_2;t_1,t_2) - I_a(y_1,y_2;t_1,t_2)|\, dt_2 dt_1 \to 0$$

as $\kappa \to \infty$ first and then $a \to \infty$.

Proof. Similar to (28), we have

$$\begin{aligned} & |I(y_1,y_2;t_1,t_2) - I_a(y_1,y_2;t_1,t_2)| \\ \le & C \int\int_{\{|\mu_1|\le a, |\mu_2|\le a t_2^H\}^c} \frac{1}{t_1^H t_2^H} \\ & \times e^{-\frac{1}{2}\left|\mu_1 u_1^{H-\frac{1}{\alpha}} + t_1^H \frac{\mu_2}{t_2^H}\left(\left(u_1+\frac{t_2}{t_1}\right)^{H-\frac{1}{\alpha}} - u_1^{H-\frac{1}{\alpha}}\right)\right|^{\alpha} - \frac{1}{2}\left|\mu_2 u_2^{H-\frac{1}{\alpha}}\right|^{\alpha}} d\mu_1 d\mu_2 \\ \le & \int\int_{\{|\mu_1|<\infty, |\mu_2|> a t_2^H\}} + \int\int_{\{|\mu_1|>a, |\mu_2|\le a t_2^H\}} \\ = & J_a^{(1)}(y_1,y_2;t_1,t_2) + J_a^{(2)}(y_1,y_2;t_1,t_2), \text{ say.} \end{aligned}$$

(In the first inequality above A^c stands for the complement of of the set A.) Using (29), we then see that

$$\begin{aligned}
&\kappa^{-(1-H)}\int_{\kappa\delta}^{\kappa}\int_0^q J_a^{(1)}\left(y_1,y_2;t_1,t_2\right)dt_2dt_1\\
\leq\ & C\left(\kappa^{-(1-H)}\int_{\kappa\delta}^{\kappa}\frac{1}{t_1^H}dt_1\right)\int_0^q\frac{1}{t_2^H}\left\{\int_{\left\{|\mu_2|>at_2^H\right\}}e^{-\frac{1}{2}\left|\mu_2u_2^{H-\frac{1}{\alpha}}\right|^{\alpha}}d\mu_2\right\}dt_2\\
\leq\ & C\int_0^{\eta}\frac{1}{t_2^H}dt_2+\int_{\eta}^{q}\frac{1}{t_2^H}\left\{\int_{\left\{|\mu_2|>at_2^H\right\}}e^{-\frac{1}{2}\left|\mu_2u_2^{H-\frac{1}{\alpha}}\right|^{\alpha}}d\mu_2\right\}dt_2\to 0,
\end{aligned}$$

as $a\to\infty$ first and then $\eta\to 0$.

Next note that when $|\mu_2|\leq at_2^H$,

$$\begin{aligned}
&t_1^H\frac{\mu_2}{t_2^H}\left|\left(u_1+\frac{t_2}{t_1}\right)^{H-\frac{1}{\alpha}}-u_1^{H-\frac{1}{\alpha}}\right|\\
\leq\ & at_1^{1/\alpha}\left|\left(t_1u_1+t_2\right)^{H-\frac{1}{\alpha}}-\left(t_1u_1\right)^{H-\frac{1}{\alpha}}\right|\\
\leq\ & Cat_1^{\frac{1}{\alpha}+H-\frac{1}{\alpha}-1}=Cat_1^{H-1}\leq Ca\left(\kappa\delta\right)^{H-1}\to 0,\quad\text{when }\kappa\delta\leq t_1\leq\kappa.
\end{aligned}$$

Therefore, noting $\frac{1}{2}\leq u_1\leq 1$, when $|\mu_2|\leq at_2^H$,

$$\begin{aligned}
&\int_{\{|\mu_1|>a\}}e^{-\frac{1}{2}\left|\mu_1u_1^{H-\frac{1}{\alpha}}+t_1^H\frac{\mu_2}{t_2^H}\left(\left(u_1+\frac{t_2}{t_1}\right)^{H-\frac{1}{\alpha}}-u_1^{H-\frac{1}{\alpha}}\right)\right|^{\alpha}}d\mu_1\\
\leq\ & \int_{\left\{|\mu_1|>\frac{a}{2}\right\}}e^{-\frac{1}{2}\left|\mu_1u_1^{H-\frac{1}{\alpha}}\right|^{\alpha}}d\mu_1.
\end{aligned}$$

Hence

$$\begin{aligned}
&\kappa^{-(1-H)}\int_{\kappa\delta}^{\kappa}\int_0^q J_a^{(2)}\left(y_1,y_2;t_1,t_2\right)dt_2dt_1\\
\leq\ & C\left(\kappa^{-(1-H)}\int_{\kappa\delta}^{\kappa}\int_0^q\frac{1}{t_1^Ht_2^H}dt_2dt_1\right)\int_{\left\{|\mu_1|>\frac{a}{2}\right\}}e^{-\frac{1}{2}\left|\mu_1u_1^{H-\frac{1}{\alpha}}\right|^{\alpha}}d\mu_1\\
\leq\ & C\int_{\left\{|\mu_1|>\frac{a}{2}\right\}}e^{-\frac{1}{2}\left|\mu_1u_1^{H-\frac{1}{\alpha}}\right|^{\alpha}}d\mu_1\to 0,
\end{aligned}$$

as $a\to\infty$. This proves the lemma. ■

In view of Lemma 15, we next consider the approximation to

$$\kappa^{-(1-H)}\int_{\kappa\delta}^{\kappa}\int_0^q I_a\left(y_1,y_2;t_1,t_2\right)dt_2dt_1,\tag{31}$$

which will lead to the Lemma 16 below. For this purpose let ν_κ be such that

$$2\nu_\kappa<\kappa\delta,\quad \nu_\kappa\to\infty\quad\text{and}\quad\frac{\nu_\kappa}{\kappa}\to 0\quad\text{as }\kappa\to\infty.$$

We have

$$\begin{aligned}
&E\left[e^{-i\frac{\mu_1}{t_1^H}\Lambda_{(0,t_1]}(t_1)-\mu_2\left(\Lambda_{(0,t_1+t_2]}(t_1+t_2)-\Lambda_{(0,t_1]}(t_1)\right)}\right]\\
=\ & E\left[e^{-i\frac{\mu_1}{t_1^H}\Lambda_{(0,t_1-\nu_\kappa]}(t_1)-i\mu_2\left(\Lambda_{(0,t_1-\nu_\kappa]}(t_1+t_2)-\Lambda_{(0,t_1-\nu_\kappa]}(t_1)\right)}\right]\\
&\times E\left[e^{-i\frac{\mu_1}{t_1^H}\Lambda_{(t_1-\nu_\kappa,t_1]}(t_1)-i\mu_2\left(\Lambda_{(t_1-\nu_\kappa,t_1+t_2]}(t_1+t_2)-\Lambda_{(t_1-\nu_\kappa,t_1]}(t_1)\right)}\right].
\end{aligned}\tag{32}$$

Here

$$\begin{aligned}
&\Lambda_{(0,t_1-\nu_\kappa]}(t_1+t_2)-\Lambda_{(0,t_1-\nu_\kappa]}(t_1)\\
=\;& \int_0^{t_1-\nu_\kappa}\left\{(t_1+t_2-u)^{H-\frac{1}{\alpha}}-(t_1-u)^{H-\frac{1}{\alpha}}\right\}Z(du)\\
=^D\;& \int_{-t_1}^{-\nu_\kappa}\left\{(t_2-u)^{H-\frac{1}{\alpha}}-(-u)^{H-\frac{1}{\alpha}}\right\}Z(du)\\
=\;& \Lambda_{(-t_1,-\nu_\kappa]}(t_2),
\end{aligned}$$

where

$$\sup_{\kappa\delta\le t_1\le\kappa,0<t_2\le q} P\left[\left|\Lambda_{(-t_1,-\nu_\kappa]}(t_2)\right|>\eta\right]\to 0 \quad \text{for all } \eta>0.$$

Further

$$\begin{aligned}
\frac{1}{t_1^H}\Lambda_{(t_1-\nu_\kappa,t_1]}(t_1) &= \frac{1}{t_1^H}\int_{t_1-\nu_\kappa}^{t_1}(t_1-u)^{H-\frac{1}{\alpha}}Z(du)\\
&=^D \frac{1}{t_1^H}\int_0^{\nu_\kappa}u^{H-\frac{1}{\alpha}}Z(du)=^D\frac{\nu_\kappa^H}{t_1^H}\int_0^1 u^{H-\frac{1}{\alpha}}Z(du),
\end{aligned}$$

where

$$\sup_{\kappa\delta\le t_1\le\kappa}\frac{\nu_\kappa}{t_1}=\frac{\nu_\kappa}{\kappa\delta}\to 0,$$

so that

$$\sup_{\kappa\delta\le t_1\le\kappa} P\left[\left|\frac{1}{t_1^H}\Lambda_{(t_1-\nu_\kappa,t_1]}(t_1)\right|>\eta\right]\to 0 \quad \text{for all } \eta>0.$$

Moreover

$$\begin{aligned}
&\Lambda_{(t_1-\nu_\kappa,t_1+t_2]}(t_1+t_2)-\Lambda_{(t_1-\nu_\kappa,t_1]}(t_1)\\
=\;& \int_{t_1-\nu_\kappa}^{t_1+t_2}(t_1+t_2-u)^{H-\frac{1}{\alpha}}Z(du)-\int_{t_1-\nu_\kappa}^{t_1}(t_1-u)^{H-\frac{1}{\alpha}}Z(du) \quad \text{because } t_1-\nu_\kappa>0,\\
=^D\;& \int_{-\nu_\kappa}^{t_2}(t_2-u)^{H-\frac{1}{\alpha}}Z(du)-\int_{-\nu_\kappa}^{0}(-u)^{H-\frac{1}{\alpha}}Z(du)\\
=^D\;& \int_{-\nu_\kappa}^{0}\left\{(t_2-u)^{H-\frac{1}{\alpha}}-(-u)^{H-\frac{1}{\alpha}}\right\}Z(du)+\int_0^{t_2}(t_2-u)^{H-\frac{1}{\alpha}}Z(du)\\
=\;& \Lambda_{(-\nu_\kappa,t_2]}(t_2),
\end{aligned}$$

where

$$\sup_{\kappa\delta\le t_1\le\kappa,0<t_2\le q} P\left[\left|\Lambda_{(-\nu_\kappa,t_2]}(t_2)-\Lambda(t_2)\right|>\eta\right]\to 0 \quad \text{for all } \eta>0.$$

In addition, because $\sup_{\kappa\delta\le t_1\le\kappa}\frac{\nu_\kappa}{t_1}=\frac{\nu_\kappa}{\kappa\delta}\to 0$,

$$\sup_{\kappa\delta\le t_1\le\kappa} P\left[\left|\frac{1}{t_1^H}\Lambda_{(0,t_1-\nu_\kappa]}(t_1)-\frac{1}{t_1^H}\Lambda_{(0,t_1]}(t_1)\right|>\eta\right]\to 0 \quad \text{for all } \eta>0.$$

Thus from the preceding four approximations it follows that (32) is approximated, uniformly over $\kappa\delta\le t_1\le\kappa, 0<t_2\le q, |\mu_1|\le a, |\mu_2|\le a$, by

$$E\left[e^{-i\frac{\mu_1}{t_1^H}\Lambda_{(0,t_1]}(t_1)}\right]E\left[e^{-i\mu_2\Lambda(s_2)}\right]$$

In addition, we have $\sup_{\kappa\delta\le t_1\le\kappa,|\mu_1|\le a}\left|F\left(\frac{\mu_1}{t_1^H}-\mu_2\right)-F\left(-\mu_2\right)\right|\to 0$.

Thus, taking in addition Lemmas 14 and 15 into account, and using the preceding approximations in (30), we have obtained the following lemma.

Lemma 16. *Let $R_\kappa(y_1, y_2, a, \delta)$ to be the difference between*

$$\kappa^{-(1-H)} \int_0^{\frac{\kappa}{m}} \int_0^q I(y_1, y_2; t_1, t_2)\, dt_2 dt_1$$

and

$$\frac{1}{\kappa^{1-H}} \int_{\frac{\kappa}{m}\delta}^{\frac{\kappa}{m}} \frac{1}{t_1^H} \int_0^q \left\{ \int\int_{\{|\mu_1|\le a, |\mu_2|\le a\}} e^{-i\frac{\mu_1}{t_1^H} y_1 - i\mu_2(y_2-y_1)} \right.$$
$$\left. \times E\left[e^{-i\frac{\mu_1}{t_1^H}\Lambda_{(0,t_1]}(t_1)}\right] E\left[e^{-i\mu_2\Lambda(t_2)}\right] |F(\mu_2)|^2 d\mu_1 d\mu_2 \right\} dt_2 dt_1. \tag{33}$$

Then,

$$\lim_{\delta\to 0} \lim_{a\to\infty} \lim \sup_{\kappa\to\infty} \left(\sup_{y_1,y_2} |R_\kappa(y_1, y_2, a)| \right) = 0.$$

We can now complete the proof of Lemma 12.

Proof of Lemma 12. The approximation in the preceding lemma 16 is uniform in y_1 and y_2, and hence it holds also when (y_1, y_2) is as in (21). Note that

$$\begin{aligned}
y_2 - y_1 &= \Lambda_{(-\infty, \kappa\frac{l-1}{m}]}\left(\kappa\frac{l-1}{m} + t_1 + t_2\right) - \Lambda_{(-\infty,\kappa\frac{l-1}{m}]}\left(\kappa\frac{l-1}{m} + t_1\right) \\
&= \int_{-\infty}^{\kappa\frac{l-1}{m}} \left\{ \left(\kappa\frac{l-1}{m} + t_1 + t_2 - u\right)^{H-1/\alpha} - \left(\kappa\frac{l-1}{m} + t_1 - u\right)^{H-1/\alpha} \right\} Z(du) \\
&=^D \int_{-\infty}^{0} \left\{ (t_1 + t_2 - u)^{H-1/\alpha} - (t_1 - u)^{H-1/\alpha} \right\} Z(du) \\
&=^D \int_{-\infty}^{-t_1} \left\{ (t_2 - u)^{H-1/\alpha} - (-u)^{H-1/\alpha} \right\} Z(du) \\
&= \Lambda_{(-\infty,-t_1]}(t_2),
\end{aligned}$$

where

$$\sup_{\frac{\kappa}{m}\delta \le t_1 \le \kappa, 0\le t_2\le q} P\left[\left|\Lambda_{(-\infty,-t_1]}(t_2)\right| > \eta\right] \to 0 \quad \text{for all } \eta > 0.$$

Thus the factor $\frac{\mu_1}{t_1^H} y_1 + \mu_2(y_2 - y_1)$ in (33), with (y_1, y_2) as in (21), can be replaced by

$$\frac{\mu_1}{t_1^H} y_1 = \frac{\mu_1}{t_1^H} \Lambda_{(-\infty,\kappa\frac{l-1}{m}]}\left(\kappa\frac{l-1}{m} + t_1\right).$$

Taking this into account and by making the transformation $\frac{1}{t_1^H}\left(\frac{\kappa}{m}\right)^H \mu_1 \longmapsto \mu_1$, the approximation (33) takes the form

$$\frac{m^H}{\kappa} \int_{\frac{\kappa}{m}\delta}^{\frac{\kappa}{m}} \int_0^q \left\{ \int\int_{\left\{|\mu_1|\le \frac{1}{t_1^H}\left(\frac{\kappa}{m}\right)^H a, |\mu_2|\le a\right\}} e^{-i\mu_1\left(\frac{\kappa}{m}\right)^{-H} \Lambda_{\left(-\infty,\kappa\frac{l-1}{m}\right]}\left(\kappa\frac{l-1}{m}+t_1\right)} \right.$$
$$\left. \times E\left[e^{-i\mu_1\left(\frac{\kappa}{m}\right)^{-H}\Lambda_{(0,t_1]}(t_1)}\right] E\left[e^{-i\mu_2\Lambda(t_2)}\right] |F(\mu_2)|^2 d\mu_1 d\mu_2 \right\} dt_2 dt_1.$$

Here note that $E\left[e^{-i\mu_1\left(\frac{\kappa}{m}\right)^{-H}\Lambda_{(0,t_1]}(t_1)}\right] = E\left[e^{-i\mu_1\left(\frac{\kappa}{m}\right)^{-H} t_1^H \Lambda_{(0,1]}(1)}\right]$. Hence making the further transformation $t_1\left(\frac{\kappa}{m}\right)^{-1} \longmapsto t_1$, and noting that $E\left[e^{-i\mu_1 t_1^H \Lambda_{(0,1]}(1)}\right] = E\left[e^{-i\mu_1\Lambda_{(0,t_1]}(t_1)}\right]$, the preceding approximation takes the form

$$\frac{1}{m^{1-H}}\int_{\delta}^{1}\int_{0}^{q}\left\{\int\int_{\left\{|\mu_1|\leq\frac{a}{t_1^H},|\mu_2|\leq a\right\}}J\left(m,\mu_1,\mu_2,t_2,t_1\right)d\mu_1 d\mu_2\right\}dt_2dt_1 \tag{34}$$

where

$$\begin{aligned}&J\left(m,\mu_1,\mu_2,t_2,t_1\right)\\ =\;& e^{-i\mu_1 m^H\kappa^{-H}\Lambda_{\left(-\infty,\kappa\frac{l-1}{m}\right]}\left(\kappa\frac{l-1}{m}+\kappa\frac{t_1}{m}\right)}E\left[e^{-i\mu_1\Lambda_{(0,t_1]}(t_1)}\right]E\left[e^{-i\mu_2\Lambda(t_2)}\right]\left|F\left(\mu_2\right)\right|^2.\end{aligned}$$

Note that

$$\frac{1}{m^{1-H}}\int_{0}^{1}\int_{0}^{q}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}J\left(m,\mu_1,\mu_2,t_2,t_1\right)d\mu_1 d\mu_2\right\}dt_2dt_1$$

is the same as (25) of Lemma 12.

Now let $K\left(a\right)$ be the difference between (34) and

$$\frac{1}{m^{1-H}}\int_{\delta}^{1}\int_{0}^{q}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}J\left(m,\mu_1,\mu_2,t_2,t_1\right)d\mu_1 d\mu_2\right\}dt_2dt_1.$$

(Here m, q and δ are fixed.) Then noting that $\left|e^{-i\lambda}\right|\leq 1$ and $\left|F\left(\mu_2\right)\right|\leq C$, we have

$$\begin{aligned}&m^{1-H}K\left(a\right)\\ \leq\;&\left(\int_0^q\int_{-\infty}^{\infty}\left|E\left[e^{-i\mu_2\Lambda(t_2)}\right]\right|d\mu_2dt_2\right)\int_\delta^1\int_{\{|\mu_1|>a\delta\}}\left|E\left[e^{-i\mu_1\Lambda_{(0,t_1]}(t_1)}\right]\right|d\mu_1dt_1\\ &+\left(\int_0^q\int_{\{|\mu_2|>a\}}\left|E\left[e^{-i\mu_2\Lambda(t_2)}\right]\right|d\mu_2dt_2\right)\int_\delta^1\int_{-\infty}^{\infty}\left|E\left[e^{-i\mu_1\Lambda_{(0,t_1]}(t_1)}\right]\right|d\mu_1dt_1\\ \leq\;&2R\left(a\right)R\left(0\right),\quad\text{(assume } q>1\text{)}\end{aligned}$$

where

$$\begin{aligned}R\left(a\right)&=\int_0^q\int_{\{|\mu_2|>a\delta\}}\left|E\left[e^{-i\mu_2\Lambda_{(0,t_2]}(t_2)}\right]\right|d\mu_2dt_2\\ &\leq C\int_0^q\frac{1}{t_2^H}\int_{\left\{|\mu_2|>a\delta t_2^H\right\}}e^{-c|\mu_2|^\alpha}d\mu_2dt_2\\ &\leq C\int_0^\eta\frac{1}{t_2^H}dt_2+C\int_{\{|\mu_2|>a\delta\eta^H\}}e^{-c|\mu_2|^\alpha}d\mu_2,\end{aligned}$$

where the right hand side converges to 0 as $a\to\infty$ first and then $\eta\to0$. Also $R\left(0\right)<\infty$. Hence

$$m^{1-H}K\left(a\right)\to0\quad\text{ as } a\to\infty.$$

Next note that

$$\begin{aligned}&\int_0^\delta\int_{-\infty}^{\infty}\left|E\left[e^{-i\mu_1\Lambda_{(0,t_1]}(t_1)}\right]\right|d\mu_1dt_1\\ \leq\;& C\left(\int_0^\delta\frac{1}{t_1^H}dt_1\right)\left(\int_{-\infty}^{\infty}e^{-c|\mu_1|^\alpha}d\mu_1\right)\leq\frac{C\delta^{1-H}}{1-H}\to0\text{ as }\delta\to0.\end{aligned}$$

This completes the proof of the lemma 12. ■

Completion of the proof of Proposition 6. We now indicate the modifications required in the above proofs in order to obtain the general statement of the Proposition 6. First consider the process, for u real,

$$t \longmapsto \kappa^{-\frac{1-H}{2}} \int_0^{\kappa t} \int_{-\infty}^{\infty} e^{-i\mu\left(\Lambda(s)-\kappa^H u\right)} F(\mu)\, d\mu ds. \tag{35}$$

In this case we take

$$\zeta_{\kappa ml} = \kappa^{-\frac{1-H}{2}} \int_{\kappa\frac{l-1}{m}}^{\kappa\frac{l}{m}} \int_{-\infty}^{\infty} e^{-i\mu\left(\Lambda(s)-\kappa^H u\right)} F(\mu)\, d\mu ds, \quad l = 1, 2, ...$$

Then it is clear that only the verification of (R2) will require modification. It is easy to see that the only change will be that the earlier (y_1, y_2) in (21) will now be replaced by $\left(y_1 - \kappa^H u, y_2 - \kappa^H u\right)$, that is, the earlier factor $e^{-i\frac{\mu_1}{t_1^H}y_1 - i\mu_2(y_2-y_1)}$ in the approximation (33) will now take the form

$$e^{-i\frac{\mu_1}{t_1^H}\left(y_1-\kappa^H u\right)-i\mu_2(y_2-y_1)}.$$

As a result $\Lambda_{\left(-\infty,\kappa\frac{l-1}{m}\right]}\left(\kappa\frac{l-1}{m} + \kappa\frac{t_1}{m}\right)$ in (25) will need to be replaced by

$$\Lambda_{\left(-\infty,\kappa\frac{l-1}{m}\right]}\left(\kappa\frac{l-1}{m} + \kappa\frac{t_1}{m}\right) - \kappa^H u =^D \kappa^H \left(\Lambda_{\left(-\infty,\frac{l-1}{m}\right]}\left(\frac{l-1}{m} + \frac{t_1}{m}\right) - u\right).$$

This means in Lemma 13, $\Lambda_{\left(-\infty,\frac{l-1}{m}\right]}\left(\frac{l-1}{m} + \frac{t_1}{m}\right)$ will need to be replaced by $\Lambda_{\left(-\infty,\frac{l-1}{m}\right]}\left(\frac{l-1}{m} + \frac{t_1}{m}\right) - u$. In this case the limit in Lemma 13 will be $\sum_{i=0}^r v_i^2 \left(L_{t_i}^u - L_{t_{i-1}}^u\right)$. Thus

$$(35) \overset{fdd}{\Longrightarrow} \sqrt{b} W^{(u)}\left(L_t^u\right),$$

where $W^{(u)}(t)$ is a standard Brownian motion independent of $\Lambda(t)$.

To consider the general case of Proposition 6, for simplicity take $q = 2$. Then consider, for reals u_1, u_2, a_1 and a_2, with u_1, u_2 distinct,

$$\kappa^{-\frac{1-H}{2}} \sum_{j=1}^{2} a_j \int_0^{\kappa t} \int_{-\infty}^{\infty} e^{-i\mu\left(\Lambda(s)-\kappa^H u_j\right)} F(\mu)\, d\mu ds. \tag{36}$$

Now take

$$\zeta_{\kappa ml} = \kappa^{-\frac{1-H}{2}} \sum_{j=1}^{2} a_j \int_{\kappa\frac{l-1}{m}}^{\kappa\frac{l}{m}} \int_{-\infty}^{\infty} e^{-i\mu\left(\Lambda(s)-\kappa^H u_j\right)} F(\mu)\, d\mu ds, \quad l = 1, 2, ...$$

Here also it is clear only the verification of (R2) will be different. The essential difference is that $E_{\kappa\frac{l-1}{m}}\left[\zeta_{\kappa ml}^2\right]$ will now involve a cross product term.

We now show that the contribution of this cross product term can be neglected asymptotically. First, this term will be of the same form as that of (22) except that the earlier (y_1, y_2) in (21) will now be replaced by $\left(y_1 - \kappa^H u_1, y_2 - \kappa^H u_2\right)$, that is, the earlier factor $e^{-i\frac{\mu_1}{t_1^H}y_1 - i\mu_2(y_2-y_1)}$ in (33) will take the form

$$e^{-i\frac{\mu_1}{t_1^H}\left(y_1-\kappa^H u_1\right)-i\mu_2(y_2-y_1)+i\mu_2\kappa^H(u_1-u_2)},$$

where now $u_1 - u_2 \neq 0$. This means $\Lambda_{\left(-\infty,\kappa\frac{l-1}{m}\right]}\left(\kappa\frac{l-1}{m} + \kappa\frac{t_1}{m}\right)$ in (25) will be replaced by $\Lambda_{\left(-\infty,\kappa\frac{l-1}{m}\right]}\left(\kappa\frac{l-1}{m} + \kappa\frac{t_1}{m}\right) - \kappa^H u_1$, and in addition the factor $\int_0^q \int_{-\infty}^{\infty} E\left[e^{-i\mu_2\Lambda(t_2)}\right] |F(\mu_2)|^2 d\mu_2 dt_2$ in (25) will be changed to

$$\begin{aligned}
&\int_0^q \int_{-\infty}^{\infty} e^{i\mu_2\kappa^H(u_1-u_2)} E\left[e^{-i\mu_2\Lambda(t_2)}\right] |F(\mu_2)|^2 d\mu_2 dt_2 \\
= &\int_0^q \frac{1}{t_2^H} \int_{-\infty}^{\infty} e^{i\frac{\mu_2}{t_2^H}\kappa^H(u_1-u_2)} E\left[e^{-i\mu_2\Lambda(1)}\right] \left|F\left(\frac{\mu_2}{t_2^H}\right)\right|^2 d\mu_2 dt_2,
\end{aligned} \tag{37}$$

where note that $\left|F\left(\frac{\mu_2}{t_2^H}\right)\right| \leq C$ and $\int_{-\infty}^{\infty}\left|E\left[e^{-i\mu_2\Lambda(1)}\right]\right| d\mu_2 \leq C$. In particular, each fixed $\eta > 0$,

$$\int_{\eta}^{q} \frac{1}{t_2^H} \int_{-\infty}^{\infty} e^{i\frac{\mu_2}{t_2^H}\kappa^H(u_1-u_2)} E\left[e^{-i\mu_2\Lambda(1)}\right] \left|F\left(\frac{\mu_2}{t_2^H}\right)\right|^2 d\mu_2 dt_2 \to 0$$

as $\kappa \to \infty$, by the Riemann-Lebesgue Lemma. Further $\int_0^{\eta} \frac{1}{t_2^H} dt_2 \to 0$ as $\eta \to 0$. Hence (37) converges to 0.

Thus the cross product term indicated above can be neglected, so that the limit in Lemma 13 will be $\sum_{j=1}^{2} a_j^2 \sum_{i=0}^{r} v_i^2 \left(L_{t_i}^{u_j} - L_{t_{i-1}}^{u_j}\right)$. This means

$$(36) \overset{fdd}{\Longrightarrow} \sqrt{b} \sum_{j=1}^{2} a_j W^{(u_j)}\left(L_t^{u_j}\right),$$

where $W^{(u_j)}(t), j = 1, 2$, are independent standard Brownian motions, independent of the process $\Lambda(t)$. This completes the proof of Proposition 6 (except for the verification of (R3) and the proof of Lemma 10). ■

Regarding the convergence of finite dimensional distributions in Theorem 1, it is a direct consequence of Proposition 6, in view of the representation (8). The same is the case for Theorem 2, because for any reals $x_1, ..., x_k$, $c_1, ..., c_k$ the linear combination $\sum_{j=1}^{k} c_j \left(L_t^{\varepsilon x_j + u_i} - L_t^{y_i}\right)$ will have the representation in Proposition 6 (see (10)), with

$$F(\mu) = \sum_{j=1}^{k} c_j \left(e^{i\mu x_j} - 1\right).$$

It thus only remains to obtain the covariance structure of the limit in Theorem 2. It is enough to obtain the limiting covariance for $L_t^{\varepsilon x + u_i} - L_t^{\varepsilon y + u_i}$ for each $t > 0$. According to the representation (11) and Proposition 6,

$$\varepsilon^{-\frac{1-H}{2H}}\left(L_t^{\varepsilon x + u_i} - L_t^{\varepsilon y + u_i}\right) \Longrightarrow \sqrt{b^*}\,|x-y|^{\frac{1-H}{2H}}\, W^{(u_i)}\left(L_t^{u_i}\right),$$

where note that the variance of $\sqrt{b^*}\,|x-y|^{\frac{1-H}{2H}}\, W^{(u_i)}(t)$ is the same as the variance of $\sqrt{b^*}\left(B_{\frac{1}{H}-1}^{(u_i)}(t,x) - B_{\frac{1}{H}-1}^{(u_i)}(t,y)\right)$. This gives the required covariance structure.

Also, according to (12),

$$|F(\mu)|^2 = \left|e^{i\mu} - 1\right|^2 = 4\sin^2\left(\frac{\mu}{2}\right).$$

3 The remaining proofs

Recall that to complete the proof of the convergence of finite dimensional distributions in Theorems 1 and 2, it remains to prove Lemma 10. This as well as the proof of Proposition 5 are given in this section. As noted earlier (R3) is a consequence of Proposition 5, but we shall indicate that the computation of the fourth moment is enough for the verification of (R3), see the Remark at the end of this section.

We shall need the following result (which is not required when $H = \frac{1}{\alpha}$).

Lemma 17. *Assume $H \neq \frac{1}{\alpha}$. Then*

$$\sup_{\frac{t}{2} \leq v \leq t, s>0, r>0} \frac{t^{1/\alpha}}{r^H} \left|(v+s+r)^{H-\frac{1}{\alpha}} - (v+s)^{H-\frac{1}{\alpha}}\right| \leq C.$$

Proof. Noting $H-1-\frac{1}{\alpha}<0$, we have when $\frac{t}{2}\le v$,

$$\begin{aligned}\left|(v+s+r)^{H-\frac{1}{\alpha}}-(v+s)^{H-\frac{1}{\alpha}}\right| &\le C\int_0^r (v+s+x)^{H-1-\frac{1}{\alpha}}\,dx\\ &\le C\left(\min(t,s)\right)^{H-1-\frac{1}{\alpha}}\int_0^r dx\\ &= C\left(\min(t,s)\right)^{H-1-\frac{1}{\alpha}}\,r. \end{aligned} \tag{38}$$

Further, when $H-\frac{1}{\alpha}<0$, we have, using $t\le 2v$,

$$\left|(v+s+r)^{H-\frac{1}{\alpha}}-(v+s)^{H-\frac{1}{\alpha}}\right|\le (v+s)^{H-\frac{1}{\alpha}}\le C\left(\min(t,s)\right)^{H-\frac{1}{\alpha}}, \tag{39}$$

and similarly when $H-\frac{1}{\alpha}>0$, we have using $v\le t$,

$$\begin{aligned}&\left|(v+s+r)^{H-\frac{1}{\alpha}}-(v+s)^{H-\frac{1}{\alpha}}\right|\\ \le\ &(v+s+r)^{H-\frac{1}{\alpha}}\le Cr^{H-\frac{1}{\alpha}}\quad \text{if } r>t\ge s \text{ or } r>s>t.\end{aligned} \tag{40}$$

We have using (38) and noting $1-H>0$,

$$\begin{aligned}\frac{t^{1/\alpha}}{r^H}\left|(v+s+r)^{H-\frac{1}{\alpha}}-(v+s)^{H-\frac{1}{\alpha}}\right| &\le Ct^{\frac{1}{\alpha}}r^{-H}t^{H-1-\frac{1}{\alpha}}r\\ &= \left(\frac{r}{t}\right)^{1-H}\le C,\ \text{ if } r\le t.\end{aligned}$$

In addition, using (39), we have

$$\begin{aligned}&\frac{t^{1/\alpha}}{r^H}\left|(v+s+r)^{H-\frac{1}{\alpha}}-(v+s)^{H-\frac{1}{\alpha}}\right|\\ \le\ &Ct^{\frac{1}{\alpha}}r^{-H}t^{H-\frac{1}{\alpha}}=Cr^{-H}t^H\le C,\quad \text{if } H-\frac{1}{\alpha}<0,\ r>t.\end{aligned}$$

Further, using (40),

$$\begin{aligned}&\frac{t^{1/\alpha}}{r^H}\left|(v+s+r)^{H-\frac{1}{\alpha}}-(v+s)^{H-\frac{1}{\alpha}}\right|\\ \le\ &Ct^{\frac{1}{\alpha}}r^{-H}r^{H-\frac{1}{\alpha}}=Cr^{-\frac{1}{\alpha}}t^{\frac{1}{\alpha}}\le C,,\quad \text{if } H-\frac{1}{\alpha}>0,\ r>t>s.\end{aligned}$$

From (38) we have,

$$\begin{aligned}\frac{t^{1/\alpha}}{r^H}\left|(v+s+r)^{H-\frac{1}{\alpha}}-(v+s)^{H-\frac{1}{\alpha}}\right| &\le Ct^{\frac{1}{\alpha}}r^{-H}s^{H-1-\frac{1}{\alpha}}r\\ &= C\left(\frac{r}{s}\right)^{1-H}\left(\frac{t}{s}\right)^{\frac{1}{\alpha}}\le C,\ \text{ if } r\le s \text{ and } t<s.\end{aligned}$$

It remains to consider the case $H-\frac{1}{\alpha}>0$, $r>t$, $t<s$, $r>s$. In this case we have from (40),

$$\frac{t^{1/\alpha}}{r^H}\left|(v+s+r)^{H-\frac{1}{\alpha}}-(v+s)^{H-\frac{1}{\alpha}}\right|\le Ct^{\frac{1}{\alpha}}r^{-H}r^{H-\frac{1}{\alpha}}=\left(\frac{t}{r}\right)^{\frac{1}{\alpha}}\le C$$

because $t<s<r$. This completes the proof of the lemma. ■

Next we present the proof of Lemma 10.

Proof of Lemma 10. We have, similar to (28), with $I_*(t_1,t_2)$ as defined in (24),

$$\begin{aligned} I_*(t_1,t_2) &\le C\frac{1}{t_1^H t_2^H}\int\int e^{-\frac{1}{2}\left|\mu_1 u_1^{H-\frac{1}{\alpha}}+t_1^H\frac{\mu_2}{t_2^H}\left(\left(u_1+\frac{t_2}{t_1}\right)^{H-\frac{1}{\alpha}}-u_1^{H-\frac{1}{\alpha}}\right)\right|^{\alpha}-\frac{1}{2}\left|\mu_2 u_2^{H-\frac{1}{\alpha}}\right|^{\alpha}} \\ &\quad\times\left|F\left(\frac{\mu_1}{t_1^H}-\frac{\mu_2}{t_2^H}\right)F\left(\frac{\mu_2}{t_2^H}\right)\right|d\mu_1 d\mu_2. \end{aligned} \tag{41}$$

Here recall that $\frac{1}{2}\le u_1\le 1$ and $\frac{1}{2}\le u_2\le 1$. Let us write

$$\mu_1 u_1^{H-\frac{1}{\alpha}}+\mu_2\frac{t_1^H}{t_2^H}\left(\left(u_1+\frac{t_2}{t_1}\right)^{H-\frac{1}{\alpha}}-u_1^{H-\frac{1}{\alpha}}\right)=\left(\mu_1+\mu_2 h(t_1,t_2)\right)u_1^{H-\frac{1}{\alpha}},$$

with

$$\begin{aligned} h(t_1,t_2) &= u_1^{-H+\frac{1}{\alpha}}\frac{t_1^H}{t_2^H}\left(\left(u_1+\frac{t_2}{t_1}\right)^{H-\frac{1}{\alpha}}-u_1^{H-\frac{1}{\alpha}}\right) \\ &= u_1^{-H+\frac{1}{\alpha}}\frac{t_1^{\frac{1}{\alpha}}}{t_2^H}\left((t_1u_1+t_2)^{H-\frac{1}{\alpha}}-(t_1u_1)^{H-\frac{1}{\alpha}}\right). \end{aligned}$$

Then, in view of the above Lemma 17 and because $\frac{1}{2}\le u_1\le 1$,

$$\sup_{t_1>0,t_2>0}|h(t_1,t_2)|\le C.$$

(Note that $h(t_1,t_2)=0$ when $H=\frac{1}{\alpha}$.) Noting, $\left|F\left(\frac{\mu_1}{t_1^H}-\frac{\mu_2}{t_2^H}\right)F\left(\frac{\mu_2}{t_2^H}\right)\right|\le C\left(\frac{|\mu_1|}{t_1^H}+\frac{|\mu_2|}{t_2^H}\right)\frac{|\mu_2|}{t_2^H}$ and making the transformation $\mu_1+\mu_2 h(t_1,t_2)\longmapsto\mu_1$, $\mu_2\longmapsto\mu_2$, we then obtain

$$\begin{aligned} I_*(t_1,t_2) &\le C\frac{1}{t_1^H t_2^H}\int\int\left(\frac{|\mu_1|+|\mu_2|}{t_1^H}+\frac{|\mu_2|}{t_2^H}\right)\frac{|\mu_2|}{t_2^H}e^{-c|\mu_1|^{\alpha}du-c|\mu_2|^{\alpha}}d\mu_1 d\mu_2 \\ &\le C\frac{1}{t_1^H t_2^H}\left(\frac{1}{t_1^H}+\frac{1}{t_2^H}\right)\frac{1}{t_2^H}=\frac{C}{t_1^{2H}t_2^{2H}}+\frac{C}{t_1^H t_2^{3H}}. \end{aligned}$$

Thus

$$\kappa^{-(1-H)}\int_1^{\frac{\kappa}{m}}\int_q^{\frac{\kappa}{m}}I_*(t_1,t_2)\,dt_2dt_1\le C\kappa^{-(1-H)}\int_1^{\frac{\kappa}{m}}\int_q^{\frac{\kappa}{m}}\left(\frac{C}{t_1^{2H}t_2^{2H}}+\frac{C}{t_1^H t_2^{3H}}\right)dt_2dt_1.$$

We have

$$\kappa^{-(1-H)}\int_1^{\frac{\kappa}{m}}\int_q^{\frac{\kappa}{m}}\frac{1}{t_1^{2H}t_2^{2H}}dt_2dt_1\le C\kappa^{-(1-H)}\kappa^{2-4H}=C\kappa^{1-3H},$$

where $1-3H<0$. Similarly

$$\kappa^{-(1-H)}\int_1^{\frac{\kappa}{m}}\int_q^{\frac{\kappa}{m}}\frac{1}{t_1^H t_2^{3H}}dt_2dt_1\le C\int_q^{\infty}\frac{1}{t_2^{3H}}dt_2\le Cq^{1-3H}\to 0\qquad\text{as } q\to\infty.$$

Now, note that we also have $I_*(t_1,t_2)\le\frac{1}{t_1^H t_2^{2H}}$, using $\left|F\left(\frac{\mu_1}{t_1^H}-\frac{\mu_2}{t_2^H}\right)\right|\le C$. Hence

$$\kappa^{-(1-H)}\int_0^1\int_q^{\frac{\kappa}{m}}I_*(t_1,t_2)\,dt_2dt_1\le C\kappa^{-(1-H)}\int_0^1\int_q^{\frac{\kappa}{m}}\frac{1}{t_1^H t_2^{2H}}dt_2dt_1\le C\kappa^{-H}.$$

This completes the proof of the Lemma 10. ■

It remains to prove Proposition 5.

Proof of Proposition 5. We consider the case $l=2$ in detail and then indicate the modification required for the general case. It is enough to take $u=0$. Then we have, assuming $t>s$,

$$\begin{aligned} & E\left[\left(\kappa^{-\frac{1-H}{2}}\int_{\kappa s}^{\kappa t}\int_{-\infty}^{\infty}e^{-i\mu\Lambda(v)}F(\mu)\,d\mu dv\right)^4\right] \\ \leq\ & 4!\kappa^{-2(1-H)}\int_{\kappa s}^{\kappa t}\int_0^{\kappa(t-s)}\int_0^{\kappa(t-s)}\int_0^{\kappa(t-s)}I_*(t_1,t_2,t_3,t_4)\,dt_4dt_3dt_2dt_1, \end{aligned} \tag{42}$$

where

$$\begin{aligned} I_*(t_1,t_2,t_3,t_4) \leq & \int\int\int\int\left|E\left[e^{-i\sum_{j=1}^4\mu_j\Lambda(t_1+\ldots+t_j)}\right]\right| \\ & \times\left|F(\mu_1)F(\mu_2)F(\mu_3)F(\mu_4)\right|d\mu_1d\mu_2d\mu_3d\mu_4. \end{aligned}$$

Here

$$\begin{aligned} & \left|E\left[e^{-i\sum_{j=1}^4\mu_j\Lambda(t_1+\ldots+t_j)}\right]\right| \\ \leq\ & \left|E\left[e^{-i\sum_{j=1}^4\mu_j\sum_{k=1}^j\Lambda_{(t_1+\ldots+t_{k-1},t_1+\ldots+t_k]}(t_1+\ldots+t_j)}\right]\right| \end{aligned}$$

(Here and below $t_1+\ldots+t_{k-1}=0$ when $k=1$.) Letting

$$\lambda_j=\mu_j+\ldots+\mu_4,$$

we have

$$\begin{aligned} & \sum_{j=1}^4\mu_j\sum_{k=1}^j\Lambda_{(t_1+\ldots+t_{k-1},t_1+\ldots+t_k]}(t_1+\ldots+t_j) \\ =\ & \sum_{k=1}^4\Bigg\{\lambda_k\Lambda_{(t_1+\ldots+t_{k-1},t_1+\ldots+t_k]}(t_1+\ldots+t_k) \\ & +\sum_{j=k+1}^4\lambda_j\left(\Lambda_{(t_1+\ldots+t_{k-1},t_1+\ldots+t_k]}(t_1+\ldots+t_j)-\Lambda_{(t_1+\ldots+t_{k-1},t_1+\ldots+t_k]}(t_1+\ldots+t_{j-1})\right)\Bigg\}. \end{aligned}$$

(Here and below $\sum_{j=k+1}^4$ is empty when $k=4$.) Here the summands on the right hand side are independent, and the distribution of the k-th summand is the same as that of

$$\lambda_k\Lambda_{(0,t_k]}(t_k)+\sum_{j=k+1}^4\lambda_j\left(\Lambda_{(0,t_k]}(t_k+\ldots+t_j)-\Lambda_{(0,t_k]}(t_k+\ldots+t_{j-1})\right).$$

We have, similar to (27) and the three steps subsequent to it,

$$\begin{aligned} & E\left[e^{i\left(\frac{\lambda_k}{t_k^H}\Lambda_{(0,t_k]}(t_k)+\sum_{j=k+1}^4\frac{\lambda_j}{t_j^H}\left(\Lambda_{(0,t_k]}(t_k+\ldots+t_j)-\Lambda_{(0,t_k]}(t_k+\ldots+t_{j-1})\right)\right)}\right] \\ \leq\ & e^{-\int_0^{t_k}\left|\frac{\lambda_k}{t_k^H}u^{H-\frac{1}{\alpha}}+\sum_{j=k+1}^4\frac{\lambda_j}{t_j^H}\left((u+t_{k+1}+\ldots+t_{j-1}+t_j)^{H-\frac{1}{\alpha}}-(u+t_{k+1}+\ldots+t_{j-1})^{H-\frac{1}{\alpha}}\right)\right|^\alpha du} \\ =\ & e^{-\int_0^1\left|\lambda_ku^{H-\frac{1}{\alpha}}+t_k^H\sum_{j=k+1}^4\frac{\lambda_j}{t_j^H}\left(\left(u+\frac{t_{k+1}+\ldots+t_j}{t_k}\right)^{H-\frac{1}{\alpha}}-\left(u+\frac{t_{k+1}+\ldots+t_{j-1}}{t_k}\right)^{H-\frac{1}{\alpha}}\right)\right|^\alpha du} \\ \leq\ & e^{-\left|\lambda_ku_k^{H-\frac{1}{\alpha}}+t_k^H\sum_{j=k+1}^4\frac{\lambda_j}{t_j^H}\left(\left(u_k+\frac{t_{k+1}+\ldots+t_j}{t_k}\right)^{H-\frac{1}{\alpha}}-\left(u_k+\frac{t_{k+1}+\ldots+t_{j-1}}{t_k}\right)^{H-\frac{1}{\alpha}}\right)\right|^\alpha} \\ =\ & J_k(t_k,\ldots,t_4;\lambda_k\ldots,\lambda_4),\ \text{say}, \end{aligned}$$

where $\frac{1}{2} \le u_k \le 1$, $1 \le k \le 4$. Further

$$\frac{t_k^H}{t_j^H}\left(\left(u_k + \frac{t_{k+1} + ... + t_j}{t_k}\right)^{H-\frac{1}{\alpha}} - \left(u_k + \frac{t_{k+1} + ... + t_{j-1}}{t_k}\right)^{H-\frac{1}{\alpha}}\right)$$

is 0 when $H = \frac{1}{\alpha}$ and, when $H \neq \frac{1}{\alpha}$, is of the form of Lemma 17 with $t = t_k$, $r = t_j$, $v = t_k u_k$ and $s = t_{k+1} + ... + t_{j-1}$. Thus, similar to (41) together with a similar use of Lemma 17 therein and using $|F(\lambda)| \le C|\lambda|$, we have (letting $\lambda_5 = 0$)

$$\begin{aligned}
& I_*(t_1, t_2, t_3, t_4) \\
\le\ & \frac{C}{t_1^H t_2^H t_3^H t_4^H} \int\int\int\int \prod_{k=1}^{4} J_k(t_k, ..., t_4; \lambda_k ..., \lambda_4) \left| F\left(\frac{\lambda_k}{t_k^H} - \frac{\lambda_{k+1}}{t_{k+1}^H}\right)\right| d\lambda_1 d\lambda_2 d\lambda_3 d\lambda_4 \\
\le\ & \frac{C}{t_1^H t_2^H t_3^H t_4^H} \int\int\int\int \left(\prod_{j=1}^{4}\left(\frac{|\lambda_j| + ... + |\lambda_4|}{t_j^H} + \frac{|\lambda_{j+1}| + ... + |\lambda_4|}{t_{j+1}^H}\right)\right) \\
& \times e^{-c\sum_{j=1}^{4}|\lambda_j|^\alpha} d\lambda_1 d\lambda_2 d\lambda_3 d\lambda_4 \\
\le\ & \frac{C}{t_1^H t_2^H t_3^H t_4^H}\left(\frac{1}{t_1^H} + \frac{1}{t_2^H}\right)\left(\frac{1}{t_2^H} + \frac{1}{t_3^H}\right)\left(\frac{1}{t_3^H} + \frac{1}{t_4^H}\right)\frac{1}{t_4^H}. \qquad (43)
\end{aligned}$$

The right hand side here is a sum of the terms of the form

$$\left(\frac{1}{t_1^{aH} t_{j_1}^{bH}}\right)\left(\frac{1}{t_{j_2}^{cH} t_{j_3}^{dH}}\right), \qquad (44)$$

for an arrangement (j_1, j_2, j_3) of $(2, 3, 4)$, where a, b, c and d are integers satisfying the following constraints:

- $1 \le a \le 2$ with $a + b = 4$, and $1 \le c \le 3$ such that $c + d = 4$.

To proceed further, first suppose that

$$\kappa(t-s) > 1. \qquad (45)$$

Then, using the bound (43), we next obtain

$$\begin{aligned}
& \kappa^{-2(1-H)} \int_{\kappa s}^{\kappa t} \int_1^{\kappa(t-s)} \int_1^{\kappa(t-s)} \int_1^{\kappa(t-s)} I_*(t_1, t_2, t_3, t_4)\, dt_4 dt_3 dt_2 dt_1 \\
\le\ & C(t-s)^{2(1-H)}. \qquad (46)
\end{aligned}$$

We have, corresponding to the second factor in (44),

$$\begin{aligned}
& \kappa^{-(1-H)} \sum_{1\le c\le 3, c+d=4} \int_1^{\kappa(t-s)} \int_1^{\kappa(t-s)} \frac{1}{t_{j_2}^{cH} t_{j_3}^{dH}} dt_{j_2} dt_{j_3} \\
\le\ & 2\kappa^{-(1-H)} \int_1^{\kappa(t-s)} \int_1^{\kappa(t-s)} \left(\frac{1}{t_{j_2}^{H} t_{j_3}^{3H}} + \frac{1}{t_{j_2}^{2H} t_{j_3}^{2H}}\right) dt_{j_2} dt_{j_3} \\
\le\ & C\kappa^{-(1-H)}\left((\kappa(t-s))^{1-H} + (\kappa(t-s))^{2-4H} + 2\right) \le C(t-s)^{1-H}. \qquad (47)
\end{aligned}$$

The last bound is obtained as follows:

$$\kappa^{-(1-H)}(\kappa(t-s))^{1-H} = (t-s)^{1-H},$$

$$\kappa^{-(1-H)}\left(\kappa\left(t-s\right)\right)^{2-4H}=\kappa^{1-3H}\left(t-s\right)^{2-4H}\leq\left(t-s\right)^{3H-1}\left(t-s\right)^{2-4H}=\left(t-s\right)^{1-H},$$

where we use (45) together with the fact $1-3H<0$, and similarly

$$\kappa^{-(1-H)}\leq\left(t-s\right)^{1-H}.$$

Next, corresponding to the first factor in (44),

$$\begin{aligned}
&\kappa^{-(1-H)}\sum_{1\leq a\leq 3,a+b=4}\int_{\kappa s}^{\kappa t}\int_{1}^{\kappa(t-s)}\frac{1}{t_1^{aH}t_{j_1}^{bH}}dt_{j_1}dt_1\\
\leq\ &\kappa^{-(1-H)}\int_{\kappa s}^{\kappa t}\int_{1}^{\kappa(t-s)}\left(\frac{1}{t_1^{H}t_{j_1}^{3H}}+\frac{1}{t_1^{2H}t_{j_1}^{2H}}\right)dt_{j_1}dt_1\\
\leq\ &C\kappa^{-(1-H)}\left(\left(\kappa t\right)^{1-H}-\left(\kappa s\right)^{1-H}+\left(\kappa\left(t-s\right)\right)^{2-4H}+1\right), \qquad (48)
\end{aligned}$$

where

$$\kappa^{-(1-H)}\left(\left(\kappa t\right)^{1-H}-\left(\kappa s\right)^{1-H}\right)=t^{1-H}-s^{1-H}\leq\left(t-s\right)^{1-H}.$$

Thus (46) holds.

Next, note that

$$\kappa^{-2(1-H)}\int_{\kappa s}^{\kappa t}\int_{0}^{1}\int_{1}^{\kappa(t-s)}\int_{1}^{\kappa(t-s)}I_*\left(t_1,t_2,t_3,t_4\right)dt_4dt_3dt_2dt_1\leq C\left(t-s\right)^{2(1-H)}. \qquad (49)$$

This follows, because, as in (43) but using $\left|F\left(\frac{\lambda_1}{t_1^H}-\frac{\lambda_2}{t_2^H}\right)\right|\left|F\left(\frac{\lambda_2}{t_2^H}-\frac{\lambda_3}{t_3^H}\right)\right|\leq C$, the left hand side of (49) is bounded by

$$\begin{aligned}
&C\left(t^{1-H}-s^{1-H}\right)\kappa^{-(1-H)}\int_{1}^{\kappa(t-s)}\int_{1}^{\kappa(t-s)}\frac{1}{t_3^Ht_4^H}\left(\frac{1}{t_3^H}+\frac{1}{t_4^H}\right)\frac{1}{t_4^H}dt_4dt_3\\
\leq\ &C\left(t^{1-H}-s^{1-H}\right)\left(t-s\right)^{(1-H)},\quad\text{using (47).} \qquad (50)
\end{aligned}$$

In the same way the bound in (49) holds when the integral $\int_{\kappa s}^{\kappa t}\int_0^1\int_1^{\kappa(t-s)}\int_1^{\kappa(t-s)}$ there is changed to $\int_{\kappa s}^{\kappa t}\int_1^{\kappa(t-s)}\int_0^1\int_1^{\kappa(t-s)}$ or to $\int_{\kappa s}^{\kappa t}\int_1^{\kappa(t-s)}\int_1^{\kappa(t-s)}\int_0^1$. Thus Proposition 5 holds when $\kappa\left(t-s\right)>1$ (and $l=2$).

Next, in the remaining case

$$\kappa\left(t-s\right)\leq 1,$$

the right hand side in (42) is bounded by, similar to (26),

$$\begin{aligned}
&C\kappa^{-2(1-H)}\int_{\kappa s}^{\kappa t}\int_{0}^{\kappa(t-s)}\int_{0}^{\kappa(t-s)}\int_{0}^{\kappa(t-s)}\frac{1}{t_1^Ht_2^Ht_3^Ht_4^H}dt_4dt_3dt_2dt_1\\
\leq\ &C\kappa^{-2(1-H)}\left(\kappa\left(t-s\right)\right)^{4(1-H)}=C\kappa^{2(1-H)}\left(t-s\right)^{4(1-H)}\\
\leq\ &C\left(t-s\right)^{-2(1-H)}\left(t-s\right)^{4(1-H)}=C\left(t-s\right)^{2(1-H)}. \qquad (51)
\end{aligned}$$

This completes the proof of Proposition 5 for $l=2$.

The proof for the general $l\geq 1$ is the same, except for notational differences. To see this, in the general case the bound analogous to that in (43) will take the form

$$C\left(\frac{1}{\prod_{j=1}^{2l}t_j^H}\right)\frac{1}{t_{2l}^H}\prod_{j=1}^{2l-1}\left(\frac{1}{t_j^H}+\frac{1}{t_{j+1}^H}\right).$$

This will be a finite sum of the products of the form

$$\frac{1}{t_1^{aH} t_{j_1}^{bH}} \prod_{r=1}^{l-1} \frac{1}{t_{j_{2r}}^{c_r H} t_{j_{2r+1}}^{d_r H}},$$

for an arrangement $(j_1, j_2, ..., j_{2l-1})$ of $(2, 3, 4, ..., 2l)$, where, as in (44), a, b, c_r and d_r are integers such that $1 \leq a \leq 2$ with $a + b = 4$, and each $1 \leq c_r \leq 3$ such that $c_r + d_r = 4$. Then it is clear that Proposition 5 holds for $l \geq 1$ also. This completes the proof. ■

Remark. We now show that the computation of the fourth moment is sufficient for the verification of (R3). Taking $s = \frac{l-1}{m}$ and $t = \frac{l}{m}$, the bound $C(t-s)^{1-H}$ in (47) becomes Cm^{H-1}. Similarly, the bound in (48) becomes

$$\left(\left(\frac{l}{m}\right)^{1-H} - \left(\frac{l-1}{m}\right)^{1-H} + m^{4H-2}\kappa^{1-3H} + \kappa^{-(1-H)} \right),$$

and the bound in (50) becomes $Cm^{H-1}\left(\left(\frac{l}{m}\right)^{1-H} - \left(\frac{l-1}{m}\right)^{1-H} \right)$, so that when $\kappa(t-s) = \frac{\kappa}{m} > 1$,

$$E\left[\zeta_{\kappa m l}^4\right] \leq Cm^{H-1} \left(\left(\frac{l}{m}\right)^{1-H} - \left(\frac{l-1}{m}\right)^{1-H} + m^{4H-2}\kappa^{1-3H} + \kappa^{-(1-H)} \right).$$

We have

$$m^{H-1} \sum_{l=1}^{[mL]} \left(\left(\frac{l}{m}\right)^{1-H} - \left(\frac{l-1}{m}\right)^{1-H} \right) = m^{H-1} \left(\frac{[mL]}{m}\right)^{1-H} \quad \text{with } H-1 < 0.$$

Further $1 - 3H < 0$. Hence (R3) follows when $\frac{\kappa}{m} > 1$.

When $\kappa(t-s) = \frac{\kappa}{m} \leq 1$, the first bound $C\kappa^{-2(1-H)} (\kappa(t-s))^{4(1-H)}$ in (51) is bounded by $C\kappa^{-2(1-H)}$, and hence (R3) holds in this case also. ■

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