isibang/ms/2005/39
August 16th, 2005
http://www.isibang.ac.in/~statmath/eprints

Structure of a Code related to Sp(4,q), q even

N.S. Narasimha Sastry and R.P. Shukla

Indian Statistical Institute, Bangalore Centre
8th Mile Mysore Road, Bangalore, 560059 India

Structure of a Code related to Sp(4,q), q even

N.S. Narasimha Sastry and R.P. Shukla

Abstract

We determine the socle and the radical series of the binary code associated with a finite regular generalized quadrangle of even order, considered as a module for the commutator of each of the orthogonal subgroups in the corresponding symplectic group.

1 Introduction and Statement of the result

Let $V(q) = \mathbb{F}_q^4$, $q = 2^n$, be the vector space of dimension four over $\mathbb{F}_q$ endowed with a non-degenerate symplectic bilinear form $\hbar$ and $W(q)$ denote the incidence system with the set P of all one dimensional subspaces of $V(q)$ as its *point-set*, the set L of all two dimensional subspaces of $V(q)$ which are isotropic with respect to $\hbar$ as its *line-set* and symmetrized proper inclusion as the *incidence*. Then, $W(q)$ is a regular generalized quadrangle of order q ([12], p.37). Further, elements of the symplectic group G defined by $\hbar$ act as incidence preserving permutations on the sets P and L.

Let k be an algebraically closed extension field of $\mathbb{F}_q$. We denote by $\mathcal{C}$ the image of the $kG-$module homomorphism from the permutation $G-$ module k^L on L over k to the permutation $G-$ module k^Pon P over k, taking $l \in L$ to $\sum_{p\in l} p \in k^P$. The endomorphism of k^P taking each element of P to the '*all-one*' vector $\mathbf{1} = \sum_{p\in P} p$ in k^P is a $G-$ module homomorphism onto the trivial $G-$submodule $k\mathbf{1}$ of k^P. If Y_P is the kernel of the 'augmentation map' from k^P to k taking $p \in P$ to $1 \in k$, then

$$k^P = k\mathbf{1} \oplus Y_P.$$

Since every element of P is incident with $q + 1$ elements of L, it follows that $\mathbf{1} \in \mathcal{C}$ and

$$\mathcal{C} = k\mathbf{1} \oplus \overline{\mathcal{C}},$$

where $\overline{\mathcal{C}} = Y_P \cap \mathcal{C}$. The Loewy structure of the $kG-$module $\overline{\mathcal{C}}$ is determined in ([14], Theorem 2, p.486). Here, we determine the Loewy structure of $\overline{\mathcal{C}}$ as a $k\Omega(f)$-module, where $\Omega(f)$ is the commutator subgroup of the orthogonal group $O(f) \subset G$ defined by a non-degenerate quadratic form f on $V(q)$ polarizing to $\hbar$. That is, $\hbar\,(x, y) = f\,(x + y) - f(x) - f(y)$ holds for all $x, y \in V(q)$. There are two such quadratic forms on $V(q)$, up to $G-$equivalence (see [3], Theorem 6, p.214). They are distinguished by the presence (hyperbolic case) or otherwise (elliptic case) of isotropic projective lines in the set $V_q(f)$ of zeroes of f in the projective $3-$ space $P(3,q)$ over $\mathbb{F}_q$; that is, they correspond to the cases when the Witt index of f is 2 or 1 ([3], 4.1, p.218). The subgroups $O(f)$ and $\Omega(f)$ of G are isomorphic to $(SL(2,q) \times SL(2,q)) \cdot 2$ and $SL(2,q) \times SL(2,q)$, respectively, when the Witt index of f is 2; and to $SL(2,q^2) \cdot 2$ and $SL(2,q^2)$, respectively, when the Witt index of f is 1.We also mention that $Sp(4,q)$ contains exactly 2 conjugacy classes of subgroups of each of the types $SL(2,q^2)$ and $SL(2,q) \times SL(2,q)$ (see [6], Corollary, p.247).

To state our theorem, we describe the simple $kG-$modules. Let $N = \{0, 1, \cdots, 2n-1\}$, with addition taken modulo $2n$. Let $V = V(q) \otimes k \simeq k^4$ and extend the symplectic form $\hbar$ to V. Then G is the subgroup of the algebraic group $Sp(V) \simeq Sp(4,k)$ fixed by the $n-$th power of the Frobenius map σ (which is the 'square-the-matrix-entries' map on $GL(4,k)$). It is well known that $Sp(V)$ has an endomorphism τ with $\tau^2 = \sigma$ ([19], Theorem 28, p.146). For any non-negative integer i, we denote by V_i the $Sp(V)$- module (i-th *Frobenius twist* of V) whose vector space structure is the same as that of V and an element g of $Sp(V)$ acts on V_i as $\tau^i(g)$ acts on V. For $I \subseteq N$, let V_I denote the kG-module $\otimes_{i \in I} V_i$ (with $V_\emptyset = k$). Then, by Steinberg's tensor product Theorem ([18], §11), $\{V_I : I \subseteq N\}$ is a complete set of inequivalent simple kG-modules.

Let $N_1 = \{0, 1, \cdots, n-1\}$, with addition taken modulo n. For $I \subseteq N$, define $I_e = \{i \in N_1 : 2i \in I\}$ and $I_o = \{i \in N_1 : 2i-1 \in I\}$. For $n > 1$, we denote by $\mathcal{N}$ the set of all subsets I of N which has no consecutive elements, that is, all I such that I_e and I_0 are disjoint and if $i \in I_e$, then $i+1 \notin I_o$. For each subset I of N_1, the subset $N_1 \backslash (I \cup \{i+1 | i \in I\})$ of N_1 will be called the *admissible complement* of I and will be denoted by I^{ac}. We observe that for $I \subseteq N_1$, the subset $K = \{2i : i \in I\} \cup \{2i-1 : i \in I^{ac}\}$ is the unique maximal subset of N such that $K_e = I$ and $K_o = I^{ac}$. Also, $I^{ac} = \emptyset$ if $|I| \geq n-1$ or if $N_1 \setminus I = \{i, j\}$, $i < j$ and $j \neq i+1$. For each $m \in \{0, 1, \cdots, n-2, n\}$, let $\mathcal{A}_m$ denote the set of all subsets I of N_1 such that $|I^{ac}| = m$. We observe that $\mathcal{A}_n = \{\emptyset\}$ and $\mathcal{A}_{n-1} = \emptyset$. For a module M admitting a chain of submodules $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_r$, we indicate the chain of factor modules $E_i = M_i/M_{i+1}$ as $E_{r-1} \backslash \cdots \backslash E_0$ or $E_0 / \cdots / E_{r-1}$. For a kG-module M, we write M as $rad^0(M)$, denote by $rad^1(M)$ the *radical of* M (that is, the smallest submodule of M with semi-simple quotient) and, for a positive integer i, define the *i-th radical* $rad^i(M)$ of M recursively as $rad^1(rad^{i-1}(M))$. Dually, we write $soc^0(M) = \{0\}$, denote by $soc^1(M)$ the *socle of* M (that is, the largest semi-simple submodule of M) and, for a positive integer i, define the *i-th socle* $soc^i(M)$ of M recursively by $soc^i(M)/soc^{i-1}(M) = soc(M/soc^{i-1}(M))$. The *radical length* (resp.ly *socle length*) of M is the positive integer r such that $rad^r(M) = 0$ but $rad^{r-1}(M) \neq 0$

(resp.ly $soc^r(M) = M$ but $soc^{r-1}(M) \neq M$). We refer to $M/rad^1(M)$ as the *head* of M and write it as $hd(M)$.

Theorem 1 *(a) Let $n = 1$. As $k\Omega(f)$ -modules,*

(i) when f is of Witt index 1, V_0 and V_1 are semi-simple; $\bar{\mathcal{C}}$ is multiplicity - free and is isomorphic to the direct sum of V_0 and a $k\Omega(f)$-module X with $head(X) \simeq V_1$ and $rad(X) \simeq k$; and

(ii) when f is of Witt index 2, $\bar{\mathcal{C}}$ is semi-simple and each composition factor of $\bar{\mathcal{C}}$ appears with multiplicity one.

(b) Let $n \geq 2$. As a $k\Omega(f)$-module,

(i) V_I is semi-simple for each $I \in \mathcal{N}$ and $\bar{\mathcal{C}}$ is multiplicity - free;

(ii) socle length of $\bar{\mathcal{C}}$ is $n+1$ and its j^{th}- socle layer $soc^j(\bar{\mathcal{C}})/soc^{j-1}(\bar{\mathcal{C}})$, $1 \leq j \leq n+1$, is isomorphic to

$$\bigoplus_{I \in \mathcal{N} \text{ and } |I_o| = j-1} V_I$$

(iii)

$$\overline{\mathcal{C}} \backsimeq \bigoplus_{m \in \{0,1,\cdots,n-2,n\}} X_m \text{ and } X_m = \bigoplus_{I \in \mathcal{A}_m} X_{m,I},$$

where $X_{m,I}$ is the unique indecomposable $k\Omega(f)$-submodule of $\overline{\mathcal{C}}$ with head V_K with K denoting the unique element of $\mathcal{N}$ such that $K_e = I$ and $K_o = I^{ac}$. The radical length of $X_{m,I}$ is $m+1$ and its j-th radical layer $rad^j(X_{m,I})/rad^{j+1}(X_{m,I})$,$0 \leq j \leq m$, is isomorphic to

$$\bigoplus_{J \in \mathcal{N},\ J_e = I,\ J_o \subseteq I^{ac} \text{ and } |J_o| = m-j} V_J$$

and

(iv) $soc^j(X_{m,I}) = rad^{m+1-j}(X_{m,I})$,$1 \leq j \leq m$.

To prove this, we use

Theorem 2 *([14], Theorem 2) The radical series of $\bar{\mathcal{C}}$ as a $kG-$ module has length $2n+1$. The radical layers are*

$$rad^j(\bar{\mathcal{C}})/rad^{j+1}(\bar{\mathcal{C}}) = \bigoplus_{I \in \mathcal{N} \text{ and } |I_e| - |I_o| + n = j} V_I \qquad (0 \leq j \leq 2n)$$

Moreover,

$$soc^j(\bar{\mathcal{C}}) = rad^{2n+1-j}(\bar{\mathcal{C}}).$$

The crucial observation is that if E is a section of $\bar{\mathcal{C}}$ which is a nonsplit extension of V_K by V_L, then E is a nonsplit $k\Omega(f)$-module if and only if $L = K \cup \{2t-1\}$ for some $t \in N_1$ (see the the first para of the proof of b(ii) in Section 3). Thus either each or none of the simple $k\Omega(f)$ −factors of V_L descends.

In Section 2, we study the structure of V_I as a $k\Omega(f)$-module and prove Theorem 1 in Section 3. We also write the socle structure of $\bar{\mathcal{C}}$ as a $k\Omega(f)$ −module for $n = 1, 2, 3, 4$. We view this work as part of the program of determining the module structure of permutation modules for finite groups of Lie type. See ([15], [16]) and references in them for a few other cases considered in the literature.

2 V_I as k$\Omega(f)$−modules

2.1.Throughout, we choose the nondegenerate symplectic bilinear form $\hbar$ on $\mathbb{F}_q^4 \times \mathbb{F}_q^4$ to be

$$\hbar((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = x_1y_4 + x_4y_1 + x_2y_3 + y_2x_3.$$

The quadratic forms f_e and f_h on $\mathbb{F}_q^4$ polarizing to $\hbar$ and of respective Witt indices 1 and 2 are chosen to be

$$\begin{aligned} f_e(x_1, x_2, x_3, x_4) &= x_1x_4 + \alpha^{q+1}x_2^2 + x_2x_3 + \alpha^{q+1}x_3^2; \text{ and} \\ f_h(x_1, x_2, x_3, x_4) &= x_1x_4 + x_2x_3, \end{aligned}$$

where α is an element of $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$ such that $\alpha + \alpha^q = 1$.

This choice of f_e and f_h entails no loss of generality in view of the uniqueness of a nondegenerate symplectic form $\hbar$ on $V(q)$, up to $GL(V(q))-$ equivalence (see [11], Corollary 8.2, p.587); and the uniqueness of a quadratic form on $V(q)$ of a given Witt index and polarizing to $\hbar$, up to $G-$ equivalence (see [3], Theorem 6, p.214 and [4], p.39).

2.2 The case f_e: The polynomial $\alpha^{q+1}x_2^2 + x_2x_3 + \alpha^{q+1}x_3^2$ is irreducible over $\mathbb{F}_q$ but factors as $(\alpha x_2 + \alpha^q x_3)(\alpha^q x_2 + \alpha x_3)$ over $\mathbb{F}_{q^2}$. So, the zero set $\mathcal{E}$ of f_e in $P(3, q)$ is an elliptic quadric and the zero set $\widehat{\mathcal{E}}$ of f_e in $P(3, q^2)$ is a hyperbolic quadric. Since f_e polarizes to $\hbar$, $\mathcal{E}$ is an ovoid in the generalized quadrangle $W(q)$ defined by $\hbar$ ([2], p.51).

(A) Recall that $\Omega(f_e) \backsimeq SL(2, q^2)$. We now interpret the $k\Omega(f_e)$ −modules V_i as $kSL(2, q^2)$-modules. For this, we construct:

(i) an isomorphism from $SL(2, q^2)$ to $\Omega(f_e)$, following ([7], Theorems 15.3.11 and 15.3.18); and

(ii) a graph automorphism τ of G, following ([5] , pp.58-60; also see [20], Chapter 12).

(i) (a) The inversive planes $(P(1,q^2),\mathcal{L})$ and $(\mathcal{E},\Phi)$

Recall that a subline of the projective line $P(1,q^2)$ over $\mathbb{F}_{q^2}$ is the set of all one dimensional subspaces of $\mathbb{F}_{q^2}^2$ generated by the $\mathbb{F}_q-$ span of a fixed basis of $\mathbb{F}_{q^2}^2$. Equivalently, it is a nonsingular Hermitian variety $V_A = V(xA(x^{(q)})^{tr})$ in $P(1,q^2)$ defined by an equation of the form $a_1x_1^{q+1} + bx_1x_2^q + b^qx_1^qx_2 + a_2x_2^{q+1}$, where $x = (x_1,x_2) \in \mathbb{F}_{q^2}^2$, $x^{(q)} = (x_1^q, x_2^q)$ and $A = \begin{pmatrix} a_1 & b \\ b^q & a_2 \end{pmatrix}$ with $a_i \in \mathbb{F}_q, b \in \mathbb{F}_{q^2}$, $a_1a_2 \neq bb^q$ ([8], Lemma 6.2, p.138). Let $\mathcal{L}$ denote the set of all sublines of $P(1,q^2)$. Then, the incidence structure $(P(1,q^2),\ \mathcal{L})$, with symmetrized inclusion as incidence is an inversive plane of order q; that is, a $3-(q^2+1,q+1,1)$ design ([2], p.257).This is isomorphic to the incidence system $(\mathcal{E},\Phi)$, where $\mathcal{E}$ is as above, Φ is the set of all planes in $P(3,q)$ which are not tangent to $\mathcal{E}$ and the incidence again is symmetrized containment ([2], p.257).

(b) An isomorphism between $(P(1,q^2),\mathcal{L})$ and $(\ \mathcal{E},\Phi)$. To construct this, we follow ([7] , see Theorem 15.3.11, p. 21). Let i be the injective map from $P(1,q^2)$ to $P(3,q^2)$ defined by

$$i(P(x_1,x_2)) = P(x_1^{q+1}, x_1x_2^q, x_1^qx_2, x_2^{q+1})$$

and ψ be the involutory projectivity of $P(3,q^2)$ defined by

$$\psi(P(x_1,x_2,x_3,x_4)) = P(x_1, \alpha x_2 + \alpha^q x_3, \alpha^q x_2 + \alpha x_3, x_4).$$

We identify $P(3,q)$ with its image in $P(3,q^2)$ under the map $\mathbb{F}_q a \mapsto \mathbb{F}_{q^2}a$; where $a = (a_1,a_2,a_3,a_4) \in \mathbb{F}_q^4$. We treat $\hbar$ chosen in (2.1) also as a nondegenerate symplectic form on $\mathbb{F}_{q^2}^4$. For x in $P(3,q^2)$, we denote by $\hbar\,(x,-)$ the plane in $P(3,q^2)$ consisting of all points y of $P(3,q^2)$ such that $\hbar\,(x,y) = 0$. Since ψ stabilizes $\hbar$, $\psi(\hbar\,(x,-)) = \hbar(\psi\,(x),-)$ for each x in $P(3,q^2)$. Let j be the map from $\mathcal{L}$ to Φ taking V_A to $\hbar(\psi\,(P(a'),-)) \cap P(3,q)$, where $a' = (a_1,b,b^q,a_2) \in \mathbb{F}_{q^2}^4$. These planes are not tangent to $\mathcal{E}$ because : f_e takes the nonzero value $a_1a_2 - bb^q$ on $\psi\,(P\,(a'))$ and so $\psi\,(P\,(a')) \notin \mathcal{E}$; it is known that the tangent planes to $\mathcal{E}$ in $P(3,q)$ are all of the form $\hbar\,(x,-) \cap P(3,q)$, $x \in \mathcal{E}$; and the correspondence $\alpha \longleftrightarrow \hbar\,(\alpha,-) \cap P(3,q)$ between the points of $P(3,q)$ and the planes of $P(3,q)$ is incidence preserving and bijective. The map $\psi \circ i$ from $P(1,q^2)$ to $\mathcal{E}$ and the map j from $\mathcal{L}$ to Φ are both bijective and form an incidence preserving pair; that is, if $p \in V_A \in \mathcal{L}$, then $\psi \circ i\ (p) \in j(V_A) \in \Phi$.

(c) The isomorphism from $SL(2,q^2)$ to $\Omega\,(f_e)$

The full automorphism group of the incidence structure $(P(1,q^2),\ \mathcal{L})$ is $P\Gamma L\,(2,q^2) \simeq SL\,(2,q^2)\,\langle\xi\rangle$, where $\xi S\xi^{-1} = S^{(2)} = (s_{ij}^2)$ for all $S = (s_{ij}) \in SL\,(2,q^2)$ ([2], 6.4.1, p.274). We give an explicit isomorphism δ from its (unique) subgroup K isomorphic to $SL\,(2,q^2)\cdot 2$ into $Sp(4,q^2)$ such that $\psi(\mathrm{im}\ \delta)\psi^{-1}$ equals the stabilizer $O\,(f_e)$ of $\mathcal{E}$ in G (see [7], Theorem 15.3.18, p.27).

For $S = (s_{ij}) \in SL(2, q^2)$, let P_S denote the projectivity of $P(1, q^2)$ taking $P(x)$ to $P(xS)$. Then, $\langle P_S : S \in SL(2, q^2)\rangle \backsimeq SL(2, q^2)$. If θ is the involutory projectivity of $P(1, q^2)$ taking $P(x_1, x_2) \mapsto P(x_1^q, x_2^q)$, then $\theta P_S \theta^{-1} = P_{S^{(q)}}$, where S is as above and $S^{(q)} = (s_{ij}^{(q)})$. So $\langle P_S : S \in SL(2, q^2)\rangle\langle\theta\rangle$ is the subgroup of $P\Gamma L(2, q^2)$ of index n.

Now P_S induces a bijection T_S on $\mathcal{L}$ which takes V_A to $V_{SA((S^{(q)})^{-1})^{tr}}$. With the correspondence $A \leftrightarrow a'$, with a' as above, the map $A \mapsto SA((S^{(q)})^{-1})^{tr}$ can be written as $a' \mapsto a'R_S$, where

$$R_S = \begin{pmatrix} s_{11}s_{11}^q & s_{11}s_{21}^q & s_{21}s_{11}^q & s_{21}s_{21}^q \\ s_{11}s_{12}^q & s_{11}s_{22}^q & s_{21}s_{12}^q & s_{21}s_{22}^q \\ s_{12}s_{11}^q & s_{12}s_{21}^q & s_{22}s_{11}^q & s_{22}s_{21}^q \\ s_{12}s_{12}^q & s_{12}s_{22}^q & s_{22}s_{12}^q & s_{22}s_{22}^q \end{pmatrix} = S^{tr} \otimes (S^{(q)})^{tr}.$$

Let $\mathcal{R}_S$ denote the projectivity of $P(3, q^2)$ taking $P(x_1, x_2, x_3, x_4)$ to $P((x_1, x_2, x_3, x_4)R_S)$.Note that : ψ and $\mathcal{R}_S$ both stabilize $\hbar$; $\mathcal{R}_S$ fixes the quadratic form $g\,(x_1, x_2, x_3, x_4) = x_1x_4 + x_2x_3$; ψ is a bijection between the varieties $V_{q^2}\,(g)$ and $V_{q^2}\,(f_e)$;and $\psi\mathcal{R}_S\psi^{-1}$ is a projectivity of $P(3, q)$ also. So, $\psi\mathcal{R}_S\psi^{-1}$ is in the stabilizer of $\mathcal{E}$ in G. If $\mathcal{R}_S$ is identity, then so is T_S. These facts imply that the map $S \mapsto \psi\mathcal{R}_S\psi^{-1}$ is a monomorphism from $SL(2, q^2)$ to $O(f_e)$. Now, the map $A \mapsto A^{(q)}$ can be written as $a' \mapsto a'\mathcal{R}$, where $\mathcal{R}$ denotes the projectivity of $P(3, q^2)$ taking $P(x_1, x_2, x_3, x_4)$ to $P(x_1, x_3, x_2, x_4)$. Further, $\mathcal{R}$ is in the stabilizer of $\mathcal{E}$ in G, ψ and $\mathcal{R}$ commute and $\mathcal{R}\mathcal{R}_S\mathcal{R}^{-1} = \mathcal{R}_{S^{(q)}}$ for each $S \in SL(2, q^2)$. So, $\langle\psi\mathcal{R}_S\psi^{-1} : S \in SL(2, q^2)\rangle\ \langle\mathcal{R}\rangle$ is a subgroup of $O\,(f_e)$ and is isomorphic to $SL_2\,(q^2) \cdot 2$. Now equality holds by order considerations and the isomorphism follows.

(ii) We now describe a graph automorphism τ of $Sp(V)$, following ([5], pp.58-60). (The argument presented in loc. cit. constructs a graph automorphism for $G = Sp(V(q))$. However the arguments are valid for $Sp(V)$ also.) Let Q denote the nondegenerate quadratic form on the exterior square Λ^2V of V defined by

$$Q\left(\Sigma_{1\leq i<j\leq 4}\lambda_{ij}e_i \wedge e_j\right) = \lambda_{12}\lambda_{34} + \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23}$$

(and whose zero set in $P(\Lambda^2V)$ is the well-known *Klein quadric*). Let β denote the polarization of Q and $\gamma = e_1 \wedge e_4 + e_2 \wedge e_3$. Then the restriction of Q to the hyperplane $U = \{x \in \Lambda^2V : \beta\,(x, \gamma) = 0\}$ of Λ^2V is a nondegenerate quadratic form; and the restriction of β to U is an alternating form with radical $k\gamma$. The alternating form $\overline{\beta}$ induced by β on $\overline{U} = \frac{U}{k\gamma}$ is nondegenerate. So the symplectic space $\left(\overline{U}, \overline{\beta}\right)$ is isometric to $(V, \hbar)$. Let $\overline{p} : \overline{U} \to V$ be the isometric isomorphism induced by the linear map $p : U \to V$ defined by

$$p\,(e_1 \wedge e_2) = e_1,\ p\,(e_1 \wedge e_3) = e_2,\ p\,(e_2 \wedge e_4) = e_3,$$
$$p\,(e_3 \wedge e_4) = e_4,\ p\,(\gamma) = 0.$$

Then the map taking $g \in Sp(V)$ to $\overline{p}(\overline{\wedge^2(g)})\,\overline{p}^{-1} \in Sp(V)$ is a graph automorphism τ of $Sp(V)$ which, on restriction to G, gives a graph automorphism of G.

(B) Let W denote the standard two dimensional simple $kSL(2,q^2)$-module. For a non-negative integer i, let W_i denote, as in Section 1, the $kSL(2,q^2)$- module whose underlying vector space structure is the same as that of W and the action of $g = (a_{l,m}) \in SL(2,q^2)$ on W_i is the usual action of $g^{(2^i)} = \left(a_{l,m}^{2^i}\right)$ on W. For any subset I of N, let W_I denote the $kSL(2,q^2)-$module $\otimes_{i\in I} W_i$. Then, by Steinberg's tensor product Theorem, $\{W_I : I \subseteq N\}$ is a complete set of inequivalent simple $kSL(2,q^2)$-modules. The following decomposition was suggested to the first named author by Peter Sin.

Lemma 3 *For $i \in N_1$,*

$$V_{2i}|_{\Omega(f_e)} \simeq W_i \otimes W_{n+i}$$

and

$$V_{2i-1}|_{\Omega(f_e)} \simeq W_i \oplus W_{n+i}.$$

Proof. We prove this by using the above interpretation of V_i as $kSL(2,q^2)$- modules. The group $H = \langle R_S : S \in SL(2,q^2)\rangle$, where R_S is as in (2.2A.i(c)) and $\Omega(f_e)$ are both isomorphic to $SL(2,q^2)$ and are conjugate in $Sp(4,k)$ (in fact in $Sp(4,q^2)$) (see the last paragraph of 2.2.A. i(c)). So, for $I \subseteq N, V_I$ considered as a $kH-$ module and V_I considered as a $k\Omega(f_e)-$module are isomorphic $SL(2,q^2)$-modules. Let (e_1, e_2, e_3, e_4) be the standard ordered basis for V and (v_1, v_2) be an ordered basis for W. For $S \in SL(2,q^2)$, $S \otimes S^{(q)}$ represents not only the action of S on V with respect to the ordered basis (e_1, e_2, e_3, e_4) of V but also the action of S on $W_0 \otimes W_n$ with respect to the ordered basis $(v_2 \otimes v_2, v_2 \otimes v_1, v_1 \otimes v_2, v_1 \otimes v_1)$ of $W_0 \otimes W_n$. So, $V_0|_{\Omega(f_e)} \simeq W_0 \otimes W_n$. Now, an application of τ^{2i} to V yields the first part of the lemma.

A simple calculation shows that R_S^τ leaves the subspaces $M^1 = ke_4 + ke_1$ and $M^2 = ke_2 + ke_3$ invariant. Further, the matrix representation of R_S^τ on M^1 with respect to the ordered basis (e_4, e_1) is S and it is $S^{(q)}$ on M^2 with respect to the ordered basis (e_3, e_2). So, an application of τ^{2i-1} to V yields the second part of the lemma. ■

For $I \subseteq N_1$, let $\mathcal{N}_I$ denote the set of $2^{|I|}$ subsets J of N of size $|I|$ such that, for each $t \in I$, only one of t and $n+t$ is in J. Let $\overline{I} = I \cup (n+I)$. Then, for $K \in \mathcal{N}$, Lemma 3 yields

$$V_K|_{\Omega(f_e)} \simeq \left(\bigoplus_{L\in\mathcal{N}_{K_0}} W_L\right) \otimes W_{\overline{K}_e} \tag{1_e}$$

Notice that each irreducible component in (1_e) determines both the sets K_e and K_o.. So we have the following

Corollary 4 *Let $K, K' \in \mathcal{N}$ be distinct. Then V_K and $V_{K'}$ are semi-simple $k\Omega(f_e)$-modules with no irreducible factors in common.*

2.3 The case f_h : We now study V_I as a $k\Omega(f_h)$ −module. The zero set of f_h in $P(3,q)$ is a quadric of Witt index 2. Since f_h polarizes to $\hbar$, the projective lines contained in this quadric are isotropic with respect to $\hbar$. Further, $\Omega(f_h) = H_1H_2, [H_1, H_2] = H_1 \cap H_2 = \langle 1 \rangle$, where H_1 (resp.ly H_2) $\backsimeq SL(2,q)$ is the stabilizer of the subspace $ku_1 + ku_2$ (resp.ly $kv_1 + kv_2$) of V. Here $u_1 = e_1 + e_3$, $u_2 = e_2 + e_4, v_1 = e_1 + e_2$ and $v_2 = e_3 + e_4$. Further, the action of H_1 (resp.ly H_2) on $ku_1 + ku_2$ (resp.ly $kv_1 + kv_2$) with respect to its ordered basis (u_1, u_2) (resp.ly (v_1, v_2)) is the standard $SL(2,q)$-action on k^2.

Let M^1 and M^2 be two copies of the standard 2- dimensional simple $kSL(2,q)$-module. For $j \in N_1$, define the j^{th}−standard Frobenius twist M^i_j of M^i as in 2.2.B . For $J \subseteq N_1$, define M^i_J as $\otimes_{j \in J} M^i_j$. Further, for I and $J \subseteq N_1$, denote by $M^1_I \# M^2_J$ the outer tensor product of M^1_I and M^2_J. That is, $M^1_I \# M^2_J$ is the k−vector space $M^1_I \otimes M^2_J$ with the action of $(h_1, h_2) \in SL(2,q) \times SL(2,q)$ on it given by

$$(h_1, h_2)(m_1 \otimes m_2) = h_1(m_1) \otimes h_2(m_2).$$

Then, $\{M^1_I \# M^2_J : I, J \subseteq N_1\}$ is a complete set of pairwise nonisomorphic simple $k(SL(2,q) \times SL(2,q))-$ modules ([9], Theorem 9.14, p.136).

Lemma 5 *For* $i \in N_1$,

$$V_{2i}|_{\Omega(f_h)} \simeq M^1_i \# M^2_i$$

and

$$V_{2i-1}|_{\Omega(f_h)} \simeq (M^1_i \# k) \oplus (k \# M^2_i).$$

Proof. Let $h_i \in H_i$ be represented by the matrix $A_i \in SL(2,q)$ with respect to the ordered bases mentioned above. The matrix $A_1 \otimes A_2$ represents the action of $h = h_1h_2$ on V with respect to its ordered basis (e_1, e_2, e_3, e_4) as well as its action on $(ku_1 + ku_2)\#(ku_3 + ku_4)$ with respect to its ordered basis $(u_1 \otimes v_1, u_2 \otimes v_1, u_1 \otimes v_2, u_2 \otimes v_2)$. So, $V_0|_{\Omega(f_h)} \simeq M^1 \# M^2$. Now, an application of τ^{2i} to V yields the first part of the lemma.

A simple calculation shows that each of the subgroups H^τ_i stabilizes the subspaces $ke_2 + ke_3$ and $ke_1 + ke_4$. Further, the action of H^τ_1 (resp.ly of H^τ_2) on $ke_2 + ke_3$ (resp.ly on $ke_1 + ke_4$) with respect to the ordered basis (e_2, e_3) (resp.ly, (e_1, e_4)) is equivalent to the standard action of $SL(2,q)$ on k^2 and the action on $ke_1 + ke_4$ (resp.ly on $ke_2 + ke_3$) is trivial. So, $V_1|_{\Omega(f_h)} \simeq (M^1_1 \# k) \oplus (k \# M^2_1)$. An application of $\tau^{2(i-1)}$ to V_1 now yields the second part of the lemma. ■

For any subset I of N_1, let $\mathcal{M}_I$ denote the set of $2^{|I|}$ ordered partitions (A,B) of I. By Lemma 5 , if $K \in \mathcal{N}$, then

$$V_K|_{\Omega(f_h)} \simeq \left(M^1_{K_e} \# M^2_{K_e}\right) \otimes \left(\oplus_{(A,B)\in\mathcal{M}_{K_0}} M^1_A \# M^2_B\right)$$
$$\simeq \left(\oplus_{(A,B)\in\mathcal{M}_{K_0}} M^1_{K_e\cup A} \# M^2_{K_e\cup B}\right). \qquad ((1_h))$$

Note that for $K \in \mathcal{N}$, K_e and K_o are disjoint. Further K is determined by each irreducible component in (1_h). So, we have

Corollary 6 *Let $K, K' \in \mathcal{N}$ be distinct. Then V_K and $V_{K'}$ are semi-simple $k\Omega(f_h)$-modules with no irreducible factors in common.*

2.4. The core of the proof of the theorem is in Lemmas 7 and 8 we now state. Let $f \in \{f_e, f_h\}$ and assume that $n \geq 2$ for Lemmas 7 and 8.

Lemma 7 *Let $J, K \in \mathcal{N}$. If the symmetric difference of J and K is not equal to $\{2t-1\}$ for some $t \in N_1$, then $Ext^1_{k\Omega(f)}(V_J, V_K) = 0$.*

Now, let $J = K \cup \{2t-1\}$ and $2t-1 \notin K$. Then, $2t-2 \notin K$ and, by ([17], Theorem, p.159), there exists a unique $kG-$ module E, up to isomorphism, which is a nonsplit extension of V_K by V_J. Further,

Lemma 8 $Soc_{k\Omega(f)}\left(E|_{\Omega(f)}\right) \simeq V_K|_{\Omega(f)}$. *In particular, as a $k\Omega(f)-$ module, E is a nonsplit extension of $V_K|_{\Omega(f)}$ by $V_J|_{\Omega(f)}$.*

Proofs of Lemmas 7 and 8 for the cases $f = f_e$ and $f = f_h$ are given separately.

(2.4.i) The case $f = f_e$

For easy reference, we collect some results due to Alperin and due to Sin. Consider the following condition on $I, J \subseteq N$:

$$I \cup J = (I \cap J) \cup \{r\} \text{ and neither } r \text{ nor } r-1 \text{ is in } I \cap J \qquad (C_I, J)$$

Lemma 9 *(a) For $I, J \subseteq N$, $Ext^1_{kSL(2,q^2)}(W_I, W_J)$ as well as $Ext^1_{kG}(V_I, V_J)$ are both k or both zero according as whether the condition $(C_{I,J})$ holds or not.*

(b) For $i \in N, W_i \otimes W_i$ is a uniserial module with composition factors $k\backslash W_{i+1}\backslash k$.

Proof. For (a), see ([1], Theorem 3, p.221) and ([17], Theorem, p.159). For (b), see ([1], Lemma 4, p.224). ■

A typographical error about the assumptions on I and J in the statement of part (a) in ([1], Theorem 3, p.221) has been corrected here (see [1], p.229).

Proof of Lemma 7: In view of the isomorphism $Ext^1_{kG}(V_J, V_K) \simeq Ext^1_{kG}(V_{J\cup K}, V_{J\cap K})$, the biadditivity of the map $(M, N) \mapsto Ext^1(M, N)$ and Lemma 9(a) above, we only need to consider the case when $J = K \cup \{r\}$, $r \notin K$. In this case, the lemma follows from (1_e) and Lemma 9(a). ■

Proof of Lemma 8: From (1_e),

$$V_K|_{\Omega(f_e)} \backsimeq \left(\oplus_{L\in\mathcal{N}_{K_o}} W_L\right) \otimes W_{\overline{K_e}}$$

and

$$V_J|_{\Omega(f_e)} \backsimeq \left[\oplus_{L\in\mathcal{N}_{K_o}} (W_{L\cup\{t\}} \oplus W_{L\cup\{n+t\}})\right] \otimes W_{\overline{K_e}}.$$

We need to show that

$$Hom_{k\Omega(f_e)}(W_{L'} \otimes W_{\overline{K_e}}, E) = 0$$

for each $L' \in \mathcal{N}_{J_o}$. Let D_t denote the unique (up to isomorphism) $kG-$ module which is a nonsplit extension of k by V_{2t-1} (see Lemma 9(a)). Then $E \backsimeq D_t \otimes V_K$ ([14], Lemma 8, p.490). Since D_t is isomorphic to a submodule of the $kG-$module $V_{2t-2} \otimes V_{2t-2}$ (see [17], Lemma 2(a), p.161), $D_t \otimes V_K$ embeds in $(V_{2t-2} \otimes V_{2t-2}) \otimes V_K$. Hence it is enough to prove that

$$Hom_{k\Omega(f_e)}(W_{L'} \otimes W_{\overline{K_e}}, V_{2t-2} \otimes V_{2t-2} \otimes V_K) = 0. \qquad (*)$$

Let $L'' \in \mathcal{N}_{K_o}$ and $L' = L \cup \{r\}$ where $L \in \mathcal{N}_{K_o}$, $r = t$ or $n + t$. Since simple $k\Omega(f_e)-$modules are self dual, by Lemma 3,

$$\begin{aligned} &Hom_{k\Omega(f_e)}(W_{L'} \otimes W_{\overline{K_e}}, V_{2t-2} \otimes V_{2t-2} \otimes W_{L''} \otimes W_{\overline{K_e}}) \\ &\simeq Hom_{k\Omega(f_e)}(W_{L'} \otimes W_{\overline{K_e\cup\{t-1\}}}, W_{L''} \otimes W_{\overline{K_e\cup\{t-1\}}}) = 0. \end{aligned}$$

So (*) holds and the lemma follows.■

(2.4.ii) The case $f = f_h$

First we recall a useful result due to A. Jones ([10], Theorem 3, p.629).

Lemma 10 *Let R be a Dedekind domain, G_1 and G_2 be finite groups, $G = G_1 \times G_2$, $\Gamma_i = RG_i$, $i = 1, 2$, and $\Gamma = RG$. Let M_i and M_i' be (left) Γ_i-modules. Then, as $R-$ modules,*

$$\begin{aligned} Ext^1_\Gamma(M_1\#M_2, M_1'\#M_2') \simeq\ & Hom_{\Gamma_1}(M_1, M_1') \otimes_R Ext^1_{\Gamma_2}(M_2, M_2') \\ &\oplus Ext^1_{\Gamma_1}(M_1, M_1') \otimes_R Hom_{\Gamma_2}(M_2, M_2') \end{aligned}$$

and

$$Hom_\Gamma(M_1\#M_2, M_1'\#M_2') \simeq Hom_{\Gamma_1}(M_1, M_1') \otimes_R Hom_{\Gamma_2}(M_2, M_2').$$

Proof of Lemma 7: As in the case $f = f_e$, we need only to consider the case when $J = K \cup \{r\}$, where $r \notin K$. If $r = 2t$ for some $t \in N_1$, then (1_h) and Lemma 10 imply $Ext^1_{k\Omega(f_h)}(V_J, V_K) = 0.$■

Proof of Lemma 8:

From (1_h),

$$V_K|_{\Omega(f_h)} \simeq \oplus_{(A,B)\in\mathcal{M}_{K_0}} \left(M^1_{K_e\cup A} \# M^2_{K_e\cup B}\right)$$

and

$$V_J|_{\Omega(f_h)} \simeq \oplus_{(A,B)\in\mathcal{M}_{K_0}} \left[(M^1_{K_e\cup A\cup\{t\}} \# M^2_{K_e\cup B}) \oplus (M^1_{K_e\cup A} \# M^2_{K_e\cup B\cup\{t\}})\right].$$

We need to show that

$$Hom_{k\Omega(f_h)}((M^1_{K_e\cup A\cup\{t\}} \# M^2_{K_e\cup B}) \oplus (M^1_{K_e\cup A} \# M^2_{K_e\cup B\cup\{t\}}), E) = 0$$

for each $(A, B) \in \mathcal{M}_{K_o}$. In view of the discussion regarding D_t in the $f = f_e$ case, we prove that

$$Hom_{k\Omega(f_h)}((M^1_{K_e\cup A\cup\{t\}} \# M^2_{K_e\cup B}) \oplus (M^1_{K_e\cup A} \# M^2_{K_e\cup B\cup\{t\}}), V_{2t-2} \otimes V_{2t-2} \otimes V_K)$$

is zero for each $(A, B) \in \mathcal{M}_{K_o}$. Now if $(A, B), (A', B') \in \mathcal{M}_{K_o}$, then

$$\begin{aligned}
&Hom_{k\Omega(f_h)}((M^1_{K_e\cup A\cup\{t\}} \# M^2_{K_e\cup B}) \oplus (M^1_{K_e\cup A} \# M^2_{K_e\cup B\cup\{t\}}),\\
&V_{2t-2} \otimes V_{2t-2} \otimes (M^1_{K_e\cup A'} \# M^2_{K_e\cup B'}))\\
&\simeq Hom_{k\Omega(f_h)}((M^1_{K_e\cup A\cup\{t\}} \# M^2_{K_e\cup B}) \otimes V_{2t-2} \oplus (M^1_{K_e\cup A} \# M^2_{K_e\cup B\cup\{t\}}) \otimes V_{2t-2},\\
&V_{2t-2} \otimes (M^1_{K_e\cup A'} \# M^2_{K_e\cup B'}))\\
&\simeq Hom_{k\Omega(f_h)}((M^1_{K_e\cup A\cup\{t\}} \# M^2_{K_e\cup B}) \otimes \left(M^1_{t-1} \# M^2_{t-1}\right) \oplus (M^1_{K_e\cup A} \# M^2_{K_e\cup B\cup\{t\}}) \otimes\\
&\left(M^1_{t-1} \# M^2_{t-1}\right), \left(M^1_{t-1} \# M^2_{t-1}\right) \otimes (M^1_{K_e\cup A'} \# M^2_{K_e\cup B'}))\\
&\simeq Hom_{k\Omega(f_h)}((M^1_{K_e\cup A\cup\{t-1,t\}} \# M^2_{K_e\cup B\cup\{t-1\}}) \oplus (M^1_{K_e\cup A\cup\{t-1\}} \# M^2_{K_e\cup B\cup\{t-1,t\}}),\\
&M^1_{K_e\cup A'\cup\{t-1\}} \# M^2_{K_e\cup B'\cup\{t-1\}}) = 0
\end{aligned}$$

by Lemma 10. Here we have used : (i) the duality between the functors 'Hom' and 'Tensor', namely for any group X and $kX-$modules U_i, we have $Hom_{kX}(U_1, U_2 \otimes U_3) \simeq Hom_{kX}(U_1 \otimes U_2^*, U_3)$, where U^* is the dual of U; and (ii) the fact that each simple module for $G = Sp(4, q)$ is self-dual . So Lemma 8 holds for $f = f_h$ also. ■

3 Proof of Theorem 1

(a) Let $n = 1$. Then $dim_k(\overline{\mathcal{C}}) = 9$ (see, for example, [13], p. 308). By ([14], Lemma 4, p.488) and Frobenius reciprocity ([11], p.689) it follows that V_1, k and V_0 (in descending order) are kG-composition factors of $\overline{\mathcal{C}}$. Semisimplicity of each composition factor of $\overline{\mathcal{C}}$ and its multiplicity freeness as a $k\Omega(f)-$ module are proved in Corollaries 4 and 6. For the remaining part of the proof, we treat the cases separately. Consider the case $f = f_e$. Since $V_0|_{\Omega(f)} \simeq W_0 \otimes W_1$ (Lemma 3) is the Steinberg module (and hence projective) for $\Omega(f)$, we need to prove that the kG-module E which is the unique (up to isomorphism) nonsplit extension of k by V_1 remains nonsplit as a $k\Omega(f)$ -module. Further, since E is isomorphic to a submodule of the kG-module $V_0 \otimes V_0$ (see [17], Lemma 2(a), p.161), $V_1|_{\Omega(f)} \simeq W_0 \oplus W_1$ (Lemma 3) and $W_0 \otimes W_1$ is the Steinberg module for $k\Omega(f)$, by ([1], Theorem 1, p. 220), we have

$$Hom_{k\Omega(f)}(V_1, V_0 \otimes V_0) = 0.$$

Now we consider the case $f = f_h$. Then M^1, M^2 and $V|_{\Omega(f)} \simeq M^1 \# M^2$ (see Lemma 5) are the Steinberg modules for H_1, H_2 and $\Omega(f)$, respectively. Hence we need to prove that E as a $k\Omega(f)$- module is split. But this is clear from Lemmas 5 and10.

(b) Let $n \geq 2$. By Theorem 2, $\{V_I\}_{I \in \mathcal{N}}$ are the $kG-$ composition factors of $\overline{\mathcal{C}}$ and they appear with multiplicity one. So, (i) follows from Corollaries 4 and 6.

We now prove (ii). Let V_K and V_J, $K, J \in \mathcal{N}$, be in the i-th and j-th kG- socle layers of $\overline{\mathcal{C}}, 1 \leq i < j \leq 2n$. Assume that there is a kG- module E which is a nonsplit extension of V_K by V_J. Then, by Lemma 9(a), $j = i+1$. Further, E is a nonsplit $k\Omega\,(f)$-extension of V_K by V_J if and only if $J = K \cup \{2t-1\}$ for some $t \in N_1$ (Lemma 7). Further, if this holds, $Soc_{k\Omega(f)}\,(E) = V_K$ (Lemma 8). That is, either all or none, of the simple $k\Omega\,(f)$- summands of V_J descend. This observation together with the kG- socle structure of $\overline{\mathcal{C}}$ yields (ii).

Now, let $m \in \{0, 1, \cdots, n-2, n\}$ and $I \in \mathcal{A}_m$. In what follows, all modules considered are over $k\Omega(f)$. Since $\overline{\mathcal{C}}$ is multiplicity free as a $k\Omega(f)$-module, it has a unique submodule $X_{m,I}$ with head V_K, where K is the unique element of $\mathcal{N}$ with $K_e = I$ and $K_o = I^{ac}$. First $X_{0,I} = V_{\{2t: t \in I\}}$ (see (1_e) and (1_h)) and is semisimple. Let $m > 0$ and write $X_{m,I}$ as L for brevity. Since $Head\,(L)$ is contained in the m-th socle layer of $\overline{\mathcal{C}}$, L is a submodule of $Soc^m\left(\overline{\mathcal{C}}\right)$. For each summand of $rad^1(L)/rad^2(L)$ of the form V_J, a nonsplit extension of V_J by $Head\,(L)$ appears as a section of L. Further, $Head\,(L) = V_K$. So, $J = K \backslash \{2t-1\}$ for some $t \in I^{ac}$ (Lemma 8). This proves that $rad^1(L)$ is contained in the unique submodule L' of $Soc^{m-1}\left(\overline{\mathcal{C}}\right)$ whose head is isomorphic to

$$\bigoplus_{J \in \mathcal{N},\ J_e = I,\ J_o \subseteq I^{ac}\ \text{and}\ |J_o| = m-1} V_J. \tag{3.1}$$

The multiplicity freeness of $\overline{\mathcal{C}}$ (as a $k\Omega(f)$-module) implies that $L \cap L' = rad^1(L)$. We now show that $L' = rad^1(L)$. Since $(L + L')/L' \backsimeq L/(L \cap L') = L/rad^1(L) \backsimeq V_K$, by uniqueness of L, $L + L' = L$. Thus $L' \subseteq L$ and $L' = L \cap L' = rad^1(L)$. Thus, $rad^1(L)/rad^2(L)$ is the module in (3.1). Now successive application of this argument to $rad^i(L)/rad^{i-1}(L)$ yields the statement about the radical structure of $X_{m,I}$ in (iii). Further, (iv) is also clear.

Since $socle(X_{m,I}) = V_{\{2t:t\in I\}}$ is simple, $X_{m,I}$ is indecomposable. For $I \neq J \in \mathcal{A}_m$, $X_{m,I} \cap X_{m,J}$ is trivial because their socles are distinct and simple. (Infact $X_{m,I}$ and $X_{m,J}$ have no composition factors in common.) Let

$$X_m = \bigoplus_{I\in\mathcal{A}_m} X_{m,I}.$$

For distinct $m, m' \in \{0, 1, \cdots, n-2, n\}$, X_m and $X_{m'}$ have no composition factors in common and every composition of $\overline{\mathcal{C}}$ occurs in some X_m. Thus sum of X_m's equals $\overline{\mathcal{C}}$, completing the proof of all parts of (iii) and of Theorem 1.■

Remark 11 *(i) If $n = 1$, as a $k\Omega^{\tau}(f)$-module, V_0 and V_1 are semi-simple and $\overline{\mathcal{C}}$ is multiplicity free. Further, if f is of Witt index 1, then $\overline{\mathcal{C}}$ is isomorphic to the direct sum of V_0 and a $k\Omega^{\tau}(f)$-module U with $head(U) \simeq k$ and $rad(U) \simeq V_1$. If f is of Witt index 2, then $\overline{\mathcal{C}}$ is semi-simple .*

(ii) If $n \geq 2$, as a $k\Omega^{\tau}(f)$-module, then V_I is semi-simple for each $I \in \mathcal{N}$ and $\overline{\mathcal{C}}$ is multiplicity free and is isomorphic to

$$\bigoplus_{m\in\{0,1,\cdots,n-2,n\}} \Big(\bigoplus_{I\in\mathcal{A}_m} U_{m,I}\Big)$$

where $U_{m,I}$ is a unique indecomposable module of radical length $m+1$. Its j-th radical layer $rad^j(U_{m,I})/rad^{j+1}(U_{m,I})$ $(0 \leq j \leq m)$ is isomorphic to

$$\bigoplus_{J\in\mathcal{N},\ J_e=I,\ J_o\subseteq I^{ac}\ \ and\ |J_o|=j} V_I.$$

Moreover, for each $m \in \{0, 1, \cdots, n-2, n\}$,

$$soc^j(U_{m,I}) = rad^{m+1-j}(U_{m,I}).$$

(i) and (ii) follow from an argument similar to the proof of Theorem 1, using Theorem 2 and Lemmas 7 and 8.

Examples: We illustrate the descent of the composition factors V_J in Theorem 2 when considered as a $k\Omega(f)$-module for the cases $n = 1, 2, 3, 4$. For $J = \{i_1, i_2, \ldots\} \subseteq N$, we write V_J as $\underline{J}$ or $\underline{i_1, i_2, \ldots}$.

If $n=1$, then
$$\overline{\mathcal{C}} = \underline{0} + (\underline{1}/k).$$
If $n=2$, then
$$\overline{\mathcal{C}} = X_0 \oplus X_2 \equiv [\ \underline{0} + \underline{2} + \underline{0,2}\ ] \oplus [\underline{1,3}/(\underline{1} \oplus \underline{3})/k].$$
If $n=3$, then
$$\overline{\mathcal{C}} = X_0 \oplus X_1 \oplus X_3 \equiv [\underline{0,2} + \underline{0,4} + \underline{2,4}] \oplus [(\underline{0,3}\ /\underline{0}) \\ \oplus (\underline{2,5}/\underline{2}) \oplus (\underline{1,4}/\underline{4})] \oplus [\underline{1,3,5}/(\underline{1,3} + \underline{1,5} + \underline{3,5})/(\underline{1} + \underline{3} + \underline{5})/k].$$
If $n=4$, then
$$\overline{\mathcal{C}} = X_0 \oplus X_1 \oplus X_2 \oplus X_4,$$
Where
$$X_0 = \bigoplus_{J \subseteq N_1, |J|=3 \text{ or } 4} \underline{2J} \oplus (\underline{0,4} \oplus \underline{2,6});$$
$$X_1 = \bigoplus_{i=0}^{3} (\underline{2i, 2i+2, 2i+5})/(\underline{2i, 2i+2});$$
$$X_2 = \begin{array}{l} (\underline{0,3,5}\ /(\underline{0,3} \oplus \underline{0,5})/\underline{0}) \oplus (\underline{2,5,7}/(\underline{2,5} \oplus \underline{2,7}\ )/\underline{2}) \oplus \\ (\underline{1,4,7}\ /(\underline{1,4} \oplus \underline{4,7})\ /\underline{4}) \oplus (\underline{6,1,3}\ /(\underline{6,1} \oplus \underline{6,3}\ )/\underline{6}) \end{array}$$
$$X_4 = A_4/A_3/A_2/A_1/A_0,$$
where A_i is the direct sum of J as J runs over the subsets of $\{1,3,5,7\}$ of size i.

Acknowledgement

We thank Professor Peter Sin for kindly pointing out an error in Theorem 1 in an earlier version of this paper. The second named author also thanks Harish-Chandra Research Institute, Allahabad (India) for kindly permitting him the use of its library and computing facilities during the period of this work.

References

[1] J.L. Alperin, Projective modules for $SL(2, 2^n)$, J. Pure and Applied algebra 15 (1979) 219-234

[2] P. Dembowski, *Finite geometries*, Classics in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 1997 (reprint of 1968 edition)

[3] R.H. Dye, Interrelations of symplectic and orthogonal groups in characteristic two, J. of Algebra 59 (1979) 202-221

[4] R.H. Dye, On the Arf Invariant, J. of Algebra 53 (1978) 36-39

[5] D.E. Flesner, The geometry of subgroups of $PSp_4(2^n)$, Illinois J. Math. 19 (1975) 48-70

[6] D.E. Flesner, Maximal subgroups of $PSp_4(2^n)$, containing central elations or non-centered skew elations, Illinois J. Math. 19 (1975) 247-268

[7] J.W.P. Hirschfeld, *Finite projective spaces of three dimensions*, Oxford Univ. Press Oxford, 1985

[8] J.W.P. Hirschfeld, *Projective geometries over finite fields*, Oxford Univ. Press, Oxford, 1998

[9] B. Huppert and N. Blackburn, *Finite groups,* Vol.2, Springer-Verlag, New York, 1982

[10] A. Jones, Integral representations of direct product of groups, Canad. J. Math. 15(1963) 625-630

[11] S. Lang, *Algebra*, Third Edition, Addison-Wessley, 1999.

[12] S.E. Payne and J.A. Thas, *Finite generalized quadrangles,* Advance Publishing Program, Pitman, Boston 1984

[13] N.S.N. Sastry, Codes and generalized polygons, in *'Proc. of Seminar on combinatorics and applications'* in honour of Prof. S.S. Shrikhande, Indian Statistical Institute, Calcutta (1982) 303-310

[14] N.S.N. Sastry and P. Sin, The code of regular generalized quadrangle of even order, Proc. Symp. Pure Math. 63 (1998) 485-496

[15] N.S.N. Sastry and P. Sin, The binary code associated with nondegenerate quadrics of a symplectic space of even order, J. Combin. Theory (A) 94 (2001) 1-14

[16] N.S.N. Sastry and P. Sin, On the doubly transitive permutation representations of $Sp(2n, \mathbb{F}_2)$, J. of Algebra 257 (2002) 509-527

[17] P. Sin, Extension of simple modules for $Sp_4(2^n)$ and $Sz(2^n)$, Bull. Lond. Math. Soc. 24 (1992) 159-164

[18] R. Steinberg, Representations of Algebraic groups, Nagoya J. Math. 22 (1963) 33-56

[19] R. Steinberg, *Lectures on Chevalley Groups*, Mimeographed Notes, Yale Univ. Math. dept., New Haven, Conn., 1968

[20] D.E. Taylor, *The geometry of the classical groups*, Sigma Series in Pure Mathematics, V.9, Helderman- Verlag, Berlin, 1992

N.S. Narasimha Sastry; Stat.- Math. Unit; Indian Statistical Institute;
8th Mile Mysore Road; R.V. College Post; Bangalore - 560 059
India
Email: nsastry@isibang.ac.in
R.P. Shukla; Department of Mathematics; University of Allahabad;
Allahabad - 211 002 India
Email: rps@mri.ernet.in