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Completely contractive maps

Tirthankar Bhattacharyya and Gadadhar Misra

Indian Statistical Institute, Bangalore Centre
8th Mile Mysore Road, Bangalore, 560059 India

CONTRACTIVE AND COMPLETELY CONTRACTIVE HOMOMORPHISMS OF PLANAR ALGEBRAS

TIRTHANKAR BHATTACHARYYA AND GADADHAR MISRA

Abstract. We consider contractive homomorphisms of a planar algebra $\mathcal{A}(\Omega)$ over a finitely connected bounded domain $\Omega \subseteq \mathbb{C}$ and ask if they are necessarily completely contractive. We show that a homomorphism $\rho : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{H})$ for which $\dim(\mathcal{A}(\Omega)/\ker\rho) = 2$ is the direct integral of homomorphisms ρ_T induced by operators on two dimensional Hilbert spaces via a suitable functional calculus $\rho_T : f \mapsto f(T),\ f \in \mathcal{A}(\Omega)$. It is well-known that contractive homomorphisms ρ_T, induced by a linear transformation $T : \mathbb{C}^2 \to \mathbb{C}^2$ are necessarily completely contractive. Consequently, using Arveson's dilation theorem for completely contractive homomorphisms, one concludes that such a homomorphism ρ_T possesses a dilation. In this paper, we construct this dilation explicitly. In view of recent examples discovered by Dritschel and McCullough, we know that not all contractive homomorphisms ρ_T are completely contractive even if T is a linear transformation on a finite-dimensional Hilbert space. We show that one may be able to produce an example of a contractive homomorphism ρ_T of $\mathcal{A}(\Omega)$ which is not completely contractive if an operator space which is naturally associated with the problem is not the MAX space. Finally, within a certain special class of contractive homomorphisms ρ_T of the planar algebra $\mathcal{A}(\Omega)$, we construct a dilation.

1. Introduction

All our Hilbert spaces are over complex numbers and are assumed to be separable. Let $T \in \mathcal{B}(\mathcal{H})$, the algebra of bounded operators on $\mathcal{H}$. The operator T induces a homomorphism $\rho_T : p \mapsto p(T)$, where p is a polynomial. Equip the polynomial ring with the supremum norm on the unit disc, that is, $\|p\| = \sup\{|p(z)| : z \in \mathbb{D}\}$. A well-known inequality due to von Neumann (cf. [18]) asserts that ρ_T is contractive, that is, $\|\rho_T\| \leq 1$ if and only if the operator T is a contraction. Thus in this case, contractivity of the homomorphism ρ_T is equivalent to the operator T being a contraction. As is well known, Sz.-Nagy [24] showed that a contraction T on a Hilbert space $\mathcal{H}$ dilates to a unitary operator U on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, that is, $Pp(U)h = p(T)h$ for all $h \in \mathcal{H}$ and any polynomial p, where $P : \mathcal{K} \to \mathcal{H}$ is the projection of $\mathcal{K}$ onto $\mathcal{H}$. The unitary operator U has a continuous functional calculus and hence induces a $*$ - homomorphism $\varphi_U : C(\sigma(U)) \to \mathcal{B}(\mathcal{K})$. It is easy to check that $P[(\varphi_U)_{|\mathcal{A}(\mathbb{D})}(f)]_{|\mathcal{H}} = \rho_T(f)$, for f in $\mathcal{A}(\mathbb{D})$, where $\mathcal{A}(\mathbb{D})$ is the closure of the polynomials with respect to the supremum norm on the disc $\mathbb{D}$.

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Let Ω be a finitely connected bounded domain in $\mathbb{C}$. We make the standing assumption that the boundary of Ω is the disjoint union of simple analytic closed curves. Let T be a bounded linear operator on the Hilbert space $\mathcal{H}$ with spectrum $\sigma(T) \subseteq \Omega$. Given a rational function $r = p/q$ with no poles in the spectrum $\sigma(T)$, there is the natural functional calculus $r(T) = p(T)q(T)^{-1}$. Thus T induces a unital homomorphism $\rho_T = r(T)$ on the algebra of rational functions $\mathrm{Rat}(\Omega)$ with poles off Ω. Let $\mathcal{A}(\Omega)$ be the closure of $\mathrm{Rat}(\Omega)$ with respect to the norm $\|r\| := \sup\{|r(z)\| : z \in \Omega\}$. Since functions holomorphic in a neighborhood of $\bar{\Omega}$ can be approximated by rational functions with poles off $\bar{\Omega}$, it follows that they belong to $\mathcal{A}(\Omega)$.

The homomorphism ρ_T is said to be *dilatable* if there exists a normal operator N on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ with $\sigma(N) \subseteq \partial\bar{\Omega}$ such that the induced homomorphism $\varphi_N : C(\sigma(N)) \to \mathcal{B}(\mathcal{K})$, via the functional calculus for the normal operator N, satisfies the relation

$$P(\varphi_N)_{|\mathcal{A}(\Omega)}(f)h = \rho_T(f)h, \tag{1.1}$$

for h in $\mathcal{H}$ and f in $\mathcal{A}(\Omega)$. Here $P : \mathcal{K} \to \mathcal{H}$ is the projection of $\mathcal{K}$ onto $\mathcal{H}$.

The observations about the disk prompt two basic questions:

(i) when is ρ_T contractive;

(ii) do contractive homomorphisms ρ_T necessarily dilate?

For the disc algebra, the answer to the first question is given by von Neumann's inequality while the answer to the second question is affirmative – Sz.-Nagy's dilation theorem. If the domain Ω is simply connected these questions can be reduced to that of the disc (cf. [23]).

If the domain Ω is the annulus, while no satisfactory answer to the first question is known, the answer to the second question was shown to be affirmative by Agler (cf. [4]).

If $\rho_T : \mathcal{A}(\Omega) \to \mathcal{M}_2$ is a homomorphism induced by an operator $T : \mathbb{C}^2 \to \mathbb{C}^2$ then it is possible to obtain a characterization of contractivity and then use it to show that the second question has an affirmative answer. We do this in Section 3.2. In Section 2, we show that a larger class of contractive homomorphisms, we call them contractive homomorphisms of *rank 2*, dilate. This is done by proving that the rank 2 homomorphisms are direct integrals of homomorphisms induced by two dimensional operators.

Arveson (cf. [5] and [6]) has shown that the existence of a dilation of a contractive homomorphism ρ of the algebra $\mathcal{A}(\Omega)$ is equivalent to complete contractivity of the homomorphism ρ. We recall some of these notions in greater detail in section 4. We then show, how one may proceed to possibly construct an example of a contractive homomorphism of the algebra $\mathcal{A}(\Omega)$ which does not dilate.

In the final section of the paper, we obtain a general criterion for contractivity. This involves a factorization of a certain positive definite kernel. More importantly, we outline a scheme for constructing the dilation of a homomorphism $\rho_T : \mathcal{A}(\Omega) \to \mathcal{M}_n$ induced by an operator T with distinct eigenvalues. This scheme is a generalization of the construction of the dilation in section 3.2.

2. Homomorphisms of Rank Two

A homomorphism $\rho : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{H})$ is said to be of *rank n* if it has the property $\dim\big(\mathcal{A}(\Omega)/\ker\rho\big) = n$. In this section, we shall begin construction of dilation for homomorphisms of rank 2.

Nakazi and Takahashi showed that contractive homomorphisms $\rho : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{H})$ of rank 2 are completely contractive for any uniform sub-algebra of the algebra of continuous functions $C(\bar{\Omega})$ (see [17]). We would like to mention here that a generalization of this result was obtained by Meyer in Theorem 4.1 of [12]. He showed that given a commutative unital closed subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{K})$ (for some Hilbert space $\mathcal{K}$) and a positive integer d, any $d-1$ contractive unital homomorphism $\rho : \mathcal{A} \to \mathcal{M}_d$ is completely contractive. In what follows, we construct explicit dilations for homomorphisms from $\mathcal{A}(\Omega)$ to $\mathcal{B}(\mathcal{H})$ of rank two.

We first show that any homomorphism ρ of rank 2 is the direct integral of homomorphisms of the form ρ_T as defined in the introduction, where $T \in \mathcal{M}_2$. The existence of dilation of a contractive homomorphism ρ_T induced by a two dimensional operator T is established in [13] by showing that the homomorphism ρ_T must be completely contractive. It then follows that every contractive homomorphism ρ of rank 2 must be completely contractive. This implies by Arveson's theorem that they possess a dilation. However, it is not always easy to construct the dilation whose existence is guaranteed by the theorem of Arveson. In this case, we shall explicitly construct the dilation of a homomorphism of rank 2. This is achieved by constructing the dilation of a contractive homomorphism of the form ρ_T for a two dimensional operator T.

Lemma 1. *If $\rho_T : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{L})$ is a homomorphism of rank two, then up to unitary equivalence, the Hilbert space $\mathcal{L}$ is a direct integral*

$$\mathcal{L} = \int_\Lambda^\oplus \mathcal{L}_\lambda d\nu(\lambda)\,,$$

where each $\mathcal{L}_\lambda$ is two-dimensional. In this decomposition, the operator T is of the form

$$T = \int_\Lambda^\oplus \begin{pmatrix} z_1(\lambda) & c(\lambda) \\ 0 & z_2(\lambda) \end{pmatrix} d\nu(\lambda).$$

Proof. To begin with, it is easy to see (see Lemma 1 of [17]) that $\mathcal{L}$ is a direct sum of two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and the operator $T : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K}$ is of the form:

$$\begin{pmatrix} z_1 I_\mathcal{H} & C \\ 0 & z_2 I_\mathcal{K} \end{pmatrix}, \text{ with } z_1,\ z_2 \in \Omega \text{ or } \begin{pmatrix} z I_\mathcal{H} & C \\ 0 & z I_\mathcal{K} \end{pmatrix}, \text{ with } z \in \Omega,$$

where C is a bounded operator from $\mathcal{K}$ to $\mathcal{H}$. Now if we put $\mathcal{K}_0 = (\ker C)^\perp$, $\mathcal{K}_1 = \ker C$, $\mathcal{H}_0 = \overline{\text{Ran}}\, C$ and $\mathcal{H}_1 = (\text{Ran}\, C)^\perp$, then with respect to the decomposition $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$ and $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, we have

$$C = \begin{pmatrix} \tilde{C} & 0 \\ 0 & 0 \end{pmatrix},$$

where the operator $\tilde{C}$ is from $\mathcal{K}_0$ to $\mathcal{H}_0$. The polar decomposition of $\tilde{C}$ then yields $\tilde{C} = VP$, where the operator V is unitary and P is positive. We apply the spectral theorem to the positive operator P and conclude that there exists a unitary operator $\Gamma : \int_\Lambda^\oplus \mathcal{H}_\lambda d\nu(\lambda) \to \mathcal{K}_0$ which intertwines the multiplication operator M on the Hilbert space $\int_\Lambda^\oplus \mathcal{H}_\lambda d\nu(\lambda)$ and P.

Now notice that the operator $T : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K}$ can be rewritten as

$$\begin{pmatrix} z_1 I_{\mathcal{H}_1} & 0 & \tilde{C}_{\mathcal{K}_0 \to \mathcal{H}_0} & 0 \\ 0 & z_1 I_{\mathcal{H}_0} & 0 & 0 \\ 0 & 0 & z_2 I_{\mathcal{K}_0} & 0 \\ 0 & 0 & 0 & z_2 I_{\mathcal{K}_1} \end{pmatrix}.$$

Interchanging the third and the second column and then the second and third row, which can be effected by a unitary operator, we see that the operator T is unitarily equivalent to the direct sum of a diagonal operator D and an operator $\tilde{T}$ of the form $\begin{pmatrix} z_1 I_{\mathcal{H}_0} & \tilde{C}_{\mathcal{K}_0 \to \mathcal{H}_0} \\ 0 & z_2 I_{\mathcal{K}_0} \end{pmatrix}$, where $\tilde{C}$ has dense range. It is clear that if we conjugate the operator $\tilde{T}$ by the operator $I_{\mathcal{H}_0} \oplus U_{\mathcal{H}_0 \to \mathcal{K}_0}$, where U is any unitary operator identifying $\mathcal{H}_0$ and $\mathcal{K}_0$ then we obtain a unitarily equivalent copy of $\tilde{T}$ (again, denoted by $\tilde{T}$) which is of the form $\begin{pmatrix} z_1 I_{\mathcal{H}_0} & \tilde{C}_{\mathcal{K}_0 \to \mathcal{H}_0} U_{\mathcal{H}_0 \to \mathcal{K}_0} \\ 0 & z_2 I_{\mathcal{K}} \end{pmatrix}$. Now, if we apply the polar decomposition to $\tilde{C}$ then we see that the off diagonal entry is a positive operator on $\mathcal{H}_0$. One then sees that $\tilde{T}$ is unitarily equivalent to $\begin{pmatrix} z_1 I_{\int_\Lambda^\oplus \mathcal{H}_\lambda d\nu(\lambda)} & M \\ 0 & z_2 I_{\int_\Lambda^\oplus \mathcal{H}_\lambda d\nu(\lambda)} \end{pmatrix}$ via conjugation using the operator $\Gamma \oplus \Gamma$. We need to conjugate this operator one more time using the unitary W that identifies $\int_\Lambda^\oplus \mathcal{H}_\lambda d\nu(\lambda) \oplus \int_\Lambda^\oplus \mathcal{H}_\lambda d\nu(\lambda)$ and $\int_\Lambda^\oplus \mathcal{H}_\lambda \oplus \mathcal{H}_\lambda d\nu(\lambda)$, where $W(s_1 \oplus s_2)(\lambda) = s_1(\lambda) \oplus s_2(\lambda)$ for $s_1 \oplus_2 \in \int_\Lambda^\oplus \mathcal{H}_\lambda d\nu(\lambda)$. It is easy to calculate $W\tilde{T}W^*$ and verify the claim. ■

In view of the Lemma above, it is now enough to consider dilations of homomorphisms ρ_T where T is a linear transformation on $\mathbb{C}^2$.

3. Dilations and Abrahamse-Nevanlinna-Pick Interpolation

3.1. Consider any reproducing kernel Hilbert space $\mathcal{H}_K$ of holomorphic functions on Ω with $K : \Omega \times \Omega \to \mathbb{C}$ as the kernel. Assume that the multiplication operator M by the independent variable z is bounded. Then $M^*(K(\cdot, z)) = \bar{z} K(\cdot, z)$ and it is clear by differentiation that $M^* \bar{\partial}_z K(\cdot, z) = K(\cdot, z) + \bar{z} \bar{\partial}_z K(\cdot, z)$.

The matrix representation of the operator M^* restricted to the subspace $\mathcal{M}$ spanned by the two vectors $K(\cdot, z_1)$ and $K(\cdot, z_2)$ has two distinct eigenvalues $\bar{z}_1$ and $\bar{z}_2$. Similarly, the operator M^* restricted to the subspace $\mathcal{N}$ spanned by the two vectors $K(\cdot, z)$ and $\bar{\partial}_z K(\cdot, z)$ has only one eigenvalue $\bar{z}$ of multiplicity 2. In the lemma below, we identify certain 2 dimensional subspaces of $\mathcal{H}_K \oplus \mathcal{H}_K$ which are invariant under the multiplication operator M^* and then find out the form of the matrix. The reproducing kernel K satisfies:

$$K(z_1, z_2) = \langle K(\cdot, z_2), K(\cdot, z_1) \rangle, \; z_1, z_2 \in \Omega, \tag{3.1a}$$

$$(\partial_z K)(z, u) = \langle K(\cdot, u), \bar{\partial}_z K(\cdot, z) \rangle, \; u, \; z \in \Omega. \tag{3.1b}$$

Using (3.1) and applying the Gram-Schmidt orthogonolization process to the set $\{K(\cdot, z_1), K(\cdot, z_2)\}$, we get the orthonormal pair of vectors

$$e(z_1) = \frac{K(\cdot, z_1)}{K(z_1, z_1)^{1/2}} \text{ and } f(z_1, z_2) = \frac{K(z_1, z_1) K(\cdot, z_2) - K(z_2, z_2) K(\cdot, z_1)}{K(z_1, z_1)^{1/2} \big(K(z_1, z_1) K(z_2, z_2) - |K(z_1, z_2)|^2 \big)^{1/2}}.$$

Now for any $\mu \in \bar{\mathbb{D}}$, the pair of vectors

$$h_1(z_1,z_2) = \begin{pmatrix} 0 \\ e(z_1) \end{pmatrix} \text{ and } h_2(z_1,z_2) = \begin{pmatrix} (1-|\mu|^2)^{1/2} e(z_2) \\ \mu f(z_1,z_2) \end{pmatrix}$$

are orthonormal in $\mathcal{H}_K \oplus \mathcal{H}_K$. Similarly, using (3.1b), orthonormalization of the pair of vectors $\{K(\cdot,z), \bar\partial_z K(\cdot,z)\}$ produces the orthonormal set $\{e(z), f(z)\}$, where

$$e(z) = \frac{K(\cdot,z)}{K(z,z)^{1/2}} \text{ and } f(z) = \frac{K(z,z)\bar\partial_z K(\cdot,z) - \langle \bar\partial_z K(\cdot,z), K(\cdot,z)\rangle K(\cdot,z)}{K(z,z)^{1/2}\big(K(z,z)\,\|\bar\partial_z K(\cdot,z)\|^2 - |\langle \bar\partial_z K(\cdot,z), K(\cdot,z)\rangle|^2\big)^{1/2}}.$$

and then for any $\lambda \in \bar{\mathbb{D}}$,

$$k_1(z) = \begin{pmatrix} 0 \\ e(z) \end{pmatrix} \text{ and } k_2(z) = \begin{pmatrix} (1-|\lambda|^2)^{1/2} e(z) \\ \lambda f(z) \end{pmatrix}$$

form a set of two orthonormal vectors in $\mathcal{H}_K \oplus \mathcal{H}_K$.

Note that from the definition of M^* it follows that $M^* e(z_1) = \bar z_1 e(z_1)$ for all $z_1 \in \Omega$. Therefore we have $(M^* \oplus M^*) h_1(z_1,z_2) = \bar z_1 h_1(z_1,z_2)$. Now,

$$\begin{aligned} M^* f(z_1,z_2) &= \frac{K(z_1,z_1)\bar z_2 K_\alpha(\cdot,z_2) - K(z_2,z_2)\bar z_1 K(\cdot,z_1)}{K(z_1,z_1)^{1/2}(K(z_1,z_1)K(z_2,z_2) - |K(z_1,z_2)|^2)^{1/2}} \\ &= \bar z_2 f(z_1,z_2) + \frac{(\bar z_2 - \bar z_1)K(z_1,z_2)}{(K(z_1,z_1)K(z_2,z_2) - |K(z_1,z_2)|^2)^{1/2}} e(z_1). \end{aligned}$$

It follows that $\mathcal{M}$ is invariant under $M^* \oplus M^*$. In particular, we have

$$\begin{aligned} (M^* \oplus M^*) h_2(z_1,z_2) &= \begin{pmatrix} (1-|\mu|^2)^{1/2} M^* e(z_2) \\ \mu M^* f(z_1,z_2) \end{pmatrix} \\ &= \begin{pmatrix} (1-|\mu|^2)^{1/2} \bar z_2 e(z_2) \\ \mu\big(\bar z_2 f(z_1,z_2) + \frac{(\bar z_2 - \bar z_1)K(z_1,z_2)}{(K(z_1,z_1)K(z_2,z_2) - |K(z_1,z_2)|^2)^{1/2}} e(z_1)\big) \end{pmatrix} \\ &= \bar z_2 \begin{pmatrix} (1-|\mu|^2)^{1/2} e(z_2) \\ \mu f(z_1,z_2) \end{pmatrix} + \begin{pmatrix} 0 \\ \mu \frac{(\bar z_2 - \bar z_1)K(z_1,z_2)}{(K(z_1,z_1)K(z_2,z_2) - |K(z_1,z_2)|^2)^{1/2}} e(z_1) \end{pmatrix} \\ &= \bar z_2 h_2(z_1,z_2) + \mu \frac{(\bar z_2 - \bar z_1)|K(z_1,z_2)|}{(K(z_1,z_1)K(z_2,z_2) - |K(z_1,z_2)|^2)^{1/2}} h_1(z_1,z_2), \end{aligned}$$

where we have absorbed the argument of $K(z_1,z_2)$ in μ.

Now recall that $(M^* - \bar z)K(\cdot,z) = 0$. Differentiating with respect to $\bar z$, we obtain, $M^* \bar\partial_z K(\cdot,z) = K(\cdot,z) + \bar z \bar\partial_z K(\cdot,z)$. Thus the subspace $\mathcal{N}$ spanned by the vectors $k_1(z), k_2(z)$ is invariant under M^*. A little more computation, similar to the one above, gives us the matrix representation of the restriction of the operator $M^* \oplus M^*$ to the subspace $\mathcal{N}$.

So, we have proved the following Lemma.

Lemma 2. *The two-dimensional space $\mathcal{M}$ spanned by the two vectors $h_1(z_1,z_2), h_2(z_1,z_2)$ is an invariant subspace for the operator $M^* \oplus M^*$ on $\mathcal{H}_K \oplus \mathcal{H}_K$ and the restriction of this operator* *to the subspace $\mathcal{M}$ has the matrix representation*

$$\begin{pmatrix} \bar{z}_1 & \frac{\mu(\bar{z}_2-\bar{z}_1)|K(z_1,z_2)|}{(K(z_1,z_1)K(z_2,z_2)-|K(z_1,z_2)|^2)^{1/2}} \\ 0 & \bar{z}_2 \end{pmatrix}.$$

Similarly, the two-dimensional space $\mathcal{N}$ spanned by the two vectors $k_1(z), k_2(z)$ is an invariant subspace for the operator $M^* \oplus M^*$ on $\mathcal{H} \oplus \mathcal{H}$ and the restriction of this operator to the subspace $\mathcal{N}$ has the matrix representation

$$\begin{pmatrix} \bar{z} & \frac{\lambda K(z,z)}{(K(z,z)\|\bar{\partial}_z K(\cdot,z)\|^2-|\langle\bar{\partial}_z K(\cdot,z),K(\cdot,z)\rangle|^2)^{1/2}} \\ 0 & \bar{z} \end{pmatrix}.$$

Let μ, λ be a pair of complex numbers and fix a pair of 2×2 matrices A_s and B_t –

$$A_s = \begin{pmatrix} z_1 & 0 \\ s\mu(z_1 - z_2) & z_2 \end{pmatrix},\ z_1,\ z_2 \in \Omega \text{ and } B_t = \begin{pmatrix} z & 0 \\ t\lambda & z \end{pmatrix},\ z \in \Omega, \tag{3.2}$$

where s, t are a pair of positive real numbers. If we choose

$$s := s_K = \frac{|K(z_1,z_2)|}{(K(z_1,z_1)K(z_2,z_2) - |K(z_1,z_2)|^2)^{1/2}}, \text{ and} \tag{3.3a}$$

$$t := t_K = \frac{K(z,z)}{(K(z,z)\|\bar{\partial}_z K(\cdot,z)\|^2 - |\langle\bar{\partial}_z K(\cdot,z), K(\cdot,z)\rangle|^2)^{1/2}}, \tag{3.3b}$$

then it follows from the Lemma that the matrix A_s (respectively, B_t) is the compression of the operator $M \oplus M$ on the Hilbert space $\mathcal{H}_K \oplus \mathcal{H}_K$ to the two dimensional subspaces $\mathcal{M}$ (respectively, $\mathcal{N}$) if and only if $|\mu| \le 1$ (respectively, $|\lambda| \le 1$).

A natural family of Hilbert spaces $H^2_{\boldsymbol{\alpha}}(\Omega)$ consisting of modulus automorphic holomorphic functions on Ω was studied in the paper [2]. This family is indexed by $\boldsymbol{\alpha} \in \mathbb{T}^m$, where m is the number of bounded connected components in $\mathbb{C} \setminus \Omega$ and $\mathbb{T}$ is the unit circle. Each $H^2_{\boldsymbol{\alpha}}(\Omega)$ has a reproducing kernel $K_{\boldsymbol{\alpha}}(z,w)$. It was shown in [2] that every pure subnormal operator with spectrum $\bar{\Omega}$ and the spectrum of the normal extension contained in $\partial\bar{\Omega}$ is unitarily equivalent to M on one of these Hilbert spaces.

In the following subsection, we will show that any contractive homomorphism of the algebra $\mathrm{Rat}(\Omega)$ is of the form ρ_{A_s} or ρ_{B_t} with $K = K_{\boldsymbol{\alpha}}$ and $|\mu| \le 1$ and $|\lambda| \le 1$ respectively. Since the operator $M \oplus M$ is subnormal, we would have exhibited the dilation.

3.2. Construction of Dilations. The generalization of Nevanlinna-Pick theorem due to Abrahamse states that given n points $w_1, w_2, \ldots, w_n$ in the open unit disk, there is a holomorphic function $f : \Omega \to \mathbb{C}$ with $f(z_i) = w_i$ for $i = 1, 2, \ldots, n$ if and only if the matrix

$$M(\underline{w}, \boldsymbol{\alpha}) \stackrel{\text{def}}{=} \big(\!\big((1 - w_i\bar{w}_j)K_{\boldsymbol{\alpha}}(z_i,z_j)\big)\!\big) \tag{3.4}$$

is positive semidefinite. A deep result due to Widom (cf. [11, page 140]) shows that the map $\boldsymbol{\alpha} \mapsto K_{\boldsymbol{\alpha}}(z,w)$ is continuous for any fixed pair (z,w) in $\Omega \times \Omega$.

In what follows, we shall first show that a homomorphism $\rho : \mathrm{Rat}(\Omega) \to \mathcal{M}_2$ is contractive if and only if it is of the form ρ_{A_s} or ρ_{B_t} with $|\mu| \leq 1$ and $|\lambda| \leq 1$, respectively and

$$s = s_\Omega(z_1, z_2) := \sup\{|r(z_1)|^2 : r \in \ \mathrm{Rat}(\Omega),\ \|r\| \leq 1 \text{ and } r(z_2) = 0\} \tag{3.5a}$$

for any fixed but arbitrary pair $z_1, z_2 \in \Omega$;

$$t = t_\Omega(z) := \sup\{|r'(z)| : r \in \ \mathrm{Rat}(\Omega),\ \|r\| \leq 1 \text{ and } r(z) = 0\} \tag{3.5b}$$

for $z \in \Omega$.

We wish to point out that the extremal quantities $s_\Omega(z_1, z_2)$ and $t_\Omega(z)$ would remain the same even if we were to replace the $\mathrm{Rat}(\Omega)$ by the holomorphic function on Ω. The solution to the first extremal problem, with holomorphic functions in place of $\mathrm{Rat}(\Omega)$, exist by a normal family argument. Let $F : \Omega \to \mathbb{D}$ be a holomorphic function with $F(z_2) = 0$ and $F(z_2) = a$, where we have set $a = s_\Omega(z_1, z_2)$, temporarily. It then follows that $M((0, a), \boldsymbol{\alpha})$ must be non negative definite for all $\boldsymbol{\alpha} \in \mathbb{T}^m$. Consequently, we have

$$\det \begin{pmatrix} K_{\boldsymbol{\alpha}}(z_1, z_1) & K_{\boldsymbol{\alpha}}(z_1, z_2) \\ K_{\boldsymbol{\alpha}}(z_2, z_1) & (1 - a^2) K_{\boldsymbol{\alpha}}(z_2, z_2) \end{pmatrix} \geq 0$$

for all $\boldsymbol{\alpha} \in \mathbb{T}^m$. This condition is equivalent to requiring

$$|a|^2 \leq 1 - \frac{|K_{\boldsymbol{\alpha}}(z_1, z_2)|^2}{K_{\boldsymbol{\alpha}}(z_1, z_1) K_{\boldsymbol{\alpha}}(z_2, z_2)} \leq 1 - \sup\Big\{\frac{|K_{\boldsymbol{\alpha}}(z_1, z_2)|^2}{K_{\boldsymbol{\alpha}}(z_1, z_1) K_{\boldsymbol{\alpha}}(z_2, z_2)} : \boldsymbol{\alpha} \in \mathbb{T}^m\Big\}. \tag{3.6}$$

As we have pointed out earlier, since $\boldsymbol{\alpha} \to K_{\boldsymbol{\alpha}}(z_i, z_j)$ is continuous for any pair of fixed indices i and j, there exists a single $\boldsymbol{\alpha}_0$ depending only on z_1, z_2 for which the supremum in the above inequality is attained. For this choice of $\boldsymbol{\alpha}_0$ and a, clearly the determinant of $M((0, a), \boldsymbol{\alpha}_0)$ is zero. It follows from [11, Theorem 4.4, pp. 135] that the solution is unique and hence is a Blaschke product [11, Theorem 4.1, pp. 130].

Similarly, the solution to the second extremal problem, with holomorphic functions in place of $\mathrm{Rat}(\Omega)$, is a function which is holomorphic in a neighborhood of $\bar{\Omega}$ [11, Theorem 1.6, pp. 114]. Hence it is the limit of functions from $\mathrm{Rat}(\Omega)$. The following Lemma first appeared as [13, Remark 2, pp. 308].

LEMMA 3. *The homomorphism ρ_{A_s} is contractive if and only if $\|r(A_s)\| \leq 1$ for all r in* $\mathrm{Rat}(\Omega)$ *with $\|r\| \leq 1$ and $r(z_1) = 0$.*

The homomorphism ρ_{B_t} is contractive if and only if $\|r(B_t)\| \leq 1$ for all r in $\mathrm{Rat}(\Omega)$ *with $\|r\| \leq 1$ and $r(z) = 0$.*

PROOF: The two proofs are similar, so we shall prove only (1). Suppose $r(A)$ is a contraction for all $r \in \ \mathrm{Rat}(\Omega)$ with $\|r\| \leq 1$ and $r(z_1) = 0$. We have to prove $r(A)$ is a contraction for all $r \in \mathrm{Rat}(\Omega)$ with $\|r\| \leq 1$. For any such rational function r, let $r(z) = u$. Put $\varphi_u(z) = \frac{z-u}{1-\bar{u}z}$ and $\psi(z) = \varphi_u(r(z))$. Then ψ is in $\mathrm{Rat}(\Omega)$, $\|\psi\| \leq 1$ and $\psi(z) = 0$. By hypothesis, $\psi(A)$ is a contraction. Now note that $\varphi_u^{-1}(z) = \frac{z+u}{1+\bar{u}z}$. Since φ_u^{-1} maps $\mathbb{D}$ into $\mathbb{D}$, by von Neumann's inequality, $\|r(A)\| = \|\varphi_u^{-1}\psi(A)\| \leq 1$. ∎

This lemma makes it somewhat simple to derive the contractivity conditions for the homomorphisms induced by A_s and B_t.

LEMMA 4. *The homomorphism ρ_{A_s} is contractive if and only if $s^2 = s_\Omega(z_1,z_2)^{-1} - 1$ and $|\mu| \le 1$. Similarly, the homomorphism ρ_{B_t} is contractive if and only if $t = t_\Omega(z)^{-1}$ and $|\lambda| \le 1$.*

PROOF: First, using the functional calculus for A_s, we see that

$$r\begin{pmatrix} z_1 & 0 \\ s\mu(z_1 - z_2) & z_2 \end{pmatrix} = \begin{pmatrix} r(z_1) & 0 \\ s\mu(r(z_1) - r(z_2)) & r(z_2) \end{pmatrix} = \begin{pmatrix} r(z_1) & 0 \\ s\mu r(z_1) & 0 \end{pmatrix},$$

assuming $r(z_2) = 0$. Therefore, contractivity of ρ_{A_s} would imply

$$s^2|\mu|^2 + 1 \le \big(\sup\{|r(z_2)|^2 : r \in \text{Rat}(\Omega),\ \|r\| \le 1 \text{ and } r(z_2) = 0\}\big)^{-1} = s_\Omega(z_1,z_2)^{-1}.$$

Or, equivalently, if we put $s = s_\Omega(z_1,z_2)^{-1} - 1$ then we must have $|\mu| \le 1$. Now an application of Lemma 3 completes the proof.

To obtain the contractivity condition for ρ_{B_t}, using the functional calculus, we see that

$$r\begin{pmatrix} z & 0 \\ t\lambda & z \end{pmatrix} = \begin{pmatrix} r(z) & 0 \\ t\lambda r'(z) & r(z) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a\lambda r'(z) & 0 \end{pmatrix}$$

assuming $r(z) = 0$.

Therefore, contractivity of ρ_{B_t} would imply that

$$t|\lambda| \le (\sup\{|r'(w)| : r \in \text{ Rat}(\Omega),\ \|r\| \le 1 \text{ and } r(w) = 0\})^{-1} = t_\Omega(z)^{-1}.$$

Or equivalently, if we put $t = t_\Omega(z)^{-1}$ then we must have $|\lambda| \le 1$. ∎

We now have enough material to construct the dilation for a homomorphism $\rho_T : \mathcal{A}(\Omega) \to \mathcal{M}_2$. In this case, T is a 2×2 matrix with spectrum in Ω. Since we can apply a unitary conjugation to make T upper-triangular, it is enough to exhibit the dilation for the two matrices $T = A_s$ and $T = B_t$.

3.3. DILATION FOR A_s. Recall that there exists an $\boldsymbol{\alpha}_0$ depending only on z_1 and z_2 such that $\det M((0,a), \boldsymbol{\alpha}_0) = 0$. For now, set $\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}$. Let the subspace $\mathcal{M}$ of $H^2_{\boldsymbol{\alpha}} \oplus H^2_{\boldsymbol{\alpha}}$ be as in the first part of Lemma 2. For brevity, let

$$m^2 = 1 - \frac{|K_{\boldsymbol{\alpha}}(z_1,z_2)|^2}{K_{\boldsymbol{\alpha}}(z_1,z_1)K_{\boldsymbol{\alpha}}(z_2,z_2)} > 0.$$

Then $\det M((0,m),\boldsymbol{\alpha}) = \begin{pmatrix} K_{\boldsymbol{\alpha}}(z_1,z_1) & K_{\boldsymbol{\alpha}}(z_1,z_2) \\ K_{\boldsymbol{\alpha}}(z_2,z_1) & (1-m^2)K_{\boldsymbol{\alpha}}(z_2,z_2) \end{pmatrix} = 0$ by definition of m. As we have pointed out earlier, there is a holomorphic function $f : \Omega \to \mathbb{D}$ such that $f(z_1) = 0$ and $f(z_2) = m$. Moreover, if g any holomorphic function from Ω to $\mathbb{D}$ such that $g(z_1) = 0$, then the matrix $M((0,g(z_2)),\boldsymbol{\alpha})$ is positive semidefinite, which implies that $|g(z_2)|^2 \le 1 - \frac{|K_{\boldsymbol{\alpha}}(z_1,z_2)|^2}{K_{\boldsymbol{\alpha}}(z_1,z_1)K_{\boldsymbol{\alpha}}(z_2,z_2)}$. Thus $m = \sup\{|g(z_2)| : g$ is a holomorphic function from Ω to $\mathbb{D}$ and $g(z_1) = 0\}$. Hence

$$s_\Omega(z_1,z_2)^2 = \frac{1}{m^2} - 1 = \frac{|K_{\boldsymbol{\alpha}}(z_1,z_2)|^2}{K_{\boldsymbol{\alpha}}(z_1,z_1)K_{\boldsymbol{\alpha}}(z_2,z_2) - |K_{\boldsymbol{\alpha}}(z_1,z_2)|^2}.$$

So by the first part of Lemma 2, we have that the matrix of restriction of the operator $M^* \oplus M^*$ to the subspace $\mathcal{M}$ in the orthonormal basis $\{h_1(z_1,z_2), h_2(z_1,z_2)\}$ has the matrix representation A_s^* with $s^2 = s_\Omega(z_1,z_2)^{-1} - 1$ whenever $|\mu| \le 1$.

3.4. Having constructed the dilation, it is natural to find out what the characteristic function is when $\Omega = \mathbb{D}$. In this case, the general form of the matrix T discussed above is

$$T := \begin{pmatrix} z_1 & 0 \\ \mu(1-|z_1|^2)^{1/2}(1-|z_2|^2)^{1/2} & z_2 \end{pmatrix}. \tag{3.7}$$

where z_1 and z_2 are two points in the open unit disk $\mathbb{D}$ and $\mu \in \mathbb{C}$. We are using the explicit value of $s_{\mathbb{D}}(z_1, z_2)$ for the unit disc.

LEMMA 5. *For $i = 1, 2$, let $\varphi_i(z) = (z - z_i)/(1 - \bar{z}_i z)$. The characteristic function of T is*

$$\theta_T(z) = \begin{pmatrix} (1-|\mu|^2)^{1/2}\varphi_2(z) & -\mu \\ \bar{\mu}\varphi_1(z)\varphi_2(z) & (1-|\mu|^2)^{1/2}\varphi_1(z) \end{pmatrix}$$

PROOF: Recall that $\mathcal{M}$ is the subspace spanned by the orthonormal vectors $h_1(z_1, z_2)$ and $h_2(z_1, z_2)$. Since the compression of $M \oplus M$ to the co-invariant subspace $\mathcal{M}$ is T, by Beurling-Lax-Halmos theorem, we need to only find up to unitary coincidence (see [25], page 192 for definition) the inner function whose range is $\mathcal{M}^\perp$. So let $\begin{pmatrix} f \\ g \end{pmatrix}$ be a vector in the orthogonal complement of $\mathcal{M}$. The condition of orthogonality to h_1 implies that $g(z_1) = 0$ which is equivalent to $g = \varphi_1 \xi$ for arbitrary $\xi \in H^2(\mathbb{D})$. Now the orthogonality condition to h_2 implies that $(1-|\mu|^2)^{1/2} f(z_2) + \mu\xi(z_2) = 0$, which is the same as

$$(1-|\mu|^2)^{1/2}\varphi_1(z_2) f(z_2) + \mu g(z_2) = 0. \tag{3.8}$$

This implies that there is an $\eta_1 \in H^2(\mathbb{D})$ such that

$$(1-|\mu|^2)^{1/2} f + \mu g' = \varphi_2 \eta_1.$$

It is obvious that conversely if $\begin{pmatrix} f \\ g \end{pmatrix}$ is a function from $H^2(\mathbb{D}) \oplus H^2(\mathbb{D})$ such that g is in range of φ and satisfies (3.8), then it is in the orthogonal complement of $\mathcal{M}$.

Now let $\eta_2 = (1-|\mu|^2)^{1/2}\xi - \bar{\mu} f$. Then

$$\theta \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} (1-|\mu|^2)^{1/2}\varphi_2\eta_1 - \mu\eta_2 \\ \bar{\mu}\varphi_1\varphi_2\eta_1 + (1-|\mu|^2)^{1/2}\varphi_1\eta_2 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Thus if $\begin{pmatrix} f \\ g \end{pmatrix}$ satisfies (3.8), then it is in the range of θ. Conversely, it is easy to see that any element in the range of θ will satisfy (3.8). Thus the orthogonal complement of $\mathcal{M}$ in $\mathcal{H}$ is the range of θ. So θ is the characteristic function of the given matrix. ■

We would like to remark here that for $z_1 = z_2$, the characteristic function $\theta_T(u)$ for $T := \begin{pmatrix} z & 0 \\ \lambda(1-|z|^2) & z \end{pmatrix}$ can be obtained directly from the definition in case $z = 0$. A little computation, using the transformation rule for the characteristic function under a biholomorphic automorphism of the unit disk [25, pp. 239 - 240], produces the formula

$$\theta_T(u) = \begin{pmatrix} (1-|\lambda|^2)^{1/2}\varphi(u) & \lambda \\ \bar{\lambda}\varphi^2(u) & (1-|\lambda|^2)^{1/2}\varphi(u) \end{pmatrix}, \quad u \in \mathbb{D}$$

in the general case.

Let T_μ be the matrix defined in (3.7). Note that if $T_{\mu'}$ and T_μ are two such matrices with $|\mu'| = |\mu|$, then

$$\begin{aligned}
\theta_{T_{\mu'}}(z) &= \begin{pmatrix} (1-|\mu'|^2)^{1/2}\varphi_2 & -\mu' \\ \bar{\mu}'\varphi_1\varphi_2 & (1-|\mu'|^2)^{1/2}\varphi_1 \end{pmatrix} = \\
&= \begin{pmatrix} (1-|\mu|^2)^{1/2}\varphi_2 & -e^{i\psi}\mu \\ e^{-i\psi}\bar{\mu}\varphi_1\varphi_2 & (1-|\mu|^2)^{1/2}\varphi_1 \end{pmatrix} \text{ for some } \psi \in [0,2\pi] \\
&= \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix} \begin{pmatrix} (1-|\mu|^2)^{1/2}\varphi_2 & -\mu \\ \bar{\mu}\varphi_1\varphi_2 & (1-|\mu|^2)^{1/2}\varphi_1 \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix} \\
&= \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix} \theta_{S_\mu}(z) \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix},
\end{aligned}$$

and hence their characteristic functions coincide. So they are unitarily equivalent. Conversely, if $T_{\mu'}$ and T_μ are unitarily equivalent, then their characteristic functions coincide and hence the singular values of the characteristic functions are same. Note that when $z_1 \neq z_2$, we have

$$\theta_{T_{\mu'}}(z_1)\theta_{T_{\mu'}}(z_1)^* = \begin{pmatrix} (1-|\mu'|^2)|\omega|^2 + |\mu'|^2 & 0 \\ 0 & 0 \end{pmatrix}$$

for some $\omega \in \mathbb{C}$ (independent of μ'). When $z_1 = z_2$, then

$$\theta_{T_{\mu'}}(z_1)\theta_{T_{\mu'}}(z_1)^* = \begin{pmatrix} 0 & |\mu'|^2 \\ 0 & 0 \end{pmatrix}.$$

In either case, coincidence of $\theta_{T_{\mu'}}$ and θ_{T_μ} mean that $|\mu'| = |\mu|$. Thus using the explicit characteristic function we have proved the following.

THEOREM 6. *Two matrices $T_{\mu'}$ and T_μ as defined in (3.7) are unitarily equivalent if and only if $|\mu'| = |\mu|$.*

3.5. DILATION FOR B_t. We now shift our attention to the construction of dilation when the homomorphism ρ_T is induced by a 2×2 matrix T with equal eigenvalues. So $\sigma(T) = \{z\}$. The domain Ω has its associated Szego kernel which is denoted by $\hat{K}_\Omega(z,w)$. Recall that a generalization due to Ahlfors to multiply connected domains of the Schwarz lemma says that

$$t_\Omega(z) := \big(\sup\{|r'(z)| : r \in \mathrm{Rat}(\Omega), \|r\| \le 1 \text{ and } r(z) = 0\}\big)^{-1} = \hat{K}_\Omega(z,z)^{-1}.$$

Let $\partial\Omega$ be the topological boundary of Ω and let $|d\nu|$ be the arc-length measure on $\partial\Omega$. Consider the measure $dm = |\hat{K}_\Omega(\nu,z)|^2|d\nu|$, and let the associated Hardy space $H^2(\Omega, dm)$ be denoted by $\mathcal{H}$. The measure dm is mutually absolutely continuous with respect to the arc length measure. Thus the evaluation functionals on $\mathcal{H}$ are bounded and hence $\mathcal{H}$ possesses a reproducing kernel K. Then it is known that K satisfies the property:

$$\frac{K(z,z)}{(K(z,z)\|\partial_{\bar{z}}K(\cdot,z)\|^2 - |\langle \partial_{\bar{z}}K(\cdot,z), K(\cdot,z)\rangle|^2)^{1/2}} = \hat{K}_\Omega(z,z)^{-1},$$

see [13, Theorem 2.2]. Now a (subnormal) dilation for $A_s := \begin{pmatrix} z & 0 \\ \lambda s_\Omega(z) & z \end{pmatrix}$, where $|\lambda| \leq 1$, is the operator $M \oplus M$ on the Hilbert space $\mathcal{H} \oplus \mathcal{H}$. This is easily verified since the restriction of $M^* \oplus M^*$ to the subspace $\mathcal{N}$ which was described in the second part of Lemma 2 is A_s^*.

Remark 7. If we choose $|\mu| = 1$ then the A_s^* is the restriction of M^* to the two dimensional subspace spanned by the vectors $K_{\boldsymbol{\alpha}}(\cdot, z_1)$ and $K_{\boldsymbol{\alpha}}(\cdot, z_2)$ in the Hardy space $H^2_{\boldsymbol{\alpha}}(\Omega)$ by our construction. Except in this case, the dilation of the homomorphism ρ_{A_s} we have constructed is a minimal subnormal dilation. (This dilation then may be extended to a minimal normal dilation.) While it is known that a minimal dilation is not unique when Ω is finitely connected, our construction gives a measure of this non-uniqueness. More explicitly, for each $\boldsymbol{\alpha}_0 \in \mathbb{T}^m$ for which

$$\sup\{\frac{|K_{\boldsymbol{\alpha}}(z_1,z_2)|^2}{K_{\boldsymbol{\alpha}}(z_1,z_1)K_{\boldsymbol{\alpha}}(z_2,z_2)} : \boldsymbol{\alpha} \in \mathbb{T}^m\} = \frac{|K_{\boldsymbol{\alpha}_0}(z_1,z_2)|^2}{K_{\boldsymbol{\alpha}_0}(z_1,z_1)K_{\boldsymbol{\alpha}_0}(z_2,z_2)},$$

the matrix representation of the operator $M^* \oplus M^*$ restricted to the 2 dimensional subspace $\mathcal{M}$ of the Hilbert space $H^2_{\boldsymbol{\alpha}_0} \oplus H^2_{\boldsymbol{\alpha}_0}$ equals A_s.

4. The Operator Space

The problem that we are considering naturally gives rise to an operator space structure. In this section, we show that. We begin by recalling basic definitions.

A vector space X is called an operator space if for each $k \in \mathbb{N}$, there are norms $\|\cdot\|_k$ on $X \otimes \mathcal{M}_k$ such that

(1) whenever $A = ((a_{ij})) \in \mathcal{M}_k$, $((x_{ij})) \in X \otimes \mathcal{M}_k$ and $B = ((b_{ij})) \in \mathcal{M}_k$, then

$$\|A \cdot ((x_{ij})) \cdot B\|_k \leq \|A\| \|((x_{ij}))\|_k \|B\|$$

where $A \cdot ((x_{ij})) \cdot B = ((\sum_{p=1}^m \sum_{l=1}^k a_{ip} x_{pl} b_{lj})) \in X \otimes \mathcal{M}_k$ and $\|A\|$ and $\|B\|$ are operator norms on $\mathcal{M}_k = \mathcal{B}(\mathbb{C}^k)$.

(2) For all positive integers m, k and for all $R \in X \otimes \mathcal{M}_k$ and $S \in X \otimes \mathcal{M}_m$, we have

$$\left\| \left(\begin{smallmatrix} R & 0 \\ 0 & S \end{smallmatrix} \right) \right\|_{m+k} = \max\{\|R\|_m, \|S\|_k\}.$$

Two such operator spaces $(X, \|\cdot\|_{X,k})$ and $(Y, \|\cdot\|_{Y,k})$ are said to be completely isometric if there is a linear bijection $\tau : X \to Y$ such that $\tau \otimes I_k : (X, \|\cdot\|_{X,k}) \to (Y, \|\cdot\|_{Y,k})$ is an isometry for every $k \in \mathbb{N}$.

Let X be an operator space and let $\rho : X \to \mathcal{B}(\mathcal{H})$ be a linear map, where $\mathcal{H}$ is a Hilbert space. If for each $k \in \mathbb{N}$, the map $\rho \otimes I_k : (X, \|\cdot\|_k) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{M}_k)$ is contractive then ρ is said to be *completely contractive*. Let $\mathcal{H}$ be finite-dimensional, let $T \in \mathcal{B}(\mathcal{H})$, let $X = \mathcal{A}(\Omega)$ and let $\rho = \rho_T$ be as defined earlier. We assume that the eigenvalues $z_1, z_2, \ldots, z_n$ of T are distinct.

To begin with, we introduce a notation. We denote by $I^k_{\underline{z}}$ the subset of $\mathbb{C}^n \otimes \mathcal{M}_k$ defined as

$$I^k_{\underline{z}} = \{(R(z_1), R(z_2), \ldots, R(z_n)) : R \in \mathcal{A}(\Omega) \otimes \mathcal{M}_k \text{ and } \|R\| \leq 1\}$$

where $\|R\| = \sup_{z \in \overline{\Omega}} \|R(z)\|$. When $k = 1$, we denote it by $I_{\underline{z}}$ rather than $I^1_{\underline{z}}$.

Lemma 8. *The set $I_{\underline{z}}$ defined above is a compact set.*

Proof. Clearly, $I_{\underline{z}}$ is a subset of $\bar{D}^n$. So it is enough to show that $I_{\underline{z}}$ is a closed set. Recall from Section 3 that the generalization of Nevanlinna-Pick theorem due to Abrahamse states that given n points $w_1, w_2, \ldots, w_n$ in the open unit disk, there is a holomorphic function $f : \Omega \to \mathbb{C}$ with $f(z_i) = w_i$ for $i = 1, 2, \ldots, n$ if and only if the matrix

$$M(\underline{w}, \boldsymbol{\alpha}) \stackrel{\text{def}}{=} \big(\!\big((1 - w_i\bar{w}_j)K_\alpha(z_i, z_j)\big)\!\big) \tag{4.1}$$

is positive semidefinite for all $\boldsymbol{\alpha} \in \mathbb{T}^m$. So

$$\begin{aligned} I_{\underline{z}} &= \{(w_1, w_2, \ldots, w_n) \in \overline{\mathbb{D}}^n : \text{ the matrix } M(\underline{w}, \boldsymbol{\alpha}) \text{ is positive semidefinite for all } \boldsymbol{\alpha} \in \mathbb{T}^m\} \\ &= \{(w_1, w_2, \ldots, w_n) \in \overline{\mathbb{D}}^n : \lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha})) \geq 0 \text{ for all } \boldsymbol{\alpha} \in \mathbb{T}^n\} \\ &= \cap_{\boldsymbol{\alpha} \in \mathbb{T}^m}\{(w_1, w_2, \ldots, w_n) \in \bar{\mathbb{D}}^n : \lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha})) \geq 0\} \\ &= \cap_{\boldsymbol{\alpha} \in \mathbb{T}^m} (\lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha})))^{-1}([0, \infty)) \end{aligned}$$

where $\lambda_{\min}(A)$ for a hermitian matrix A denotes its smallest eigenvalue. It is a continuous function on the set of hermitian matrices (see for example, [7, Corollary III.2.6]). Thus $\underline{w} \to \lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha}))$ is a continuous function on $\mathbb{C}^n$. Since arbitrary intersection of closed sets is closed, $I_{\underline{z}}$ is a closed set. ■

It is easy to see that the set $I^k_{\underline{z}}$ is convex and balanced, so it is the closed unit ball of some norm on $\mathbb{C}^n \otimes \mathcal{M}_k$. The sets of the form $I^k_{\underline{z}}$ were first studied, in the case $k = 1$, by Cole and Wermer [8]. The sets $I^k_{\underline{z}}$ are examples of *matrcially hyperconvex* sets studied by Paulsen in [21]. Paulsen points out that the sequence of sets $I^k_{\underline{z}} \subseteq \mathbb{C}^n \otimes \mathcal{M}_k$ determines an operator space structure on $\mathbb{C}^m$, that is, the set $I^k_{\underline{z}}$ determines a norm $\|\cdot\|_{\underline{z},k}$ in $\mathbb{C}^n \otimes \mathcal{M}_k$ such that $I^k_{\underline{z}}$ is the closed unit ball in this norm and the sequence $\{\mathbb{C}^n \otimes \mathcal{M}_k, \|\cdot\|_{\underline{z},k}\}$ satisfies the conditions (1) and (2) above. We denote this operator space by $\mathrm{HC}_{\Omega,\underline{z}}(\mathbb{C}^n)$. Paulsen also notes that this operator space is completely isometric to a quotient of a function algebra. Indeed, it is not difficult to see that $\mathrm{HC}_{\Omega,\underline{z}}(\mathbb{C}^n)$ is completely isometrically isomorphic to the quotient C^*-algebra $C(\overline{\Omega})/\mathcal{Z}$ where $\mathcal{Z} = \{f \in \mathcal{A}(\Omega) : f(z_1) = f(z_2) = \cdots = f(z_n) = 0\}$. If $k = 1$, we will write $\|\cdot\|_{\underline{z}}$ rather than $\|\cdot\|_{\underline{z},1}$.

Lemma 9. *There are n matrices $V_1, V_2, \ldots, V_n \in \mathcal{M}_n$ such that the map $\rho_T \otimes I_k : \mathcal{A}(\Omega) \otimes \mathcal{M}_k \to \mathcal{M}_n \otimes \mathcal{M}_k$ is of the form*

$$(\rho_T \otimes I_k)R = V_1 \otimes R(z_1) + V_2 \otimes R(z_2) + \cdots + V_n \otimes R(z_n)$$

for any $R \in \mathcal{A}(\Omega) \otimes \mathcal{M}_k$ and any $k \in \mathbb{N}$. The matrices V_i depend on the set $\{z_1, z_2, \ldots, z_n\}$.

Proof: If F and G are two elements of $\mathcal{A}(\Omega) \otimes \mathcal{M}_k$ which agree on the set $\{z_1, z_2, \ldots, z_n\}$, then define $H \in \mathcal{A}(\Omega) \otimes \mathcal{M}_k$ by $H = F - G$. Then H vanishes at the points $z_1, z_2, \ldots z_n$ and hence $H(z) = (z - z_1)(z - z_2) \ldots (z - z_n)W(z)$ for some W in $\mathcal{A}(\Omega) \otimes \mathcal{M}_k$. By the functional calculus,

$$(\rho_T \otimes I_k)H = (T - z_1)(T - z_2) \ldots (T - z_n)W(T).$$

Note that $(z - z_1)(z - z_2) \ldots (z - z_n)$ is the characteristic polynomial of T and by Cayley-Hamilton theorem, $(T - z_1)(T - z_2) \ldots (T - z_n) = 0$. Thus $(\rho_T \otimes I_k)H = 0$. So if $F, G \in \mathcal{A}(\Omega) \otimes \mathcal{M}_k$ are such that $F(z_i) = G(z_i)$ for all $i = 1, 2, \ldots, n$, then $(\rho_T \otimes I_k)F = (\rho_T \otimes I_k)G$.

Now define $V_1, V_2, \ldots, V_n$ by

$$V_i = \rho_T\left(\frac{(z-z_1)\ldots(z-z_{i-1})(z-z_{i+1})\ldots(z-z_n)}{(z_i-z_1)\ldots(z_i-z_{i-1})(z_i-z_{i+1})\ldots(z_i-z_n)}\right)$$

for $i = 1, 2, \ldots, n$. Given $R \in \mathcal{A}(\Omega) \otimes \mathcal{M}_k$, it agrees with the function

$$\tilde{R}(z) = \sum_{i=1}^{n} \frac{(z-z_1)\ldots(z-z_{i-1})(z-z_{i+1})\ldots(z-z_n)}{(z_i-z_1)\ldots(z_i-z_{i-1})(z_i-z_{i+1})\ldots(z_i-z_n)} R(z_i)$$

on the set $\{z_1, z_2, \ldots, z_n\}$ and hence

$$\begin{aligned}(\rho_T \otimes I_k)R &= (\rho_T \otimes I_k)\tilde{R} \\ &= \sum_{i=1}^{n} \rho_T\left(\frac{(z-z_1)\ldots(z-z_{i-1})(z-z_{i+1})\ldots(z-z_n)}{(z_i-z_1)\ldots(z_i-z_{i-1})(z_i-z_{i+1})\ldots(z_i-z_n)}\right) \otimes R(z_i) \\ &= \sum_{i=1}^{n} V_i \otimes R(z_i)\end{aligned}$$

completing the proof of the Lemma. ■

At this point, we note that $\mathcal{A}(\Omega)$ being a closed sub-algebra of the commutative C^*-algebra of all continuous functions on the boundary of Ω inherits a natural operator space structure, denoted by $\mathrm{MIN}(\mathcal{A}(\Omega))$. Recall that a celebrated theorem of Arveson says that a contractive homomorphism $\rho_T : \mathcal{A}(\Omega) \to \mathcal{L}(\mathcal{H})$ dilates if and only if it is completely contractive when $\mathcal{A}(\Omega)$ is equipped with the MIN operator space structure. The contractivity and complete contractivity of the homomorphism ρ_T amount to respectively

$$\sup\{\|w_1V_1 + w_2V_2 + \cdots + w_nV_n\| : \underline{w} = (w_1, w_2, \ldots, w_n) \in I_{\underline{z}}\} \leq 1 \tag{4.2}$$

where $\|\cdot\|$ is the operator norm on $\mathcal{M}_n$ and

$$\sup\{\|\sum_{i=1}^{n} V_i \otimes W_i\| : W_i \in \mathcal{M}_k \text{ and } W = (W_1, W_2, \ldots, W_n) \in I_{\underline{z}}^k \text{ for } k \geq 1\} \leq 1 \tag{4.3}$$

where $\|\cdot\|$ is the operator norm on $\mathcal{M}_n \otimes \mathcal{M}_k$. Now, we state the following theorem whose proof is evident from the discussion above.

THEOREM 10. *The contractive homomorphism $\rho_T : \mathcal{A}(\Omega) \to \mathcal{M}_n$ is completely contractive with respect to the MIN operator space structure on $\mathcal{A}(\Omega)$ if and only if the contractive linear map $L_T : (\mathbb{C}^n, \|\cdot\|_{\underline{z}}) \to \mathcal{M}_n$ defined by $L_T(\underline{w}) = w_1V_1 + w_2V_2 + \cdots + w_nV_n$ is completely contractive on the operator space $\mathrm{HC}_{\Omega,\underline{z}}(\mathbb{C}^n)$.*

The theorem above brings us to our concluding remarks of this section. Given a Banach space, there are two extremal natural operator space structures on it, denoted by $\mathrm{MAX}(X)$ and $\mathrm{MIN}(X)$. We refer the reader to [20] for definitions and basic details. However, this theorem shows that if $\mathrm{HC}_{\Omega,\underline{z}}(\mathbb{C}^n)$ was completely isometric to $\mathrm{MAX}(\mathbb{C}^n, \|\cdot\|_{\underline{z}})$, then every contractive homomorphism ρ_T of the algebra $\mathcal{A}(\Omega)$, induced by an n - dimensional linear transformation T with distinct eigenvalues in Ω, will necessarily dilate. This gives rise to the question of determining when $\mathrm{HC}_{\Omega,\underline{z}}(\mathbb{C}^n)$ is the same as $\mathrm{MAX}(\mathbb{C}^n, \|\cdot\|_{\underline{z}})$ which is interesting in its own right. Agler [4] proved that all contractive homomorphisms of the algebra $\mathcal{A}(\mathbb{A})$, where

$\mathbb{A} = \{z \in \mathbb{C} : r < |z| < 1\} \subseteq \mathbb{C}$ is the annulus for a fixed r in $(0,1)$, are completely contractive. This implies that $\mathrm{HC}_{\mathbb{A},\underline{z}}(\mathbb{C}^n)$ is completely isometric to $\mathrm{MAX}(\mathbb{C}^n, \|\cdot\|_{\underline{z}})$ for every $n \in \mathbb{N}$.

In [20], Paulsen related a problem similar to the one that we are considering to certain questions in the setting of operator spaces and thereby solved it. For $n \geq 1$, let G be a closed unit ball in $\mathbb{C}^n$ corresponding to a norm $\|\cdot\|_G$ on $\mathbb{C}^n$. Let $\mathcal{A}(G)$ denote the closure of polynomials in $C(G)$, the algebra of all continuous functions on G equipped with the sup norm. It is easy to see that there is a unital contractive homomorphism $\rho : \mathcal{A}(G) \to \mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$ which is not completely contractive if and only if $\mathrm{MIN}(\mathbb{C}^n, \|\cdot\|_G)$ is not completely isometric to $\mathrm{MAX}(\mathbb{C}^n, \|\cdot\|_G)$. Paulsen proved the remarkable result that for $n \geq 5$,

$$\mathrm{MIN}(\mathbb{C}^n, \|\cdot\|_G) \text{ is not completely isometric to } \mathrm{MAX}(\mathbb{C}^n, \|\cdot\|_G), \tag{4.4}$$

for any closed unit ball G. For $n = 2$, Ando's theorem implies that $\mathrm{MIN}(\mathbb{C}^2, \|\cdot\|_{\mathbb{D}^2})$ is completely isometric to $\mathrm{MAX}(\mathbb{C}^2, \|\cdot\|_{\mathbb{D}^2})$. The fact that (4.4) holds for $n \geq 3$ and any closed unit ball G is pointed out in [22, Exercise 3.7]. In the same spirit, a similar question about a class of homomorphisms, first introduced by Parrott [19] (see also [14], [15] and [16]), led Paulsen to define a natural operator space which he called COT. Let G be a unit ball and let $\underline{w}$ be a point in the interior of G. Let X be the Banach space $X = (\mathbb{C}^n, \|\cdot\|_{G,\underline{w}})$, where $\|\cdot\|_{G,\underline{w}}$ is the Caratheodory norm of G at the point $\underline{w}$. The question of whether $\mathrm{COT}_{\underline{w}}(X)$ is completely isometric to $\mathrm{MIN}(X^*)$ for $\underline{w} \in G$ was first raised in [20]. He showed that the answer is affirmative when $\underline{w} = 0$. Later in an unpublished note, it was shown by Dash [9] that $\mathrm{COT}_{\underline{w}}(G)$ and $\mathrm{MIN}(X^*)$ are not necessarily completely isometric. The question of deciding whether a contractive homomorphism $\rho_T : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{H})$ is completeley contractive or not is similar in nature. It amounts to deciding if $\mathrm{HC}_{\Omega,\underline{z}}(\mathbb{C}^n)$ is completely isometric to $\mathrm{MIN}(\mathbb{C}^n, \|\cdot\|_z)$ or $\mathrm{MAX}(\mathbb{C}^n, \|\cdot\|_z)$. It is likely that the operator space $\mathrm{HC}_{\Omega,\underline{z}}(\mathbb{C}^n)$ is completely isometric to $\mathrm{MIN}(\mathbb{C}^n, \|\cdot\|_z)$ for every $n \geq 3$. We pose this as an open problem whose solution defies us at the moment.

5. A Factorization Condition

Let T be a linear transformation on an n dimensional Hilbert space space V with distinct eigenvalues $z_1, z_2, \ldots, z_n$ in Ω. Let $v_1, v_2, \ldots, v_n$ be the n linearly independent eigenvectors of T^*. If $\sigma = \{z_1, z_2, \ldots, z_n\}$, then define a positive definite function $K : \sigma \times \sigma \to \mathbb{C}$ by setting

$$\big(\!\big(K(z_j, z_i)\big)\!\big)_{i,j}^n := \big(\!\big(\langle v_i, v_j\rangle\big)\!\big)_{i,j=1}^n. \tag{5.1}$$

As before, let $\rho_T : \mathcal{A}(\Omega) \to \mathcal{L}(V)$ be the homomorphism induced by T. Suppose there exists a dilation of the homomorphism ρ_T. Then it follows from [1, Theorem 2] that there is a flat unitary vector bundle $\mathcal{E}$ of rank n (see [2] for definitions and complete results on model theory in multiply connected domains) such that $\rho_T(f)$ is the compression of the subnormal operator M_f on the Hardy space $H^2_{\mathcal{E}}(\Omega)$ to a semi-invariant subspace in it. Consequently, there exists a homomorphism

$$\rho_M : \mathcal{A}(\Omega) \to \mathcal{B}(H^2_{\mathcal{E}}(\Omega)) \tag{5.2}$$

dilating ρ_T. The homomorphism ρ_M is induced by the multiplication operator M on $H^2_{\mathcal{E}}(\Omega)$ which is subnormal. Thus the homomorphism $\rho_N : C(\partial\Omega) \to \mathcal{B}(\mathcal{H})$ induced by the normal extension N on the Hilbert space $\mathcal{H} \supseteq H^2_{\mathcal{E}}(\Omega)$ of the operator M is a dilation of the homomorphism ρ_T in the sense of (1.1). The multiplication operator M on $H^2_{\mathcal{E}}$ is called a bundle shift. We recall [2, Theorem 3] that $\dim\ker(M-z)^* = n$. Let $K^{\mathcal{E}}_z(i)$, $i = 1, 2, \ldots, n$ be a basis (not necessarily orthogonal) of $\ker(M-z)^*$. We set

$$K^{\mathcal{E}}(z_j, z_i) := \big(\!\!\big(\langle K^{\mathcal{E}}_{z_i}(\ell), K^{\mathcal{E}}_{z_j}(p)\rangle\big)\!\!\big)_{\ell,p=1}^n, \text{ for } 1 \le i, j \le n. \tag{5.3}$$

If ρ_T dilates then the linear transformation T can be realized as the compression of the operator M on $H^2_{\mathcal{E}}(\Omega)$ to an n-dimensional co-invariant subspace, say $\mathfrak{M} \subseteq H^2_{\mathcal{E}}(\Omega)$. The subspace $\mathfrak{M}$ must consist of eigenvectors of the bundle shift M. Let x_i, $1 \le i \le n$, be a set of n vectors in $\mathbb{C}^n$ and $\mathfrak{M} = \{\sum_{\ell=1}^n x_i(\ell) K^{\mathcal{E}}_{z_i}(\ell) : 1 \le i \le n\}$. The map which sends v_i to $\sum_{\ell=1}^n x_i(\ell) K^{\mathcal{E}}_{z_i}(\ell)$, $1 \le i \le n$, intertwines T^* and the restriction of M^* to $\mathfrak{M}$. For this map to be an isometry as well, we must have

$$\langle v_i, v_j\rangle = \langle K^{\mathcal{E}}(z_j, z_i) x_i, x_j\rangle,\ x_i \in \mathbb{C}^n,\ 1 \le i \le n. \tag{5.4}$$

Conversely, if there is a flat unitary vector bundle $\mathcal{E}$ and n vectors $x_1, x_2, \ldots, x_n$ in $\mathbb{C}^n$ satisfying (5.4), then ρ_T obviously dilates. So we have proved the following theorem.

Theorem 11. *The homomorphism ρ_T is dilatable to a homomorphism $\tilde\rho$ if and only if the kernel K, as defined in (5.1), can be written as*

$$K(z_j, z_i) = \langle K^{\mathcal{E}}(z_j, z_i) x_i, x_j\rangle, \text{ for some choice of } x_1, \ldots, x_n \in \mathbb{C}^n,$$

where $K^{\mathcal{E}}(z_i, z_j)$ is defined in (5.3).

It is interesting to see how contractivity of ρ_T is related to the above theorem. Note that ρ_T is contractive if and only if $\|f(T)^*\| \le \|f\|$ by definition of ρ_T. Since $T^* v_i = \bar z_i v_i$ we note that $f(T)^* v_i = \overline{f(z_i)} v_i$, for $1 \le i \le n$ and $f \in \mathrm{Rat}(\Omega)$. It then follows that

$$\begin{aligned}\|\rho_T(f)^*\|^2 &= \sup\{\|f(T)^*\big(\sum_{i=1}^n \alpha_i v_i\big)\|^2 : \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}\}\\ &= \sup\{\|\sum_{i=1}^n \alpha_i \overline{f(z_i)} v_i\|^2 : \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}\}\\ &= \sup\{\sum_{i,j=1}^n \alpha_i \bar\alpha_j \overline{f(z_i)} f(z_j)\langle v_i, v_j\rangle : \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}\}.\end{aligned}$$

Therefore, $\|f(T)^*\| \le \|f\|$ if and only if

$$\sum_{i,j=1}^n \alpha_i \bar\alpha_j \overline{f(z_i)} f(z_j)\langle v_i, v_j\rangle \le \sum_{i,j=1}^n \alpha_i \bar\alpha_j \langle v_i, v_j\rangle,$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ and all $f \in \mathrm{Rat}(\Omega)$ with $\|f\| \le 1$. Thus contractivity of ρ_T is equivalent to non-negative definiteness of the matrix

$$\big(\!\!\big((1 - \overline{f(z_i)} f(z_j)) K(z_j, z_i)\big)\!\!\big)_{i,j=0}^n, \tag{5.5}$$

for all $f \in \mathrm{Rat}(\Omega)$, $\|f\| \le 1$. If ρ_T is dilatable then the theorem above tells us that

$$(5.6)\qquad \big(\!\big((1-\overline{f(z_i)}f(z_j))K(z_j,z_i)\big)\!\big)_{i,j=0}^n = \big(\!\big((1-\overline{f(z_i)}f(z_j))\langle K_{\mathcal{E}}(z_j,z_i)x_i,x_j\rangle\big)\!\big)_{i,j=0}^n.$$

The last matrix is non-negative definite because M on $H^2_{\mathcal{E}}(\Omega)$ induces a contractive homomorphism. We therefore see, in this concrete fashion, that if the homomorphism ρ_T was dilatable then it would be contractive.

The interesting point to note here is that our construction of the dilation of ρ_T when T is a 2×2 matrix proves that the general dilation in that case is of the form $H^2_{\boldsymbol{\alpha}}(\Omega)\otimes\mathbb{C}^2$.

Suppose that the homomorphism ρ_T admits a dilation of the form

$$(5.7)\qquad \rho_{M\otimes I} : \mathcal{A}(\Omega) \to \mathcal{B}(H^2_{\boldsymbol{\alpha}}(\Omega)\otimes\mathbb{C}^n)$$

for some $\boldsymbol{\alpha}\in\mathbb{T}^m$, that is, the multiplication operator $M\otimes I$ on $H^2_{\boldsymbol{\alpha}}(\Omega)\otimes\mathbb{C}^n$ is a dilation of T. Since the eigenvectors $\{v_1, v_2,\dots,v_n\}$ for T^* span V and the set of eigenvectors of $M^*\otimes I : H^2_{\boldsymbol{\alpha}}(\Omega)\otimes\mathbb{C}^n \to H^2_{\boldsymbol{\alpha}}(\Omega)\otimes\mathbb{C}^n$ at z_i is the set of vectors $\{K_{\boldsymbol{\alpha}}(\cdot,z_i)\otimes a_j : a_j\in\mathbb{C}^n, 1\le j\le n\}$ for $1\le i\le n$, it follows that any map $\Gamma : V\to H^2_{\boldsymbol{\alpha}}(\Omega)$ that intertwines T^* and M^* must be defined by $\Gamma(v_i) = K_{\boldsymbol{\alpha}}(\cdot,z_i)\otimes a_i$ for some choice of a set of n vectors $a_1,a_2,\dots,a_n$ in $\mathbb{C}^n$. Now Γ is isometric if and only if

$$(5.8)\qquad \big(\!\big(K(z_j,z_i)\big)\!\big) = \big(\!\big(\langle v_i,v_j\rangle\big)\!\big) = \big(\!\big(K_{\boldsymbol{\alpha}}(z_j,z_i)\langle a_i,a_j\rangle\big)\!\big).$$

Clearly, this means that $\big(\!\big(K(z_j,z_i)\big)\!\big)$ admits $\big(\!\big(K_{\boldsymbol{\alpha}}(z_j,z_i)\big)\!\big)$ as a factor in the sense that $\big(\!\big(K(z_j,z_i)\big)\!\big)$ is the Schur product of $\big(\!\big(K_{\boldsymbol{\alpha}}(z_j,z_i)\big)\!\big)$ and a positive definite matrix, namely, the matrix $A = \big(\!\big(\langle a_i,a_j\rangle\big)\!\big)$.

Conversely, the contractivity assumption on ρ_T does not necessarily guarantee that $K_{\boldsymbol{\alpha}}$ is a factor of K. However, if we make this stronger assumption, that is, we assume there exists a positive definite matrix A such that $\big(\!\big(K(z_j,z_i)\big)\!\big) = \big(\!\big(K_{\boldsymbol{\alpha}}(z_j,z_i)a_{ij}\big)\!\big)$, where $A = \big(\!\big(a_{ij}\big)\!\big)$. Since A is positive, it follows that $A = \big(\!\big(\langle a_i,a_j\rangle\big)\!\big)$ for some set of n vectors $a_1,\dots,a_n$ in $\mathbb{C}^n$. Therefore if we define the map $\Gamma : V\to H^2_{\boldsymbol{\alpha}}(\Omega)\otimes\mathbb{C}^n$ to be $\Gamma(v_i) = K_{\boldsymbol{\alpha}}(\cdot,z_i)\otimes a_i$ for $1\le i\le n$ then Γ is clearly unitary and is an intertwiner between T and M^*. Thus the theorem above has the corollary:

Corollary 12. *The homomorphism ρ_T is dilatable to a homomorphism $\tilde\rho$ of the form (5.7) if the kernel K, as defined in (5.1), is the Schur product of a positive definite matrix A and the restriction of $K_{\boldsymbol{\alpha}}$ to the set $\sigma\times\sigma$ for some $\boldsymbol{\alpha}\in\mathbb{T}^m$.*

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References

[1] M. B. Abrahamse and R. G. Douglas, *Operators on multiply connected domains*, Proc. Roy. Irish Acad., Sect. A 74 (1974), 135 - 141.

[2] M. B. Abrahamse and R. G. Douglas, *A class of subnormal operators related to multiply connected domains*, Adv. Math., 19 (1976), 106 - 148.

[3] M. B. Abrahamse, *The Pick interpolation theorem for finitely connected domains*, Michigan Math. J. 26 (1979), 195 - 203.

[4] J. Agler, *Rational dilation on an annulus*, Ann. of Math. 121 (1985), 537 - 563.

[5] W. Arveson, *Subalgebras of C^* - algebras*, Acta Math., 123 (1969) 141 - 224.

[6] W. Arveson, *Subalgebras of C^* - algebras II*, Acta Math. 128 (1969), 271 - 308.

[7] R. Bhatia, *Matrix Analysis*, Springer, 1996.

[8] B. J. Cole and J. Wermer, *Pick interpolation, von Neumann inequalities, and hyperconvex sets*, Complex potential theory (Montreal, PQ, 1993), 89 - 129, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 439, Kluwer Acad. Publ., Dordrecht, 1994.

[9] M. Dash, *COT is not necessarily a MIN operator space*, Preprint (1997).

[10] M. A. Dritschel and S. McCullough, *The failure of rational dilation on a triply connected domain*, J. Amer. Math. Soc., to appear.

[11] S. D. Fisher, *Function theory on planar domains*, John Wiley and Sons, 1983.

[12] R. Meyer, *Adjoining a unit to an operator algebra*, J. Operator Th. 46 (2001), 281 - 288.

[13] G. Misra, *Curvature inequalities and extremal properties of bundle shifts*, J. Operator Th. 11 (1984), 305-317.

[14] G. Misra, *Completely contractive Hilbert modules and Parrott's example*, Acta Math. Hungar. 63 (1994), 291- 303.

[15] G. Misra and V. Pati, *Contractive and completely contractive modules, matricial tangent vectors, and distance decreasing metrics*, J. Operator Th. 30 (1993), 353- 380.

[16] G. Misra and N. S. N. Sastry, *Completely bounded modules and associated extremal problems*, J. Funct. Anal. 91 (1990), 213- 220.

[17] T. Nakazi and K Takahashi, *Two-dimensional representations of uniform algebras*, Proc. Amer. Math. Soc. 123 (1995), 2777 - 2784.

[18] J. von Neumann, *Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes*, Math. Nachr. 4 (1951), 258 - 281.

[19] S. K. Parrott, *Unitary dilation for commuting contractions*, Pacific J. Math. 34 (1970), 481 - 490.

[20] V.I. Paulsen, *Representations of function algebras, abstract operator spaces, and Banach space Geometry,*, J. Funct. Anal. 109 (1992), 113 - 129.

[21] V. I. Paulsen, *Matrix-valued interpolation and hyperconvex sets*, Integr. Equat. Oper. Th. 41 (2001), 38 - 62.

[22] G. Pisier, *Introduction to Operator Space Theory*, London Mathematical Society Lecture Note Series, 294, Cambridge University Press, 2003.

[23] D. Sarason, *On spectral sets having connected complement*, Acta. Sci. Math. 26 (1965), 289 - 299.

[24] B. Sz.-Nagy, *Sur les contractions dans ĺespace de Hilbert*, Acta Sci. Math. 15 (1953), 87 - 92.

[25] B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on a Hilbert Space*, North-Holland Publishing Company, 1970.

Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
E-mail address: `tirtha@math.iisc.ernet.in`

Indian Statistical Institute, R. V. College Post, Bangalore 560 059, India
E-mail address: `gm@isibang.ac.in`