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Equivalence of quotient Hilbert modules – II

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EQUIVALENCE OF QUOTIENT HILBERT MODULES – II

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ABSTRACT. For any open, connected and bounded set $\Omega \subseteq \mathbb{C}^m$, let \mathcal{A} be a natural function algebra consisting of functions holomorphic on Ω . Let \mathcal{M} be a Hilbert module over the algebra \mathcal{A} and $\mathcal{M}_0 \subseteq \mathcal{M}$ be the submodule of functions vanishing to order k on a hypersurface $\mathcal{Z} \subseteq \Omega$. Recently the authors have obtained an explicit complete set of unitary invariants for the quotient module $\mathcal{Q} = \mathcal{M} \ominus \mathcal{M}_0$ in the case of $k = 2$. In this paper, we relate these invariants to familiar notions from complex geometry. We also find a complete set of unitary invariants for the general case. We discuss many concrete examples in this setting. As an application of our equivalence results, we characterise homogeneous modules over the bi-disc algebra.

1. INTRODUCTION

One source of fascination in the study of operator theory is the wide variety of connections made with other branches of mathematics. Techniques from algebra, topology, geometry and analysis are used to understand bounded linear operators on Hilbert space. And, in many instances, the behavior and properties of the operators can be used to illustrate critical features and aspects of the other fields. This is particularly true in the case of multivariate operator theory, that is, when several operators or an algebra of operators is studied. Here the setting and results from these other areas can be quite sophisticated and the techniques used to understand multivariate operator theory often require additional development. Such is the focus of this paper.

Although the spectral theorem is a key tool in the study of self-adjoint and normal operators, there are large and important classes of naturally occurring operators to which this theory doesn't apply. Examples illustrating such phenomena can be obtained by considering multiplication operators on spaces of holomorphic functions on some domain in \mathbb{C}^m . For domains in \mathbb{C} , one is in the realm of single operator theory while it is multivariate operator theory for $m > 1$. If one considers the unit ball \mathbb{B}^m in \mathbb{C}^m and the Bergman space $A^2(\mathbb{B}^m)$ for it, one obtains a module over the polynomial algebra $\mathbb{C}[z]$, where $z = (z_1, \dots, z_m)$.

Techniques from complex geometry were shown in [6], [7], and [8] to be useful in studying such Hilbert modules. Closed submodules related to polynomial ideals were shown to reflect properties of the ideals and results in a rigidity phenomenon for such submodules [16]. In [10], it was shown that the study of quotient modules, determined by polynomial ideals, could

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also be reduced to the earlier work involving complex geometry if the ideal is principal and prime. As might be expected, the non-prime case is more complicated (cf. [14]) and some real technical difficulties arise in the complex geometry needed to handle its study. Overcoming these problems by developing new results and techniques in complex geometry is the main goal of this paper.

In a basic construction, Hilber modules, such as $A^2(\mathbb{B}^m)$, can be shown to yield a hermitian holomorphic vector bundle over the domain and this bundle characterizes the module up to unitary equivalence. For quotient modules by multiplicity-free principal ideals, a bundle still exists but over the intersection of the domain with the zero variety of the ideal. In [14] it was shown that for quotient modules obtained using submodules of higher multiplicity, there is still a higher rank hermitian holomorphic bundle but its bundle structure is not enough to characterize the quotient module. One must also involve the flag structure in the bundle defined by the module action which now involves nilpotents. But even that is not enough. In particular, we must consider the nilpotent structure itself. Classifying such objects requires introducing new ideas and techniques extending older ones from complex geometry which involve jet bundles and moving frames.

In the case of multiplicity two, earlier, somewhat ad hoc, results for classification were obtained in [11] but the treatment here is not only more conceptual, it works for all multiplicities, not just the case of multiplicity two. Moreover, we are able to reinterpret the third invariant obtained in [11] for the latter case, using the second fundamental form. However, the invariants for $k > 2$ seem to involve completely new ideas in complex geometry. Other possible interpretations of these results involve connections with integrability theorems and Chern -Moser theory and will be described in more detail in the final section of the paper.

1.1. Terminology and Setting. For any bounded open connected subset Ω of \mathbb{C}^m , let $\mathcal{A}(\Omega)$ be the completion, with respect to the supremum norm on the closure $\overline{\Omega}$ of the domain Ω , of functions holomorphic in a neighbourhood of $\overline{\Omega}$. The Hilbert space \mathcal{M} is said to be a *Hilbert module* over $\mathcal{A}(\Omega)$ if \mathcal{M} is a module over $\mathcal{A}(\Omega)$ with module map $\mathcal{A}(\Omega) \times \mathcal{M} \rightarrow \mathcal{M}$ defined by pointwise multiplication such that

$$\|f \cdot h\|_{\mathcal{M}} \leq C \|f\|_{\mathcal{A}(\Omega)} \|h\|_{\mathcal{M}} \quad \text{for } f \in \mathcal{A}(\Omega) \quad \text{and } h \in \mathcal{M},$$

for some positive constant C independent of f and h . It is said to be *contractive* if we also have $C \leq 1$.

The most natural example of a Hilbert module over an algebra $\mathcal{A}(\Omega)$ is a Hilbert space \mathcal{M} which (a) consists of holomorphic functions on Ω , (b) possesses a reproducing kernel K and (c) is closed under multiplication by $\mathcal{A}(\Omega)$. The crucial property of such a Hilbert space is that the evaluation functionals for points in Ω are continuous in the Hilbert space norm.

The Hardy spaces and the Bergman spaces over bounded domains Ω in \mathbb{C}^m form natural families of examples over $\mathcal{A}(\Omega)$. Other natural examples are obtained by considering submodules of such kernel Hilbert modules. Specifically, we recall Beurling's characterisation of a submodule \mathcal{M}_0 of the Hardy module $\mathcal{M} = H^2(\mathbb{D})$ which states that the submodule \mathcal{M}_0 must be of the form

$\theta\mathcal{M}$, where θ is an inner function. It is then not hard to see, using the von Neumann - Wold decomposition for isometries, that all of these submodules of the Hardy module are isomorphic. Never the less, the quotient modules $\mathcal{M} \ominus \mathcal{M}_0$, $\mathcal{M}_0 \subseteq \mathcal{M}$, are not all isomorphic. Indeed, one way of describing the model theory of Sz.-Nagy and Foias (cf. [25]) is to say that there is a large class of contractive modules that are isomorphic to such a quotient module. It is then natural to ask when two of these quotient modules are isomorphic and seek a complete set of unitary invariants. This is done in the work of Sz.-Nagy and Foias in the context of contractive modules over the disc algebra. In contrast to what happens in the case of the Hardy module for the disc, a great profusion of different examples of submodules of the Hardy module of the poly-disc \mathbb{D}^m as well as the Euclidian ball \mathbb{B}^m are obtained in this manner as the rigidity phenomenon demonstrates [16].

In a recent paper [13], we pointed out that much of what we proved there was valid for modules over algebras of holomorphic functions that are not complete; for example, $\mathbb{C}[z]$ or the functions holomorphic on the closure of the domain. This continues to be the case in this paper as well. However, we are actually forced to consider modules over non-complete algebras and module actions which are not even bounded in the supremum norm. In particular, we must localize some of our computations to smaller open subsets of the domain Ω . However, we will continue to state our principal results for Hilbert modules over the complete function algebra $\mathcal{A}(\Omega)$ even though a more general statement would be possible.

We will study quotient modules for a special class of Hilbert modules which includes the Hardy and Bergman modules. Recall that \mathcal{M} is said to be a *quasi-free Hilbert module of rank n* , $1 \leq n < \infty$, for $\mathcal{A}(\Omega)$, if it is a Hilbert space completion of $\mathcal{A}(\Omega) \otimes_{\text{alg}} \mathbb{C}^n$ (the algebraic tensor product) such that

- (1) evaluation e_w , $e_w(f) = f(w)$, is locally uniformly bounded for $w \in \Omega$,
- (2) pointwise multiplication by functions in $\mathcal{A}(\Omega)$ defines a bounded operator on \mathcal{M} , and
- (3) a sequence $\{f_i\}$ contained in $\mathcal{A}(\Omega)$, converges to 0 in \mathcal{M} in the norm of \mathcal{M} if and only if $\{e_w(f_i)\}$ converges to 0 in \mathbb{C}^n for all w in Ω (cf. [12], [13]).

These assumptions ensure, among other things, via the Riesz representation theorem, that there is a unique vector $K(\cdot, w) \in \mathcal{M}$ satisfying the reproducing property, that is,

$$h(w) = \langle h, K(\cdot, w) \rangle, \quad h \in \mathcal{M}, \quad w \in \Omega.$$

Clearly, the map $w \mapsto e_w$, which is defined on Ω and takes values in \mathcal{M} , is weakly holomorphic. Hence, e_w is locally uniformly bounded in norm and $K(w, w) = \langle e_w, e_w \rangle$ is locally uniformly bounded.

1.2. We are interested in submodules \mathcal{M}_0 contained in \mathcal{M} which consist of all functions in \mathcal{M} that vanish to some fixed order k on a hypersurface \mathcal{Z} contained in Ω . Before we can make this notion precise, however, we need some definitions.

DEFINITION 1.1 ([21, Definition 8, p. 17]).

- (1) A hypersurface is a complex sub-manifold of complex dimension $m - 1$, that is, a subset $\mathcal{Z} \subseteq \Omega$ is a hypersurface if for any fixed $z \in \mathcal{Z}$, there exists a neighbourhood $U \subseteq \Omega$ of z and a local defining function φ for $U \cap \mathcal{Z}$.
- (2) A local defining function φ is a holomorphic map $\varphi : U \rightarrow \mathbb{C}$ such that $U \cap \mathcal{Z} = \{z \in U : \varphi(z) = 0\}$ and $\frac{f}{\varphi}$ is holomorphic on U whenever $f|_{U \cap \mathcal{Z}} = 0$. In particular, this implies that the gradient of φ doesn't vanish on \mathcal{Z} and that any two defining functions for \mathcal{Z} must differ by a unit.
- (3) We say that the function f vanishes to order k on the hypersurface \mathcal{Z} if $f = \varphi^n g$ for some $n \geq k$, a holomorphic function g on U and a defining function φ of \mathcal{Z} . The order of vanishing on \mathcal{Z} of a holomorphic function $f : \Omega \rightarrow \mathbb{C}$ does not depend on the choice of the local defining function. This definition can also be framed in terms of the partial derivatives normal to \mathcal{Z} .

It is clear that if there exists a global defining function φ for the hypersurface \mathcal{Z} , then a valid choice of a normal direction is the gradient of the function φ , which is defined on all of Ω . In general, it may not be possible to find a global defining function for the hypersurface \mathcal{Z} . However, if the second Cousin problem is solvable for Ω , then there exists a global defining function (which we will again denote by φ) for the hypersurface \mathcal{Z} . This is pointed out in the remark preceding Corollary 3 in [21, p. 34].

Even if we don't impose the condition of "solvability of the second Cousin problem" on Ω , we may restrict the holomorphic functions in the algebra $\mathcal{A}(\Omega)$ and the module \mathcal{M} to the open set U without loss of generality. We will have more to say about this matter in Section .

1.3. Standing Assumption. For any fixed but arbitrary $u \in \Omega$, we may pick a small enough open neighborhood $U \subseteq \Omega$ of u such that $U \cap \mathcal{Z}$ admits a defining function, say φ , with gradient of φ not zero on $U \cap \mathcal{Z}$. Since U is open in Ω , it follows that the module \mathcal{M} and the submodule $\mathcal{M}_0 \subseteq \mathcal{M}$ are isomorphic to the module $\mathcal{M}|_{\text{res } U}$ and the submodule $(\mathcal{M}_0)|_{\text{res } U} \subseteq \mathcal{M}|_{\text{res } U}$ of functions in $\mathcal{M}|_{\text{res } U}$ which vanish on $U \cap \mathcal{Z}$, respectively. Consequently, if we choose to work with the later pair of modules, then the corresponding quotient module $\mathcal{M}|_{\text{res } U} \ominus (\mathcal{M}_0)|_{\text{res } U}$ is isomorphic to the quotient module $\mathcal{M} \ominus \mathcal{M}_0$. *Therefore we may cut down, if necessary, the domain Ω to a suitable small open subset $U \subseteq \Omega$ and work with the smaller open set U and the hypersurface $\mathcal{Z} \cap U \subseteq U$ without loss of generality.*

1.4. Now fix a hypersurface $\mathcal{Z} \subseteq \Omega$ and let $\mathcal{M}_0 \subseteq \mathcal{M}$ be the submodule of the quasi-free Hilbert module \mathcal{M} of rank 1 consisting of those functions in \mathcal{M} which vanish to order k on the hypersurface \mathcal{Z} . The quotient $\mathcal{Q} = \mathcal{M} \ominus \mathcal{M}_0$ is a Hilbert module over $\mathcal{A}(\Omega)$, where the module action is naturally defined as $f \cdot (h + \mathcal{M}_0) = f \cdot h + \mathcal{M}_0$ (cf. [15, Definition 2.2]). In other words,

$$(1.1) \quad 0 \longleftarrow \mathcal{Q} \longleftarrow \mathcal{M} \longleftarrow \mathcal{M}_0 \longleftarrow 0,$$

is an exact sequence of Hilbert modules over $\mathcal{A}(\Omega)$, where \longleftarrow is the quotient map and \hookleftarrow is the inclusion map. It is then possible to obtain geometric invariants for the quotient module \mathcal{Q} using $\mathcal{M} \longleftarrow \mathcal{M}_0$ (cf. [10, Theorem 1.4]).

In [10] the Hilbert module \mathcal{M} is assumed to be in the class $B_1(\Omega^*)$ (see section 2.1), where $\Omega^* = \{w \in \mathbb{C}^m : \bar{w} \in \Omega\}$. The submodule \mathcal{M}_0 is taken to be the maximal set of functions vanishing on an analytic hypersurface $\mathcal{Z} \subseteq \Omega$. This ensures that there is a reproducing kernel for \mathcal{M} which in turn defines a holomorphic hermitian line bundle E on the domain Ω^* . The equivalence class of the bundle E and that of the module \mathcal{M} determine each other which is one of the main results in [6]. Two descriptions are then provided for the quotient module $\mathcal{Q} = \mathcal{M} \ominus \mathcal{M}_0$. The first approach characterizes it as the reproducing kernel Hilbert space of holomorphic functions obtained by restricting the functions on Ω to the hypersurface \mathcal{Z} . In the second approach, a hermitian holomorphic line bundle on \mathcal{Z}^* is associated with the quotient module – this is the restriction of the bundle E to the hypersurface \mathcal{Z}^* . A complete unitary invariant for the quotient module is then the curvature of this line bundle which is easily seen to be the restriction and projection of the curvature for E to \mathcal{Z}^* .

The submodule \mathcal{M}_0 in [14] is taken to be the (maximal) set of functions which vanish to some given order k on the hypersurface \mathcal{Z} . As in the previous case, two descriptions are provided for the quotient module. A matrix – valued kernel function must now be used and, in the vector bundle picture, we have a rank k hermitian, anti-holomorphic vector bundle over \mathcal{Z}^* . Some invariants for the quotient module (though not a complete set) are described in [14].

Finally, in the paper [11], a complete set of unitary invariants is obtained for the particular case where \mathcal{M}_0 consists of functions vanishing to order 2 on the hypersurface. While two of the three invariants obtained there consist of coefficients of the curvature form for the jet bundle for E , the third, which we called the “angle”, seemed to be not so familiar. In this note, we show that the angle invariant can be replaced by the second fundamental form corresponding to the inclusion of the bundle E in the jet bundle $J^{(2)}E$.

One of our main goals is to obtain invariants that are complete, computable and natural in complex geometry – for general k , not just in the case $k = 2$. The fundamental class of the variety \mathcal{Z} is one such invariant. In [10, Theorem 1.4], the fundamental class of the hypersurface \mathcal{Z} was expressed using the curvatures of the pair of modules \mathcal{M} and the submodule \mathcal{M}_0 and the localization of the inclusion map $\mathcal{M}_0 \hookrightarrow \mathcal{M}$. Here we give a complete set of invariants for the equivalence of the quotient modules. We provide a number of applications of our results to the context of homogeneous operators. Moreover, we discuss various aspects of global versus local differences in connection with the jet bundle construction. Finally, we describe some relations between the new invariants we introduce and several other topics relating to the moving frames of Cartan and integrability conditions in Chern-Moser theory.

2. REPRODUCING KERNELS AND THE MULTIVARIATE CLASS B_k

2.1. Let $\mathcal{L}(\mathbb{F})$ be the Banach space of all linear transformations on a Hilbert space \mathbb{F} of dimension n for some $n \in \mathbb{N}$. Let \mathcal{H} be a Hilbert space of functions from Ω to \mathbb{F} . For $w \in \Omega$, let $e_w : \mathcal{H} \rightarrow \mathbb{F}$ be defined by $e_w(f) = f(w)$. The Hilbert space \mathcal{H} is called a (vector-valued) functional Hilbert space if e_w is bounded for each $w \in \Omega$. In this case, the function

$K : \Omega \times \Omega \rightarrow \mathcal{L}(\mathbb{F})$ defined by $K(z, w) = e_z e_w^*$, $z, w \in \Omega$, is called the reproducing kernel of \mathcal{H} . We recall some of the basic properties of a reproducing kernel following [1].

First, the kernel K has the reproducing property:

$$(2.1) \quad \langle f, K(\cdot, w)\eta \rangle_{\mathcal{H}} = \langle f(w), \eta \rangle_{\mathbb{F}} \text{ for } \eta \in \mathbb{F}, w \in \Omega, f \in \mathcal{H}.$$

In particular, taking $f = K(\cdot, w)\zeta$ for $w \in \Omega$, $\zeta \in \mathbb{F}$, we see that K satisfies

$$(2.2) \quad \langle K(\cdot, w)\zeta, K(\cdot, z)\eta \rangle = \langle K(z, w)\zeta, \eta \rangle \text{ for } \zeta, \eta \in \mathbb{F}, z, w \in \Omega.$$

This shows for $p \geq 1$, $w_1, \dots, w_p \in \Omega$ that the block operator $\left(K(w_i, w_j) \right)_{1 \leq i, j \leq p}$ on $\mathbb{F} \oplus \dots \oplus \mathbb{F}$ (p copies) is positive. Conversely, if $K : \Omega \times \Omega \rightarrow \mathcal{L}(\mathbb{F})$ satisfies this positivity requirement for all p -tuples in Ω , one can see that there is a unique functional Hilbert space with reproducing kernel K . (It is the completion of the linear span of the functions $K(\cdot, z)\eta$ for $z \in \Omega, \eta \in \mathbb{F}$, with inner product given by (2.2).)

For $1 \leq i \leq m$, suppose that the operators $M_i : \mathcal{H} \rightarrow \mathcal{H}$ defined by $(M_i f)(z) = z_i f(z)$, $z \in \Omega, f \in \mathcal{H}$ are bounded. Then it is easy to verify that for each fixed $w \in \Omega$, and $1 \leq i \leq m$,

$$(2.3) \quad M_i^* K(\cdot, w)\eta = \bar{w}_i K(\cdot, w)\eta \text{ for } \eta \in \mathbb{F}.$$

Differentiating (2.1) we also obtain the following extension of the reproducing property:

$$(2.4) \quad \langle (\partial_i^j f)(w), \eta \rangle = \langle f, \bar{\partial}_i^j K(\cdot, w)\eta \rangle \text{ for } 1 \leq i \leq m, j \geq 0, w \in \Omega, \eta \in \mathbb{F}, f \in \mathcal{H}.$$

Let \mathbf{M} denote this commuting m -tuple of multiplication operators defined by the coordinate functions z_1, \dots, z_m and \mathbf{M}^* be the m -tuple (M_1^*, \dots, M_m^*) . It then follows from (2.3) that the joint eigenspace of the m -tuple \mathbf{M}^* at $w \in \Omega^* \subseteq \mathbb{C}^m$, where as before, $\Omega^* = \{w \in \mathbb{C}^m : \bar{w} \in \Omega\}$, contains the n -dimensional subspace $\text{ran } K(\cdot, \bar{w}) \subseteq \mathcal{H}$.

Suppose K is the reproducing kernel of a Hilbert space \mathcal{H} consisting of \mathbb{F} -valued *analytic functions on Ω* . Then K is analytic in the first argument (and hence co-analytic in the second argument). We now obtain a holomorphic vector bundle E on the base space Ω^* by requiring that $\{K(\cdot, w)v : v \in B\} \subseteq \mathcal{H}$, where B is an orthonormal basis for \mathbb{F} , be a frame at $\bar{w} \in \Omega^*$. We will also assume that $K(w, w)$ is an invertible operator for each $w \in \Omega$. Then $K(w, w)$ defines a hermitian metric for the bundle E . The assumption that $K(w, w)$ is invertible is automatic if \mathcal{H} is a quasi-free Hilbert module of finite rank n .

Before proceeding any further, we recall the class $B_k(\Omega)$ for $\Omega \subset \mathbb{C}$ and $k \in \mathbb{N}$, which was introduced in [6]. It consists of those operators T on a Hilbert space \mathcal{H} for which each $w \in \Omega$ is an eigenvalue of uniform multiplicity k , the eigenvectors span the Hilbert space \mathcal{H} and $\text{ran}(T - wI_{\mathcal{H}}) = \mathcal{H}$ for $w \in \Omega$. Later the definition was adapted to the case of an m -tuple of commuting operators \mathbf{T} acting on a Hilbert space \mathcal{H} , first in the paper [7] and then in the paper [9] from a slightly different point of view which emphasized the role of the reproducing kernel.

For $w \in \Omega \subseteq \mathbb{C}^m$, the m -tuple \mathbf{T} is in $B_k(\Omega)$ if

- (1) $\text{ran } D_{\mathbf{T}-w}$ is closed for all $w \in \Omega$, where $D_{\mathbf{T}} : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathbb{C}^k$ is defined by $D_{\mathbf{T}}h = (T_1 h, \dots, T_m h)$, $h \in \mathcal{H}$;

- (2) $\text{span } \{\ker D_{\mathbf{T}-w} : w \in \Omega\}$ is dense in \mathcal{H} and
- (3) $\dim \ker D_{\mathbf{T}^*-w} = k$ for all $w \in \Omega$.

It was then shown that each of these operator m -tuples \mathbf{T} determines a hermitian holomorphic vector bundle E of rank k on Ω and that two m -tuples of operators in $B_k(\Omega)$ are unitarily equivalent if and only if the corresponding bundles are locally equivalent. In the case $k = 1$, this is a question of equivalence of hermitian holomorphic line bundles. It is, of course, well-known that two such line bundles are equivalent if and only if their curvatures are equal. However, no such simple characterization is available if $\text{rank } E = k > 1$. We explain this difference in section 2.3 below.

We now recall that for the module \mathcal{M} over the algebra $\mathcal{A}(\Omega)$, the coordinate functions define an m -tuple of bounded multiplication operators \mathbf{M} . We have already observed in (2.3), that the joint eigenspace of \mathbf{M}^* at \bar{w} , $w \in \Omega$ includes the subspace $\{K(\cdot, w)\zeta : w \in \Omega, \zeta \in \mathbb{C}^m\} \subseteq \mathcal{H}$. Consequently, the dimension of the joint eigenspace of \mathbf{M}^* at \bar{w} , $w \in \Omega$, is at least k and together all the eigenspaces span the Hilbert space \mathcal{M} . Moreover, if \mathcal{M} is a quasi-free Hilbert module of rank k , then the joint eigenspaces have dimension k for $w \in \Omega$. Finally, the latter statement in the definition of quasi-free, however, implies that the span of the ranges of the operators $D_{(\mathbf{M}-w)^*}$ is dense for $w \in \Omega$. All of these facts don't ensure that the m -tuple \mathbf{M}^* lies in the class $B_k(\Omega^*)$ since we don't know the range of $D_{(\mathbf{M}-w)}$ is closed. This issue is discussed in a recent paper [13] where the notion of a quasi-free module was introduced. While most of our examples lie in the class $B_k(\Omega)$, our methods work even if we make the weaker hypothesis that the modules we consider are quasi-free of rank k over the algebra $\mathcal{A}(\Omega)$.

2.2. We recall some basic notions from complex geometry following Wells [26, Chapter III]. If E is a hermitian holomorphic vector bundle, then there is a canonical connection D on the bundle E which is compatible with both the holomorphic and hermitian structures. The curvature \mathcal{K} of the bundle E is then simply defined to be $D \circ D$. Let us provide some more details.

Let E be a hermitian holomorphic vector bundle of rank k over a complex manifold M and $C_p^\infty(M, E)$ be the space of smooth p -forms on M whose coefficients are smooth sections of E . A connection on the bundle E is a differential operator $D : C_p^\infty(M, E) \rightarrow C_{p+1}^\infty(M, E)$ of order 1 satisfying

$$(2.5) \quad D(f \wedge s) = df \wedge s + (-1)^p f \wedge Ds$$

for any $f \in C_p^\infty(M, \mathbb{C})$ and $s \in C_p^\infty(M, E)$, where df stands for the usual exterior derivative of f .

Assume that $\theta : E|_U \rightarrow U \times \mathbb{C}^k$ is a trivialization of E over some open subset U of M . Let (s_1, \dots, s_k) be the corresponding frame of $E|_U$. Then any $s \in C_p^\infty(M, E)$ can be written uniquely as

$$s = \sum_j \sigma_j \otimes s_j, \quad \sigma_j \in C_p^\infty(U, \mathbb{C}), \quad 1 \leq j \leq k.$$

Using the hermitian structure h of E , we can define a natural sesqui-linear map

$$\begin{aligned} C_p^\infty(M, E) \times C_q^\infty(M, E) &\rightarrow C_{p+q}^\infty(M, \mathbb{C}) \\ (s_1, s_2) &\mapsto \{s_1, s_2\} \end{aligned}$$

combining the wedge product of forms with the hermitian metric on E . If $s = \sum_j \sigma_j \otimes s_j$ and $\tilde{s} = \sum_j \tilde{\sigma}_j \otimes s_j$, then

$$\{s, \tilde{s}\} = \sum_{j, \ell} \sigma_j \wedge \bar{\sigma}_\ell h(s_j, s_\ell).$$

The curvature tensor \mathcal{K} associated with the canonical connection D is in $C_{1,1}^\infty(M, \text{herm}(E, E))$. (Here $C_{p,q}^\infty$ represents the space of forms of degree p in the holomorphic differentials and q in the anti-holomorphic ones.) Moreover, if h is a local representation of the metric in some open set, then $\mathcal{K} = \bar{\partial}(h^{-1}\partial h)$ [26, page 82].

2.3. We digress a little to explain the difference between the equivalence of line bundles and bundles of rank $k > 1$ in our context. Suppose that E is a holomorphic line bundle over the domain $\Omega \subseteq \mathbb{C}^m$ with a hermitian metric $g(w) = \langle \gamma_w, \gamma_w \rangle$, where γ is a holomorphic frame. The curvature \mathcal{K}_E is given by the formula $\mathcal{K}_E(w) = (\frac{\partial^2}{\partial w \partial \bar{w}} \log g)(w)$, for $w \in \Omega$ (see (2.6)). Clearly, in this case, $\mathcal{K}(w) \equiv 0$ on Ω is the same as saying that $\log g$ is harmonic on Ω . Let F be a second line bundle over the same domain Ω with the metric h with respect to a holomorphic frame η . Suppose that the two curvatures \mathcal{K}_E and \mathcal{K}_F are equal. It then follows that $u = \log(g/h)$ is harmonic on Ω and thus there exists a harmonic conjugate v of u on any simply connected open subset Ω_0 of Ω . For $w \in \Omega_0$, define $\tilde{\eta}_w = e^{(u(w)+iv(w))/2} \eta_w$. Then clearly, $\tilde{\eta}_w$ is a new holomorphic frame for F , which we can use without loss of generality. Consequently, we have the metric $h(w) = \langle \tilde{\eta}_w, \tilde{\eta}_w \rangle$ for F and we see that

$$\begin{aligned} h(w) &= \langle \tilde{\eta}_w, \tilde{\eta}_w \rangle \\ &= \langle e^{(u(w)+iv(w))/2} \eta_w, e^{(u(w)+iv(w))/2} \eta_w \rangle \\ &= e^{u(w)} \langle \eta_w, \eta_w \rangle \\ &= g(w). \end{aligned}$$

This calculation shows that the map $\eta_w \mapsto \gamma_w$ defines an isometric holomorphic bundle map between E and F .

Now, if the bundles arise from Hilbert modules, then they are defined as pullbacks from Grassmanians. Thus the Rigidity Theorem [6, Theorem 2.2] ensures that it extends to a unitary map of the corresponding Hilbert spaces which yields an isometric isomorphism of the Hilbert modules. Conversely, if the Hilbert modules are equivalent by means of a unitary module map V and γ is a frame for the first one, then $V\gamma$ defines a holomorphic frame for the second one. Since the corresponding metrics are equal, the curvatures agree also.

In case the rank of the bundle E is $k > 1$, the curvature \mathcal{K}_E is given by the formula (2.6), that is, $\mathcal{K} = \bar{\partial}\{\varrho^{-1}\partial\varrho\}$, where $\varrho(w) = (\langle \gamma_j(w), \gamma_i(w) \rangle)_{i,j=1}^k$ is the matrix of the metric with respect to the holomorphic frame $\{\gamma_1(w), \dots, \gamma_k(w)\}$ for w in some open subset Ω_0 of Ω . In this case, if the two bundles E and F are equivalent then it is not hard to see that the coefficient

matrices $((\mathcal{K}_E)_{ij}(z))$ and $((\mathcal{K}_F)_{ij}(z))$ for $dz_i \wedge d\bar{z}_j$ of the corresponding curvatures are similar, that is, $((\mathcal{K}_E)_{ij}(z)) = L^{-1}((\mathcal{K}_F)_{ij}(z))L$ for some invertible $k \times k$ matrix L . However, the converse question is more complicated and is discussed in the papers [6, Theorem 3.17] and [8, Theorem II, page 76].

2.4. To study quotient Hilbert modules, we must consider the behavior of the holomorphic tangent bundle to Ω relative to an analytic hypersurface. The holomorphic tangent bundle $T\Omega|_{\text{res } \mathcal{Z}}$ naturally splits as $T\mathcal{Z} \oplus N\mathcal{Z}$, where $N\mathcal{Z}$ is the normal bundle. It can be identified with the quotient $T\Omega|_{\text{res } \mathcal{Z}}/T\mathcal{Z}$. The co-normal bundle $N^*\mathcal{Z}$ is the dual of $N\mathcal{Z}$; it is the sub-bundle of $T^*\Omega|_{\text{res } \mathcal{Z}}$ consisting of cotangent vectors that vanish on $T\mathcal{Z} \subseteq T\Omega|_{\text{res } \mathcal{Z}}$. Indeed, there is an easy formula for the co-normal bundle of a smooth hypersurface which we describe now following [20, page 145].

Suppose \mathcal{Z} is given by local defining functions φ_z on $U_z \subseteq \Omega$, $z \in \Omega$, as in Definition 1.1. The line bundle $[\mathcal{Z}]$ defined on Ω is then given by transition functions $\{\psi_{zw} = \frac{\varphi_z}{\varphi_w} : z, w \in \Omega\}$ on $U_z \cap U_w$. By definition, $\varphi_z \equiv 0$ on $U_z \cap \mathcal{Z}$. It follows that the differential $d\varphi_z$ is a section of the co-normal bundle $N^*\mathcal{Z}$. Besides, $d\varphi_z$ is holomorphic and nonzero everywhere. On $U_z \cap U_w \cap \mathcal{Z}$, we have $d\varphi_z = \psi_{zw}d\varphi_w$, that is, $d\varphi_z$ defines a nonzero global section of the bundle $N^*\mathcal{Z} \otimes [\mathcal{Z}]$. Thus $N^*\mathcal{Z} \otimes [\mathcal{Z}]$ is the trivial bundle which gives the formula $N^*\mathcal{Z} = [-\mathcal{Z}]|_{\text{res } \mathcal{Z}}$, where $[-\mathcal{Z}]$ is the inverse of the line bundle $[\mathcal{Z}]$. This is the Adjunction Formula I [20, page 146].

In the following calculation, we assume that $\mathcal{Z} = \{z_1 = 0\}$. We show in subsection 3.1 that there is no loss of generality in doing so. Let $P_1 : T^*\Omega|_{\text{res } \mathcal{Z}} \rightarrow N^*\mathcal{Z}$ be the bundle map which is the projection onto $N^*\mathcal{Z}$ and $P_2 = (1 - P_1) : T^*\Omega|_{\text{res } \mathcal{Z}} \rightarrow T^*\mathcal{Z}$ be the bundle map which is the projection onto $T^*\mathcal{Z}$. Now, we have a splitting of the $(1, 1)$ forms as follows:

$$\wedge^{(1,1)} T^*\Omega|_{\text{res } \mathcal{Z}} = \sum_{i,j=1}^2 P_i \left(\wedge^{(1,0)} T^*\Omega|_{\text{res } \mathcal{Z}} \right) \wedge P_j \left(\wedge^{(0,1)} T^*\Omega|_{\text{res } \mathcal{Z}} \right).$$

Accordingly, we have the component of the curvature along the transverse direction to \mathcal{Z} which we denote by $\mathcal{K}_{\text{trans}}$. Clearly, $\mathcal{K}_{\text{trans}} = (P_1 \otimes I)\mathcal{K}|_{\text{res } \mathcal{Z}}$. Similarly, let the component of the curvature along tangential directions to \mathcal{Z} be \mathcal{K}_{tan} . Again, $\mathcal{K}_{\text{tan}} = (P_2 \otimes I)\mathcal{K}|_{\text{res } \mathcal{Z}}$. (Here I is the identity map on the vector space $\text{herm}(E, E)$.) In local coordinates, the curvature of E at $w \in \Omega^*$ is given by

$$\begin{aligned} \mathcal{K}(w) &= - \sum_{i,j=1}^m \bar{\partial}_i \left(K(w, w)^{-1} \partial_j K(w, w) \right) d\bar{w}_i \wedge dw_j \\ &= \mathcal{K}_{11}(w) d\bar{w}_1 \wedge dw_1 + \sum_{j=2}^m \mathcal{K}_{1j}(w) d\bar{w}_1 \wedge dw_j \\ &\quad + \sum_{i=2}^m \mathcal{K}_{i1}(w) d\bar{w}_i \wedge dw_1 + \sum_{i,j=2}^m \mathcal{K}_{ij}(w) d\bar{w}_i \wedge dw_j \\ (2.6) \quad &= \begin{pmatrix} \mathcal{K}_{\text{tan}}(w) & \mathcal{S}(w) \\ -\bar{\mathcal{S}}(w) & \mathcal{K}_{\text{trans}}(w) \end{pmatrix} \begin{pmatrix} d\bar{w}' \\ d\bar{w}_1 \end{pmatrix} \wedge \begin{pmatrix} dw' \\ dw_1 \end{pmatrix}, \end{aligned}$$

where $\partial_i = \frac{\partial}{\partial w_i}$, $\bar{\partial}_j = \frac{\partial}{\partial \bar{w}_j}$ and $dw' = \begin{pmatrix} dw_m \\ \vdots \\ dw_2 \end{pmatrix}$. Also, we let $\mathcal{S}(w)$, which appears in the $(1, 2)$ position of the decomposition for the curvature, denote the $(1, 1)$ form $\sum_{j=2}^m \mathcal{K}_{1j}(w) d\bar{w}_1 \wedge dw_j$.

2.5. Earlier, we showed how for the case of line bundles we could calculate the curvature from the modulus of a section and hence determine the line bundle up to equivalence. For quotient modules obtained from multiplicity-free submodules, we are interested in the line bundle restricted to the hypersurface. The approach needed here, while related to the earlier one, is somewhat different. Let us describe it briefly here. A generalization of it will be investigated later.

Recall that if $(\mathcal{K}_E)_{ij}(w) = (\mathcal{K}_F)_{ij}(w)$ for $w \in \mathcal{Z}$ and $2 \leq i, j \leq m$, then the restrictions of the two bundles E and F to the hypersurface \mathcal{Z} are equivalent [10, Theorem 1.3]. In other words, $g(w) = |u(w)|^2 h(w)$ for some holomorphic function u on \mathcal{Z} and $w \in \mathcal{Z}$. Let \mathcal{Q} be the quotient module, as in (1.1), corresponding to the submodule \mathcal{M}_0 consisting of all those functions in \mathcal{M} which vanish on the hypersurface \mathcal{Z} . We showed in [10] that the restriction of just the tangential curvature \mathcal{K}_{tan} to the hypersurface \mathcal{Z} determines the quotient module up to unitary equivalence. If we make the stronger assumption of equality of all coefficients of the curvature on \mathcal{Z} , then the quotient modules can again be shown to be equivalent in the case of $k = 2$ as is pointed out in Remark 5.1 below. (This is a key step in the reformulation of our earlier equivalence result for this case.) In this paper, we generalize this result to the case $k > 2$ by introducing a certain $k \times k$ matrix D_k of partial differential operators so that the restriction of $D_k \log \varrho$ to \mathcal{Z} determines the equivalence of the corresponding quotient modules. However, at this point we have no standard complex geometric interpretation of this characterization.

3. JET BUNDLES RELATIVE TO A HYPERSURFACE

3.1. We now consider in some detail the construction of the jet bundles needed to characterize quotient modules of higher multiplicity.

Suppose, to begin with, we have a quasi-free Hilbert module \mathcal{M} of finite rank k over $\mathcal{A}(\Omega)$ with kernel function K on Ω and corresponding hermitian holomorphic vector bundle $E_{\mathcal{M}}$. It is easy to see that if U is any open set in Ω and $\mathcal{A}_{|\text{res } U}(\Omega) = \{f|_{\text{res } U} : f \in \mathcal{A}(\Omega)\}$ is the restriction algebra, then $\mathcal{M}_{|\text{res } U}$, the restriction of the functions in the module \mathcal{M} to the open set U , is naturally a module over the algebra $\mathcal{A}_{|\text{res } U}(\Omega)$ and this module is isomorphic to \mathcal{M} . (Note, in general, $\mathcal{A}_{|\text{res } U}(\Omega)$ is not a function algebra since it may not be complete.) So we can restrict all our discussion, without loss of generality, to any open subset U of Ω . In particular, we don't need to take the domain to be as large as possible. Thus our treatment will be local. In Section 7 we will discuss some of the global aspects of our treatment of jet bundles.

The jet bundle construction introduced in [14] involves the kernel function K and differentiation along the normal to the hypersurface $\mathcal{Z} \subseteq \Omega$. We will attempt to recall the essential ideas involved as succinctly as possible but still our description of the jet bundle will require us to repeat substantial material from the earlier paper.

Let us construct the jet bundle $J^{(k)}E$ over some open subset U of Ω which intersects \mathcal{Z} . Here we assume that \mathcal{M} is rank one and make essential use of the fact that the line bundle E is given as a pullback from the Grassmanian defined by \mathcal{M} . In particular, this means that a holomorphic section for E over Ω can be viewed as arising from a holomorphic function from Ω to \mathcal{M} . Now one takes U so small that $U \cap \mathcal{Z}$ equals the zero set of a holomorphic function φ on U and the gradient of φ doesn't vanish on U .

A normal direction to \mathcal{Z} in $U \cap \mathcal{Z}$ is then given by the gradient of φ . By choosing to reorder the coordinates and possibly cutting down the size of U , we can assume that $\frac{\partial}{\partial z_1}\varphi \neq 0$ on U . It then follows that $\lambda_1 = \varphi(z)$, $\lambda_2 = z_2, \dots, \lambda_m = z_m$ for $z \in \mathcal{Z}$ defines a local coordinate system for U . As pointed out in [14, p. 368 - 369], $\frac{\partial^\ell f}{\partial z_1^\ell}(z) = 0$ for $z \in U \cap \mathcal{Z}$, $0 \leq \ell \leq k-1$ if and only if $\frac{\partial^\ell f}{\partial \lambda_1^\ell}(\lambda) = 0$ for $\lambda \in V \cap \phi(\mathcal{Z})$, $0 \leq \ell \leq k-1$, where $\phi(z) = (\varphi(z), z_2, \dots, z_m)$ and $V = \phi(U)$. Then the submodule \mathcal{M}_0 (cf. [14, (1.5)]) consisting of those functions in \mathcal{M} which vanish to order k on the hypersurface \mathcal{Z} may be described as

$$\mathcal{M}_0 = \{f \in \mathcal{M} : \frac{\partial^\ell f}{\partial z_1^\ell}(z) = 0, \quad z \in U \cap \mathcal{Z}, \quad 0 \leq \ell \leq k-1\}.$$

In the new coordinate system $\phi(z) = (\varphi(z), z_2, \dots, z_m)$, differentiation along the normal to the hypersurface coincides with $\partial_1 = \frac{\partial}{\partial z_1}$. To construct the jet bundle $J^{(k)}E$ on U , let us take a frame for E , that is, a non-zero holomorphic section s for the line bundle E on U . Thus we can assume without loss of generality, that s is defined and non-zero on all of U . (Recall we can view s as a holomorphic function from U to \mathcal{M} .) The jet bundle $J^{(k)}E$ over U is now simply the bundle determined by the holomorphic frame $\{s, \partial_1 s, \dots, \partial_1^{k-1} s\}$ on the open set U . (Here, the section s is viewed as a holomorphic function from U to \mathcal{M} and the differentiation of s is the usual differentiation of the holomorphic function s .)

It is clear that the normal direction we pick in this manner is not unique. Thus the construction of the jet bundle $J^{(k)}E$, even on an open subset U of Ω , depends on the choice of a normal direction, and hence on the defining function. It follows that the normal directions obtained from the different defining functions for $U \cap \mathcal{Z}$ give rise to distinct jet bundles. However, [14, Proposition 2.4] shows that these bundles coincide on $U \cap \mathcal{Z}$ modulo holomorphic hermitian equivalence. Hence, we may proceed by assuming, without loss of generality, that the normal direction to the hypersurface \mathcal{Z} is z_1 by making a holomorphic change of coordinates mapping the set $U \cap \mathcal{Z}$ to $\{z_1 = 0\} \subseteq V \cap \Omega$ for some open subset $V \subseteq \Omega$. Now on U , and using φ , we can define a kernel function

$$JK(z, w) = (\langle (\partial_1^i s)(z), (\partial_1^j s)(w) \rangle_{\mathcal{M}})_{i,j=0}^{k-1}$$

on U . However, the kernel function $J^{(k)}K$ depends on the choice of U and φ , but the relationship between the kernel functions obtained for different choices of sub-domains and defining functions is particularly simple on \mathcal{Z} . Further, if we restrict JK to \mathcal{Z} , or actually the intersection of this set with U , then we obtain a kernel function which defines a Hilbert space canonically isomorphic to the quotient space \mathcal{Q} . In Section 7, we will discuss further the global versus local nature of the jet bundle.

3.2. In this subsection, we first recall the “change of variable formula” for the jet bundle. We then define an action of the algebra $\mathcal{A}(\Omega)$ on the holomorphic sections of the jet bundle. We use the hermitian structure of the jet bundle to define an inner product on the linear space of holomorphic sections of the jet bundle. This is then identified as a positive definite kernel on Ω . We then discuss a notion of equivalence of the jet bundles along with a similar notion of equivalence for the corresponding module of holomorphic sections.

Let us examine more closely the relationship between the jet bundle defined by different choices of defining function on an open set U . First, We recall (cf. [14]) the construction of the jet bundle $J^{(k)}E$ starting with a holomorphic hermitian line bundle E over U . Let s_0 and s_1 be holomorphic frames for E on the coordinate patches $U_z \subseteq U$ and $U_w \subseteq U$, respectively. That is, s_0 (respectively s_1) is a non-vanishing holomorphic section of E on U_z (respectively U_w). Then there is a non-vanishing holomorphic function g on $U_z \cap U_w$ such that $s_0 = gs_1$ there. Let ε_p , $p = 1, \dots, k$ be the standard basis vectors for \mathbb{C}^k . For $\ell = 0, 1$, we let $J(s_\ell) = \sum_{j=0}^{k-1} \frac{\partial_1^j s_\ell}{\partial_1 w^j} \varepsilon_{j+1}$. An easy computation shows that $J s_0$ and $J s_1$ transform on $U_z \cap U_w$ by the rule $J(s_0) = (\mathcal{J}g)J(s_1)$, where \mathcal{J} is the lower triangular operator matrix

$$(3.1) \quad \mathcal{J} = \begin{pmatrix} 1 & \dots & \dots & \dots & \dots & 0 \\ \partial_1 & 1 & & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & \binom{\ell}{j} \partial_1^{\ell-j} & & 1 & & \vdots \\ \vdots & & & & \ddots & 0 \\ \partial_1^{k-1} & \dots & \dots & \dots & \dots & 1 \end{pmatrix}$$

with $0 \leq \ell, j \leq k-1$.

The components of $J s$, that is, $s, \partial_1 s, \dots, \partial_1^{k-1} s$, determine a frame for a rank k holomorphic vector bundle $J^{(k)}E$ on U . The transition function with respect to this frame is represented by the matrix $(\mathcal{J}g)^{\text{tr}}$, which is just the transpose of the matrix $(\mathcal{J}g)$. We will refer to this bundle $J^{(k)}E$ over U as the k th order jet bundle of the bundle E . The hermitian metric $\varrho(w) = \langle s(w), s(w) \rangle_E$ on E with respect to the frame s on E induces a hermitian metric $J\varrho$ on $J^{(k)}E$ such that with respect to the frame $J s$,

$$(3.2) \quad (J\varrho)(w) = \begin{pmatrix} \varrho(w) & \dots & (\partial^{k-1} \varrho)(w) \\ & \ddots & \\ \vdots & (\partial_1^\ell \bar{\partial}_1^m \varrho)(w) & \vdots \\ & & \ddots & \\ (\bar{\partial}_1^{k-1} \varrho)(w) & \dots & (\partial_1^{k-1} \bar{\partial}_1^{k-1} \varrho)(w) \end{pmatrix}.$$

Now, for any Hilbert module \mathcal{M} over the function algebra $\mathcal{A}(\Omega)$ and $h \in \mathcal{M}|_U$, let

$$\mathbf{h} = \sum_{\ell=0}^{k-1} \partial_1^\ell h \otimes \varepsilon_{\ell+1}$$

and $J(\mathcal{M}|_U) = \{\mathbf{h} : h \in \mathcal{M}|_U\} \subseteq \mathcal{M} \otimes \mathbb{C}^k$. Consider the map $J : \mathcal{M}|_U \rightarrow \mathcal{M} \otimes \mathbb{C}^k$ defined by $Jh = \mathbf{h}$, for $h \in \mathcal{M}|_U$. Let $J^{(k)}\mathcal{M}$ denote the module $J(\mathcal{M}|_U)$. Since J is injective, we can define an inner product on $J^{(k)}\mathcal{M}$

$$\langle J(g), J(h) \rangle_{J(\mathcal{M})} = \langle g, h \rangle_{\mathcal{M}}$$

so as to make J unitary. We point out that the module action on $J^{(k)}\mathcal{M}$ is no longer pointwise multiplication but the one that ensures J is a module map.

PROPOSITION 3.1 ([14, page 378]). *The reproducing kernel $JK : U \times U \rightarrow \mathcal{M}_k(\mathbb{C})$ for the Hilbert space $J^{(k)}\mathcal{M}$ is given by the formula:*

$$(JK)_{\ell,j}(z, w) = (\partial_1^\ell \bar{\partial}_1^j K)(z, w), \quad z, w \in U, \quad 0 \leq \ell, j \leq k-1,$$

where $\bar{\partial}_1 = \frac{\partial}{\partial \bar{w}_1}$ and $\partial_1 = \frac{\partial}{\partial z_1}$ as before.

To complete the description of the Hilbert module $J^{(k)}\mathcal{M}$, we will have to transport the action of the algebra $\mathcal{A}(\Omega)$ from \mathcal{M} to $J^{(k)}\mathcal{M}$ via the map J . The resultant action is described in [14, Lemma 3.2] which we recall now.

LEMMA 3.2. *Let \mathcal{M} be a Hilbert module of holomorphic functions on Ω over the algebra $\mathcal{A}(\Omega)$ with reproducing kernel K . Let $J^{(k)}\mathcal{M}$ be the associated module of jets with reproducing kernel JK . The adjoint of the module action J_f on $JK(\cdot, w)\mathbf{x}$, $\mathbf{x} \in \mathbb{C}^k$, is given by*

$$J_f^* JK(\cdot, w) \cdot \mathbf{x} = JK(\cdot, w)(\mathcal{J}f)(w)^* \cdot \mathbf{x}, \quad f \in \mathcal{A}(U), \quad w \in U.$$

The module $J^{(k)}\mathcal{M}$ may be thought of as the k th order jet module of the given module \mathcal{M} relative to the hypersurface \mathcal{Z} . For $w \in U$ and $1 \leq \ell \leq k$, let $s_\ell(w) = K(\cdot, w)\varepsilon_\ell$. The vectors $s_\ell(w)$ span the range \mathcal{E}_w of $K(\cdot, w) : \mathbb{C}^k \rightarrow \mathcal{M}$. The holomorphic frame $w \rightarrow \{s_1(\bar{w}), \dots, s_k(\bar{w})\}$ determines a holomorphically trivial vector bundle \mathcal{E} over U^* . The fiber of \mathcal{E} over w is $\mathcal{E}_w = \text{span}\{K(\cdot, \bar{w})\varepsilon_\ell : 1 \leq \ell \leq k\}$, $w \in U^*$. An arbitrary section of this bundle is of the form $s = \sum_{\ell=1}^k a_\ell s_\ell$, where $a_\ell, \ell = 1, \dots, k$, are holomorphic functions on U^* . The norm at $w \in U^*$ is determined by

$$(3.3) \quad \|s(w)\|^2 = \left\langle \sum_{\ell=1}^k a_\ell(w) s_\ell(w), \sum_{\ell=1}^k a_\ell(w) s_\ell(w) \right\rangle_{\mathcal{M}} = \langle K(w, w)^{\text{tr}} a(w), a(w) \rangle_{\mathbb{C}^k},$$

where $a(w) = \sum_{\ell=1}^k a_\ell(w)\varepsilon_\ell$ and $K(w, w)^{\text{tr}}$ denotes the transpose of the matrix $K(w, w)$. Since $K(w, w)$ is positive definite and $w \mapsto K(w, w)$ is real analytic, it follows that $K(w, w)$ determines a hermitian metric for the vector bundle \mathcal{E} . It is easy to verify that if the module \mathcal{M} is quasi-free and E is the corresponding bundle whose hermitian structure is determined by the kernel function K , then the bundle \mathcal{E} along with the hermitian structure induced by the kernel JK is the one we would have obtained by applying the jet construction to the bundle E .

Suppose \mathcal{M} is a Hilbert module over the function algebra $\mathcal{A}(\Omega)$ which is in $B_1(\Omega)$. Then one may identify the Hilbert space \mathcal{M} with a space of holomorphic functions on Ω possessing a complex-valued reproducing kernel K . This determines a line bundle $E_{\mathcal{M}}$ on Ω^* whose fiber at $\bar{w} \in \Omega^*$ is spanned by the vector $K(\cdot, \bar{w})$. The jet bundle of rank k is determined by

the holomorphic frame $\{K(\cdot, w), \bar{\partial}_1 K(\cdot, w), \dots, \bar{\partial}_1^{k-1} K(\cdot, w)\}$. The metric for the bundle with respect to this frame is given by the formula (compare (3.2)):

$$\left\langle \sum_{j=0}^{k-1} a_j \partial_1^j K(\cdot, w), \sum_{j=0}^{k-1} a_j \partial_1^j K(\cdot, w) \right\rangle = \sum_{j,\ell=0}^{k-1} a_j \bar{a}_\ell \langle \partial_1^j K(\cdot, w), \partial_1^\ell K(\cdot, w) \rangle.$$

Clearly, the action of the algebra $\mathcal{A}(\Omega)$ on the module $J^{(k)}\mathcal{M}$ given in Lemma 3.2 defines a holomorphic bundle map θ_f on the holomorphic frame $\{JK(\cdot, w) \cdot \epsilon_i : 1 \leq i \leq k, w \in \Omega\}$, of the jet bundle $J^{(k)}E_{\mathcal{M}}$ for each $f \in \mathcal{A}(\Omega)$. Hence the algebra $\mathcal{A}(\Omega)$ acts on the holomorphic sections of the jet bundle $J^{(k)}E_{\mathcal{M}}$ as well making it into a module equivalent to the module $J^{(k)}\mathcal{M}$. This is the jet bundle $J^{(k)}E_{\mathcal{M}}$ associated with $E_{\mathcal{M}}$.

On the other hand, the Hilbert space $J^{(k)}\mathcal{M}$ together with its kernel function JK defined in Proposition 3.1 defines a rank k hermitian holomorphic bundle on Ω^* (see discussion preceding equation (3.3)). That these two constructions yield equivalent hermitian holomorphic bundles is a consequence of the fact that J is a unitary map from \mathcal{M} onto $J^{(k)}\mathcal{M}$.

REMARK 3.1. Therefore we see that the question of determining the equivalence class of the module $J^{(k)}\mathcal{M}$ is the same as determining the equivalence class of the jet bundle $J^{(k)}E_{\mathcal{M}}$ assuming that the map implementing the equivalence is also a module map on holomorphic sections. Thus it is natural to make the following Definition.

DEFINITION 3.1. Two jet bundles are said to be equivalent if there is an isometric holomorphic bundle map which induces a module isomorphism of the holomorphic sections.

4. THE MAIN RESULT FOR JET BUNDLES

4.1. Let E be a holomorphic line bundle over Ω^* equipped with a hermitian metric G . For $\mathcal{Z}^* \subseteq \Omega^*$, let us expand the real analytic function G using the coordinates $(z_1, z') \in \Omega^*$ with $z' = (z_2, \dots, z_m) \in \mathcal{Z}^*$:

$$G(z_1, z') = \sum_{m,n=0}^{\infty} G_{m,n}(z') z_1^m \bar{z}_1^n.$$

(Note that G and $G_{m,n}$ are merely real analytic functions. Therefore, they depend on the variables we have indicated along with their conjugates.)

Suppose we start with a resolution of the form (1.1). Then we have at our disposal the domain $\Omega \subseteq \mathbb{C}^m$ and the hypersurface $\mathcal{Z} \subseteq \Omega$. We recall from [14, Theorem 3.4] that the quotient module \mathcal{Q} can be identified with the module $J^{(k)}\mathcal{M}_{|\text{res } \mathcal{Z}}$. The module action J_f on the quotient $J^{(k)}\mathcal{M}_{|\text{res } \mathcal{Z}}$, for $f \in \mathcal{A}(\Omega)$, is defined via the restriction of the map

$$(4.1) \quad (J_f^* s_\ell)(w) = JK(\cdot, w)(\mathcal{J}f)(w)^* \varepsilon_\ell$$

to \mathcal{Z} and \mathcal{J} is defined in equation (3.1).

Let φ be a local defining function for \mathcal{Z} , that is, for some open subset $U \subseteq \Omega$, we have $\mathcal{Z} \cap U = \{z \in U : \varphi(z) = 0\}$. If necessary, by restricting to a smaller open subset of U , which we continue to denote by U , we may assume that φ is in $\mathcal{A}(U)$. We recall that we may assume

$\Omega = U$ and $\mathcal{Z} = \mathcal{Z} \cap U$ without loss of generality. Now, we see that φ induces a nilpotent action on each fiber of the jet bundle $J^{(k)}E|_{\text{res } \mathcal{Z}}$ via the restriction of the map J_φ^* to \mathcal{Z} .

Therefore in this picture, with the assumptions we have made along the way, we see that the *quotient modules* \mathcal{Q} satisfies the requirement listed in (i) – (iii) of the following Definition.

DEFINITION 4.1 ([11], pp. 284). We will say that the module \mathcal{Q} over the algebra $\mathcal{A}(\Omega)$ is a *quotient module* in the class $B_k(\Omega, \mathcal{Z})$ if

- (i) there exists a resolution of the module \mathcal{Q} as in equation (1.1), where the module \mathcal{M} appearing in the resolution is quasi-free of rank 1 over the algebra $\mathcal{A}(\Omega)$;
- (ii) for $f \in \mathcal{A}(\Omega)$, the restriction of the map J_f to the hypersurface \mathcal{Z} defines the module action on $J^{(k)}\mathcal{M}|_{\text{res } \mathcal{Z}}$ which is an isomorphic copy of \mathcal{Q} ; and
- (iii) the quotient module \mathcal{Q} is quasi-free of rank k over the algebra $\mathcal{A}|_{\text{res } \mathcal{Z}}(\Omega)$.

Our main theorem is easily stated using the $k \times k$ array of differential operators:

$$(4.2) \quad \mathcal{D}_k = \begin{pmatrix} \bar{\partial}' \partial' & \bar{\partial}' \partial_1 & \bar{\partial}' \partial_1^2 & \dots & \bar{\partial}' \partial_1^{k-1} \\ \bar{\partial}_1 \partial' & \bar{\partial}_1 \partial_1 & \bar{\partial}_1 \partial_1^2 & \dots & \bar{\partial}_1 \partial_1^{k-1} \\ \bar{\partial}_1^2 \partial' & \bar{\partial}_1^2 \partial_1 & \bar{\partial}_1^2 \partial_1^2 & \dots & \bar{\partial}_1^2 \partial_1^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{\partial}_1^{k-1} \partial' & \bar{\partial}_1^{k-1} \partial_1 & \bar{\partial}_1^{k-1} \partial_1^2 & \dots & \bar{\partial}_1^{k-1} \partial_1^{k-1} \end{pmatrix},$$

where ∂' denotes the differential operator $(\partial_m, \dots, \partial_2)$ and $\bar{\partial}' = \begin{pmatrix} \bar{\partial}_m \\ \vdots \\ \bar{\partial}_2 \end{pmatrix}$. We point out that the

(1, 1) position of the matrix \mathcal{D}_k consists of a $m-1 \times m-1$ block and that each entry of the first row (respectively column) is a column (respectively, row) vector of size $m-1$.

DEFINITION 4.2. Let ϱ and $\tilde{\varrho}$ be two positive real analytic functions on a domain Ω . We say that the ϱ and $\tilde{\varrho}$ are *equivalent to order k on \mathcal{Z}* if $\mathcal{D}_k(\log \frac{\tilde{\varrho}}{\varrho}) = 0$ on \mathcal{Z} .

THEOREM 1. Suppose that $\mathcal{Q} = \mathcal{M} \ominus \mathcal{M}_0$ and $\tilde{\mathcal{Q}} = \tilde{\mathcal{M}} \ominus \tilde{\mathcal{M}}_0$ is a pair of quotient of Hilbert modules over the algebra $\mathcal{A}(\Omega)$ that are in the class $B_k(\Omega, \mathcal{Z})$. Then the quotient module \mathcal{Q} and $\tilde{\mathcal{Q}}$ are isomorphic if and only if $\tilde{\varrho}$ and ϱ are equivalent to order k , where $\tilde{\varrho}$ and ϱ are the hermitian metrics for the line bundles corresponding to the two modules \mathcal{M} and $\tilde{\mathcal{M}}$ respectively.

We begin the proof of the theorem after proving a couple of results of a general nature. Indeed, the lemma below is a function theoretic result and the proposition which follows is algebraic in nature. These two results, more or less, yield immediately a proof of the theorem.

DEFINITION 4.3. Let r be a positive real analytic function defined on Ω . Let

$$r(z_1, z') = \sum_{\ell, m=0}^{\infty} r_{\ell, m}(z') z_1^\ell \bar{z}_1^m$$

be the expansion of r in the variables z_1, \bar{z}_1 around a small neighborhood of $(0, 0)$. We will say that r is *holomorphic to order k along \mathcal{Z}* if the coefficients $r_{\ell, 0}$, $\ell \leq k$ are holomorphic and

$r_{0,m} = \bar{r}_{m,0}$, $m \leq k$ are anti-holomorphic while all the coefficients $r_{\ell,m} = 0$ for $0 < \ell, m \leq k$ then we say that r is *holomorphic to order k along \mathcal{Z}* .

Since $\mathcal{D}_2(\log \frac{\tilde{\varrho}}{\varrho}) = 0$ on all of Ω is the same as saying that

$$\sum_{i,j=1}^m \bar{\partial}_i \partial_j \log \frac{\tilde{\varrho}}{\varrho} d\bar{z}_i \wedge dz_j = 0$$

on all of Ω , it follows that $\tilde{\varrho} = |\psi|^2 \varrho$ for some holomorphic function ψ on Ω . The following Lemma is a generalization of this statement in two directions. On the one hand, we allow higher order differentiation and on the other hand, we require equality only on \mathcal{Z} .

LEMMA 4.1. *Two positive real analytic functions ϱ and $\tilde{\varrho}$ on Ω are equivalent to order k on \mathcal{Z} if and only if $\tilde{\varrho} = |\psi|^2 \varrho$, where ψ is some real analytic function for which $\log |\psi|^2$ is holomorphic to order k along \mathcal{Z} .*

Proof. Since $\frac{\tilde{\varrho}}{\varrho}$ is a positive real analytic function on Ω , it follows that we may write $\frac{\tilde{\varrho}}{\varrho} = |\psi|^2$ for some real analytic function $\psi : \Omega \rightarrow \mathbb{C}$. Let us expand the real analytic function $\log |\psi|^2$ in the variables z_1 and \bar{z}_1

$$\log |\psi|^2(z_1, z') = \sum_{\ell, m=0}^{\infty} \psi_{\ell, m}(z') z_1^{\ell} \bar{z}_1^m,$$

where the coefficients $\psi_{\ell, m}$ are real analytic functions of $z' \in \mathcal{Z}$ for $\ell, m \geq 0$. (Strictly speaking, we should have said $(0, z')$ is in \mathcal{Z} and not $z' \in \mathcal{Z}$.)

For $k = 1$, to say that $\mathcal{D}_1(\log \frac{\tilde{\varrho}}{\varrho}) = 0$ on \mathcal{Z} is the same as saying $\partial' \bar{\partial}' \log \frac{\tilde{\varrho}}{\varrho} = 0$ on \mathcal{Z} . This, in turn, is equivalent to $\partial' \bar{\partial}' (\log \frac{\tilde{\varrho}}{\varrho})|_{\mathcal{Z}} = 0$. As is well-known, $\partial' \bar{\partial}' (\log \frac{\tilde{\varrho}}{\varrho})|_{\mathcal{Z}} = 0$ if and only if $\tilde{\varrho} = |\psi_0|^2 \varrho$ for some holomorphic function ψ on \mathcal{Z} . This proves the Lemma for $k = 1$.

The proof in the forward direction is by induction. We have already verified the statement for $k = 1$. Now, assume that it is valid for k , that is, $\psi_{\ell, 0}$ is holomorphic for $\ell \leq k-1$, $\psi_{0, m} = \bar{\psi}_{m, 0}$ for $m \leq k-1$ and $\psi_{\ell, m} = 0$ for all $0 < \ell, m \leq k-1$. We will show that the same conditions are forced on the coefficients even when we replace $k-1$ by k as long as we assume $\mathcal{D}_k \log |\psi|^2 = 0$ on \mathcal{Z} . Thus we have that $\bar{\partial}' \partial_1^k \log |\psi|^2|_{\mathcal{Z}} = 0$ which forces $\bar{\partial}' \psi_{k, 0} = 0$ on \mathcal{Z} . Similarly, $\partial' \psi_{0, k} = 0$ on \mathcal{Z} making $\psi_{0, k}$ anti-holomorphic on \mathcal{Z} . The condition that $\bar{\partial}_1^{\ell} \partial_1^k \log |\psi|^2|_{\mathcal{Z}} = 0$ is clearly equivalent to $\psi_{\ell, k} = 0$ for $\ell \leq k$. Again, we have $\bar{\partial}_1^k \partial_1^m \log |\psi|^2|_{\mathcal{Z}} = 0$ is clearly equivalent to $\psi_{k, m} = 0$ for $m \leq k$. Since $|\psi|^2 = \frac{1}{2}(|\psi|^2 + \overline{|\psi|^2})$, we see that $\psi_{\ell, 0} = \frac{1}{2}(\psi_{\ell, 0} + \bar{\psi}_{0, \ell})$ and $\psi_{0, \ell} = \bar{\psi}_{\ell, 0}$.

The proof in the other direction is a straightforward verification – $\mathcal{D}_k \log |\psi|^2 = 0$ on \mathcal{Z} assuming that $\psi_{\ell, 0}$, $\ell \leq k$ are holomorphic, $\psi_{0, m} = \bar{\psi}_{m, 0}$, $m \leq k$ are anti-holomorphic and the coefficients $\psi_{\ell, m} = 0$ for $0 < \ell, m \leq k$. \square

Let $\mathbb{C}^{k \times k}$ be the algebra of all $k \times k$ complex matrices and $\mathcal{T}^{k \times k} \subseteq \mathbb{C}^{k \times k}$ be the sub-algebra of lower triangular Toeplitz matrices, that is, those lower triangular matrices A for which $A(\ell + p, \ell) = A(p)$ for $0 \leq \ell, p \leq k$, $\ell + p \leq k$.

In the proof of the following Proposition we use the fact that if $|\psi|^2$ is holomorphic to order k along \mathcal{Z} , then the coefficient function $\alpha_{\ell,0}$, in the expansion $|\psi|^2(z') = \sum_{\ell,m=0}^{\infty} \alpha_{\ell,m}(z') z_1^\ell \bar{z}_1^m$, is a holomorphic function for $\ell \leq k$.

PROPOSITION 4.2. *Suppose $\tilde{\varrho}, \varrho$ are two positive real analytic functions on Ω with $\tilde{\varrho} = |\psi|^2 \varrho$. Then the function $\log |\psi|^2$, which is necessarily real analytic, is holomorphic to order k along \mathcal{Z} if and only if there exists some holomorphic function $\Psi : \mathcal{Z} \rightarrow \mathcal{T}^{k+1 \times k+1}$ with ψ_p at the $(\ell+p, \ell)$ position. satisfying*

$$(J\tilde{\varrho})(z') = \Psi(z')(J\varrho)(z')\Psi(z')^*, \quad z' \in \mathcal{Z}.$$

Proof. Assume that $\tilde{\varrho} = |\psi|^2 \varrho$ and $|\psi|^2$ is holomorphic to order k along \mathcal{Z} . Let us compute the derivatives

$$\begin{aligned} \bar{\partial}_1^i \partial_1^j \tilde{\varrho} &= \bar{\partial}_1^i \left(\bar{\psi} \sum_{n_1=0}^j \binom{j}{n_2} \psi^{(n_2)} \varrho^{(j-n_2)} \right) \\ &= \sum_{n_1=0}^i \sum_{n_1=0}^j \binom{i}{n_1} \binom{j}{n_2} \overline{\psi^{(n_1)}} \psi^{(n_2)} \varrho^{(j-n_2, i-n_1)}, \end{aligned}$$

where $\varrho^{(j-n_2, i-n_1)} = \partial_1^{j-n_2} \bar{\partial}_1^{i-n_1} \varrho$. If we restrict this equation to \mathcal{Z} , we see that

$$\tilde{\varrho}_{j,i} = \sum_{n_1=0}^i \sum_{n_1=0}^j \bar{\psi}_{n_1} \psi_{n_2} \varrho_{j-n_2, i-n_1},$$

where $\tilde{\varrho}_{j,i}$, $\varrho_{j-n_2, i-n_1}$ and ψ_{n_2} , ψ_{n_1} are the coefficients in the expansion of the respective real analytic functions around $z_1^{(0)} = 0$ in the variable z_1 . However, this says that $J\tilde{\varrho} = \Psi(J\varrho)\Psi^*$, where Ψ is the lower triangular matrix with the holomorphic function ψ_p at the $(\ell+p, p)$ position.

Conversely, suppose $(J\tilde{\varrho})(z') = \Psi_k(z')(J\varrho)(z')\Psi_k(z')^*$, $z' \in \mathcal{Z}$, for some holomorphic function $\Psi : \mathcal{Z} \rightarrow \mathcal{T}^{(k+1) \times (k+1)}$. We have to show that $\tilde{\varrho} = |\psi|^2 \varrho$ for some real analytic function $|\psi|^2$ which is holomorphic to order k along \mathcal{Z} . Clearly, on the hypersurface \mathcal{Z} , we have

$$\tilde{\varrho}_{j,i} = \sum_{n_1=0}^i \sum_{n_2=0}^j \bar{\psi}_{n_1} \psi_{n_2} \varrho_{j-n_2, i-n_1},$$

where ψ_{n_1} is the holomorphic function on \mathcal{Z} which occurs in the n_1 sub-diagonal of the function Ψ . Now, we apply the preceding Lemma to infer that $\tilde{\varrho}$ and ϱ are equivalent to order k on \mathcal{Z} completing the proof. \square

COROLLARY 4.3. *Let (E, ϱ) and $(\tilde{E}, \tilde{\varrho})$ be two hermitian holomorphic line bundles on $\Omega \subseteq \mathbb{C}^m$. Let $J^{(k)}E$ and $J^{(k)}\tilde{E}$ be the jet bundles of E and \tilde{E} , respectively, equipped with the natural action of the algebra $\mathcal{A}(\Omega)$, that is, $f \mapsto (\partial f) \cdot s$, $f \in \mathcal{A}(\Omega)$, for a holomorphic section s . The restriction to the hypersurface \mathcal{Z} of the two jet bundles $J^{(k)}E$ and $J^{(k)}\tilde{E}$ are equivalent if and only if ϱ and $\tilde{\varrho}$ are equivalent to order k on \mathcal{Z} .*

Proof. The equivalence of the two jet bundles in the sense of Definition 3.1 amounts to the existence of a holomorphic map $\Psi : \mathcal{Z} \rightarrow \mathcal{T}^{k \times k}$ which intertwines the module action, that is, $\Psi \mathcal{J}f = \mathcal{J}f \Psi$. This intertwining property is easily verified –

$$\begin{aligned}
\Psi^*(\mathcal{J}f)^*(i, j) &= (0, \dots, 0, \psi_0, \dots, \psi_{k-1-i}) (\partial^j f, \dots, f, 0, \dots, 0)^{tr} \\
&= \binom{j}{i} \partial^{j-i} \psi_0 + \dots + \psi_{j-i} f \\
&= \psi_{j-i} f + \dots + \binom{j}{i} \partial^{j-i} \psi_0 \\
&= (0, \dots, 0, f, \dots, \partial^{k-1-i} f) (\psi_j, \dots, \psi_0, 0, \dots, 0)^{tr} \\
&= (\mathcal{J}f)^* \Psi^*(i, j)
\end{aligned}$$

completing the proof of the Corollary. \square

Proof of Theorem 1. We have pointed out in Remark 3.1 that the equivalence of the jet bundles in the sense of Definition 3.1 is the same as that of the corresponding modules. Therefore, the Corollary given above completes the proof of the Theorem 1. \square

5. THE SECOND FUNDAMENTAL FORM

We let $\mathcal{M}_0 \subseteq \mathcal{M}$ be the submodule of all functions which vanish to order 2 on the hypersurface \mathcal{Z} . As before, let $\{s, \partial_1 s\}$ be a frame for the jet bundle $J^{(2)}E$ of rank 2 corresponding to the module \mathcal{M} . In this case, under some mild hypothesis on the quotient module \mathcal{Q} , we know [11, pp. 289] that $\mathcal{K}_{\text{trans}}, \mathcal{K}_{\text{tan}}$ and the angle $\langle \partial s, s \rangle$ restricted to the hypersurface \mathcal{Z} determine the unitary equivalence class of \mathcal{Q} . Let us explain the nature of this hypothesis.

In subsection 5.2, we show that the angle invariant, which together with the transverse and the tangential curvatures forms a complete set of unitary invariants for the quotient module \mathcal{Q} can be replaced by the second fundamental form \mathbb{I} for the inclusion $E \subseteq J^{(2)}E$. In view of the equation (5.10), we have stated the theorem in terms of the restriction of the curvature. One of the disadvantages in using the angle as an invariant for the isomorphism class of the quotient module is that for it to make sense we must introduce normalized reproducing kernels (cf. [9, Remark 4.7 (b)]). To avoid this ad hoc normalization, we replace it with the second fundamental form which is a more natural geometric invariant.

5.1. Let Ω be an open connected and bounded subset of \mathbb{C}^m and $\mathcal{Z} \subseteq \Omega$ be a hypersurface, that is, a complex sub-manifold of co-dimension 1. Let ∂_1 denote differentiation along the normal direction to \mathcal{Z} . Let E be a hermitian holomorphic line bundle on Ω . Let s be a holomorphic frame for E and h be the hermitian metric. One sees that $\{s, \partial_1 s\}$ is a holomorphic frame for the jet bundle $J^{(2)}E$ of rank 2 in the normal direction to \mathcal{Z} . Then

$$(5.1) \quad (J^{(2)}h)(w) = \begin{pmatrix} h(w) & (\partial_1 h)(w) \\ (\bar{\partial}_1 h)(w) & (\bar{\partial}_1 \partial_1 h)(w) \end{pmatrix}$$

defines a metric for the jet bundle $J^{(2)}E$. One obtains an orthonormal frame, say $\{e_1, e_2\}$ from the holomorphic frame by the usual Gram-Schmidt process:

$$\begin{aligned}
 e_1 &= h^{-1/2}s, \\
 e_2 &= \frac{\partial_1 s - \langle \partial_1 s, e_1 \rangle e_1}{\|\partial_1 s - \langle \partial_1 s, e_1 \rangle e_1\|} \\
 (5.2) \quad &= \frac{\partial_1 s - \langle \partial_1 s, e_1 \rangle e_1}{h^{1/2}(\partial_1 \bar{\partial}_1 \log h)^{1/2}},
 \end{aligned}$$

where we see that $\|\partial_1 s - \langle \partial_1 s, e_1 \rangle e_1\| = h^{1/2}(\partial_1 \bar{\partial}_1 \log h)^{1/2}$ as in [6, 1.17.1]. Let D be the canonical connection and $\bar{\partial}$ be the operator $\bar{\partial}f = \sum_1^m \bar{\partial}_j f d\bar{z}_j$. Since s is holomorphic, $\bar{\partial}s = 0$ and it follows that

$$(5.3) \quad \bar{\partial}e_1 = -\frac{1}{2}h^{-3/2}\bar{\partial}h \cdot s = -\frac{1}{2}h^{-1}\bar{\partial}h \cdot e_1 = -\frac{1}{2}\bar{\partial}(\log h) \cdot e_1.$$

Similarly, differentiating (5.2), we have

$$\begin{aligned}
 \bar{\partial}e_2 &= \bar{\partial}\left(\frac{1}{h^{1/2}(\partial_1 \bar{\partial}_1 \log h)^{1/2}}\right)(\partial_1 s - \langle \partial_1 s, e_1 \rangle e_1) + \frac{\bar{\partial}(\partial_1 s - \langle \partial_1 s, e_1 \rangle e_1)}{h^{1/2}(\partial_1 \bar{\partial}_1 \log h)^{1/2}} \\
 &= -\frac{1}{2}\frac{\bar{\partial}(h\partial_1 \bar{\partial}_1 \log h)}{(h\partial_1 \bar{\partial}_1 \log h)^{3/2}} \cdot (\partial_1 s - \langle \partial_1 s, e_1 \rangle e_1) + \frac{-\bar{\partial}(h^{-1}\partial_1 h) \cdot s}{(h\partial_1 \bar{\partial}_1 \log h)^{1/2}} \\
 (5.4) \quad &= -\frac{1}{2}\frac{\bar{\partial}(h(\partial_1 \bar{\partial}_1 \log h))}{h\partial_1 \bar{\partial}_1 \log h} \cdot e_2 - \frac{\bar{\partial}(\partial_1 \log h)}{(\partial_1 \bar{\partial}_1 \log h)^{1/2}} e_1
 \end{aligned}$$

Let us calculate the canonical hermitian holomorphic connection D in $J^{(2)}(E)$ with respect to the metric (5.1). We have

$$\begin{aligned}
 De_1 &= D^{1,0}e_1 + D^{0,1}e_1 \\
 &= \alpha_{11}e_1 + \alpha_{21}e_2 + \bar{\partial}e_1 \\
 &= (\alpha_{11} - 1/2\bar{\partial}\log h)e_1 + \alpha_{21}e_2 \text{ by (5.3)} \\
 (5.5) \quad &= \theta_{11}e_1 + \theta_{21}e_2,
 \end{aligned}$$

where α_{11}, α_{21} is a pair of $(1, 0)$ forms. Similarly, we have

$$\begin{aligned}
 De_2 &= D^{1,0}e_2 + D^{0,1}e_2 \\
 &= \alpha_{12}e_1 + \alpha_{22}e_2 + \bar{\partial}e_2 \\
 &= \left(\alpha_{12} - \frac{\bar{\partial}\partial_1 \log h}{(\partial_1 \bar{\partial}_1 \log h)^{1/2}}\right)e_1 + \left(\alpha_{22} - \frac{1}{2}\frac{\bar{\partial}(h\partial_1 \bar{\partial}_1 \log h)}{h\partial_1 \bar{\partial}_1 \log h}\right)e_2 \text{ by (5.4)} \\
 (5.6) \quad &= \theta_{12}e_1 + \theta_{22}e_2,
 \end{aligned}$$

where α_{12}, α_{22} is another pair of $(1, 0)$ forms. Since we are working with an orthonormal frame, the compatibility with the metric (2.5) amounts to the requirement

$$\begin{aligned}
 \{De_1, e_2\} + \{e_1, De_2\} &= \theta_{ij} + \bar{\theta}_{ij} \\
 (5.7) \quad &= 0 \quad \text{for } 1 \leq i, j \leq 2.
 \end{aligned}$$

For $1 \leq i, j \leq 2$, equating $(1, 0)$ and $(0, 1)$ forms separately to zero in the equations $\theta_{ij} + \bar{\theta}_{ij} = 0$, we obtain $\alpha_{11} = \frac{1}{2}\partial \log h$, $\alpha_{12} = 0$, $\alpha_{21} = \frac{\partial(\bar{\partial}_1 \log h)}{(\partial_1 \bar{\partial}_1 \log h)^{1/2}}$ and $\alpha_{22} = \frac{1}{2} \frac{\partial(h \partial_1 \bar{\partial}_1 \log h)}{h \partial_1 \bar{\partial}_1 \log h}$. It therefore follows that

$$(5.8) \quad \theta = \begin{pmatrix} \frac{1}{2}(\partial - \bar{\partial}) \log h & -\frac{\bar{\partial}(\partial_1 \log h)}{(\partial_1 \bar{\partial}_1 \log h)^{1/2}} \\ \frac{\bar{\partial}(\partial_1 \log h)}{(\partial_1 \bar{\partial}_1 \log h)^{1/2}} & \frac{1}{2} \frac{(\partial - \bar{\partial})(h \partial_1 \bar{\partial}_1 \log h)}{h(\partial_1 \bar{\partial}_1 \log h)} \end{pmatrix}$$

is the matrix representation of the canonical connection D on $J^{(2)}E$ with respect to the orthonormal frame $\{e_1, e_2\}$. Thus the second fundamental form \mathbb{I} for the inclusion $E \subseteq J^{(2)}E$ is

$$(5.9) \quad \langle De_1, e_2 \rangle = \theta_{21} = -\frac{\bar{\partial}(\partial_1 \log h)}{(\partial_1 \bar{\partial}_1 \log h)^{1/2}}.$$

Let E be a holomorphic hermitian vector bundle over Ω . We can easily express the second fundamental form \mathbb{I} on \mathcal{Z} in terms of the coefficients of the full curvature (2.6) on \mathcal{Z} :

$$(5.10) \quad \mathbb{I}(z) = (\mathbb{I}_1(z)dz_1, \dots, \mathbb{I}_m(z)dz_m) = (\mathcal{K}_{\text{trans}}(z))^{-1/2} \begin{pmatrix} \mathcal{K}_{\text{trans}}(z) & \mathcal{S}(z) \end{pmatrix} \cdot \begin{pmatrix} d\bar{z}_1 \wedge dz_1 \\ dz' \wedge dz' \end{pmatrix}$$

for $z = (z_1, z') \in \Omega$.

REMARK 5.1. It follows that if we fix the transverse curvature $\mathcal{K}_{\text{trans}}$ of a line bundle E , then the second fundamental form \mathbb{I} for the inclusion $E \subseteq J_1^{(2)}E$ and the coefficient \mathcal{S} of the curvature \mathcal{K}_E determine each other. Consequently, the restriction to the hypersurface \mathcal{Z} of $\mathcal{K}_{\text{trans}}$, \mathcal{K}_{tan} and the second fundamental form \mathbb{I} of two holomorphic hermitian bundles are equal if and only if the restriction to the hypersurface \mathcal{Z} of all the coefficients of the curvature \mathcal{K} are equal.

5.2. In case $k = 2$, we may restate the Theorem 1 in terms of the tangential and the transverse curvatures along with the second fundamental form. The validity of such a statement will follow from Theorem 1 which holds for an arbitrary k . We consider the case of $k = 2$ separately for comparison with our previous result which was limited to this case only. In the statement of the theorem below, we use the invariants *tan* and *trans* which stand for the tangential and transverse curvatures. These invariants occurred in [11, Theorem, page 289]. We emphasize that the third invariant that appears in the theorem below is the second fundamental form. Therefore, this theorem is different from that of [11]. We provide an independent proof using explicit computations.

THEOREM 2. Suppose that $\mathcal{Q} = \mathcal{M} \ominus \mathcal{M}_0$ and $\tilde{\mathcal{Q}} = \tilde{\mathcal{M}} \ominus \tilde{\mathcal{M}}_0$ is a pair of quotient modules of Hilbert modules over the algebra $\mathcal{A}(\Omega)$ and that they are both in the class $\mathcal{B}_2(\Omega, \mathcal{Z})$. Then the quotient modules \mathcal{Q} and $\tilde{\mathcal{Q}}$ are isomorphic if and only if the restriction of the corresponding curvatures to the hypersurface \mathcal{Z} coincide, that is,

$$\begin{aligned} \text{tan: } \mathcal{K}_{\text{tan}} &= \tilde{\mathcal{K}}_{\text{tan}} \\ \text{trans: } \mathcal{K}_{\text{trans}} &= \tilde{\mathcal{K}}_{\text{trans}} \\ \text{angle: } \mathcal{S} &= \tilde{\mathcal{S}} \end{aligned}$$

are equal on \mathcal{Z} .

The proof of the theorem is facilitated by the following Lemma. We let $\mathcal{K}(z)$ denote the $(1, 1)$ form $\sum_{i,j=1}^m \bar{\partial}_i \partial_j (\log h)(z) d\bar{z}_i \wedge dz_j$, $z = (z_1, \dots, z_m) \in \Omega$, for some positive real analytic function h on the domain Ω .

LEMMA 5.1. *Let h and \tilde{h} be two positive real analytic functions on a domain Ω . The restrictions to the hypersurface $\mathcal{Z} \subseteq \Omega$ of the corresponding $(1, 1)$ forms \mathcal{K} and $\tilde{\mathcal{K}}$ are equal if and only if there exist holomorphic functions α, β on the hypersurface \mathcal{Z} such that*

$$\begin{aligned} \tilde{h}_{00} &= |\alpha|^2 h_{00} \\ \tilde{h}_{10} &= h_{10} + \beta |\alpha|^2 h_{00} \\ \tilde{h}_{01} &= h_{01} + \bar{\beta} |\alpha|^2 h_{00} \\ \tilde{h}_{11} &= |\alpha|^2 h_{11} + \beta |\alpha|^2 h_{01} + \bar{\beta} |\alpha|^2 h_{10} + |\beta|^2 |\alpha|^2 h_{00}, \end{aligned}$$

where $h(z_1, z') = \sum_{i,j=0}^{\infty} h_{ij}(z') z_1^i \bar{z}_1^j$ and $\tilde{h}(z_1, z') = \sum_{i,j=0}^{\infty} \tilde{h}_{ij}(z') z_1^i \bar{z}_1^j$ are the power series expansions of the real analytic functions h and \tilde{h} .

Proof. Let us put $\gamma = \tilde{h}/h$ and $\Gamma = \log \gamma$. Let us expand Γ in a power series:

$$(5.11) \quad \Gamma(z_1, z') = \Gamma_{00}(z') + z_1 \Gamma_{10}(z') + \bar{z}_1 \Gamma_{01}(z') + \dots,$$

where $(z_1, z') \in \Omega$. (We will suppress the dependence of the coefficients on z' whenever there is no possibility of confusion.) Recall that $\partial' = (\partial_2, \dots, \partial_m)$. The assumption that the restrictions to the hypersurface $\mathcal{Z} \subseteq \Omega$ of \mathcal{K} and $\tilde{\mathcal{K}}$ are equal amounts to saying that $\bar{\partial} \partial \Gamma = 0$. We split this condition into four separate ones. The first of these is the requirement that $(\bar{\partial}' \partial' \Gamma)|_{\mathcal{Z}} = 0$. The second and the third are similar: $(\partial_1 \bar{\partial}' \Gamma)|_{\mathcal{Z}} = 0$ and $(\bar{\partial}_1 \partial' \Gamma)|_{\mathcal{Z}} = 0$. The final one is $(\bar{\partial}_1 \partial_1 \Gamma)|_{\mathcal{Z}} = 0$.

In view of the expansion (5.11), the first condition is clearly the same as the requirement that $\bar{\partial}' \partial' \Gamma_{00} = 0$. Therefore it follows that $\Gamma_{00} = \alpha_1 + \bar{\alpha}_2$ for some holomorphic functions α_1, α_2 on the hypersurface \mathcal{Z} . Since Γ_{00} is positive, we also have $\Gamma_{00} = \bar{\alpha}_1 + \alpha_2$. Hence $\Gamma_{00} = \frac{\alpha_1 + \alpha_2}{2} + \frac{\bar{\alpha}_1 + \bar{\alpha}_2}{2}$. Consequently, $\gamma|_{\mathcal{Z}} = \exp(\Gamma|_{\mathcal{Z}}) = |\alpha|^2$, where $\alpha = \exp(\frac{\alpha_1 + \alpha_2}{2})$, is a holomorphic function defined on the hypersurface \mathcal{Z} .

The second condition $(\partial_1 \bar{\partial}' \Gamma)|_{\mathcal{Z}} = 0$ can be restated using the power series expansion (5.11) which is $\bar{\partial}' \Gamma_{10} = 0$. Hence Γ_{10} is holomorphic on \mathcal{Z} . Similarly, Γ_{01} is easily seen to be anti-holomorphic on \mathcal{Z} .

Finally, the condition $(\bar{\partial}_1 \partial_1 \Gamma)|_{\mathcal{Z}}$ is clearly equivalent to the vanishing of the coefficient Γ_{11} in the expansion (5.11), that is, $\Gamma_{11} = 0$.

Now, we put all of the above together and modify the expansion (5.11)

$$(5.12) \quad \Gamma(z_1, z') = \alpha_1 + \beta_1 z_1 + \eta_1 z_1^2 + \overline{\alpha_2 + \beta_2 z_1 + \eta_2 z_1^2} + \dots$$

It is not hard to see that we can have $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Indeed, $\Gamma = \frac{\Gamma + \bar{\Gamma}}{2}$, which allows us to take the common value $\frac{\alpha_1 + \alpha_2}{2}$, and similarly $\frac{\beta_1 + \beta_2}{2}$, as the coefficient of both z_1 and \bar{z}_1 . While similar considerations apply to the coefficient of z_1^2 , we have to remember that in that

case, and for all the other coefficients, these are not holomorphic functions. Therefore, we see that

$$\begin{aligned}
\gamma &= \exp \Gamma \\
&= |\exp(\frac{\alpha_1 + \alpha_2}{2})|^2 |\exp(\frac{\beta_1 + \beta_2}{2} z_1)|^2 |\exp(\frac{\eta_1 + \eta_2}{2} z_1^2)|^2 \dots \\
&= |\alpha|^2 |(1 + \beta z_1 + \beta^2 z_1^2 + \dots)|^2 |(1 + \eta^2 z_1^2 + \dots)|^2 \dots \\
&= |\alpha|^2 (1 + \beta z_1 + \bar{\beta} \bar{z}_1 + |\beta|^2 \bar{z}_1 z_1 + \dots),
\end{aligned}$$

where $\alpha = \exp(\frac{\alpha_1 + \alpha_2}{2})$ and $\beta = \frac{\beta_1 + \beta_2}{2}$. It now follows that

$$\begin{aligned}
\tilde{h} &= \tilde{h}_{00} + \tilde{h}_{10} z_1 + \tilde{h}_{01} \bar{z}_1 + \tilde{h}_{11} \bar{z}_1 z_1 + \dots \\
&= \gamma h \\
&= (h_{00} + h_{10} z_1 + h_{01} \bar{z}_1 + h_{11} \bar{z}_1 z_1 + \dots) (|\alpha|^2 (1 + \beta z_1 + \bar{\beta} \bar{z}_1 + |\beta|^2 \bar{z}_1 z_1 + \dots)) \\
&= |\alpha|^2 (h_{00} + (h_{10} + \beta h_{00}) z_1 + (h_{01} + \bar{\beta} h_{00}) \bar{z}_1 + (h_{11} + \bar{\beta} h_{10} + \beta h_{01} + h_{00} |\beta|^2) \bar{z}_1 z_1 + \dots).
\end{aligned}$$

Equating the coefficients in this equation, we clearly have the following relationship:

$$\begin{aligned}
\tilde{h}_{00} &= |\alpha|^2 h_{00} \\
\tilde{h}_{10} &= |\alpha|^2 (\tilde{h}_{10} + \beta h_{00}) \\
\tilde{h}_{01} &= |\alpha|^2 (\tilde{h}_{01} + \bar{\beta} h_{00}) \\
(5.13) \quad \tilde{h}_{11} &= |\alpha|^2 (h_{11} + \bar{\beta} h_{10} + \beta h_{01} + h_{00} |\beta|^2).
\end{aligned}$$

Conversely, we see that $\mathcal{K}_{22} = \frac{h_{11} h_{00} - |h_{10}|^2}{h_{00}^2}$ on \mathcal{Z} . If we assume the relationships between h and \tilde{h} as in (5.13) then on the hypersurface \mathcal{Z} ,

$$\begin{aligned}
\tilde{\mathcal{K}}_{22} &= \frac{|\alpha|^4 (h_{11} + \bar{\beta} h_{10} + \beta h_{01} + |\beta|^2 h_{00}) h_{00} - |\alpha|^2 (h_{10} + \beta h_{00}) (h_{01} + \bar{\beta} h_{00})}{|\alpha|^4 h_{00}^2} \\
&= \frac{h_{11} h_{00} - |h_{10}|^2}{h_{00}^2}.
\end{aligned}$$

It therefore follows that $\tilde{\mathcal{K}}_{22} = \mathcal{K}_{22}$ on \mathcal{Z} . Similarly, again restricted to \mathcal{Z} , we have $\mathcal{K}_{12} = \frac{h_{00} \partial h_{01} - h_{01} \partial h_{00}}{h_{00}^2}$. We see that $\mathcal{K}_{12} = \frac{h_{00} \partial' h_{01} - h_{01} \partial' h_{00}}{h_{00}^2}$ on \mathcal{Z} . Hence a calculation, using (5.13), shows that

$$\begin{aligned}
\tilde{\mathcal{K}}_{12} &= \frac{(\partial h_{01} + \bar{\beta} \partial h_{00}) h_{00} - (h_{01} + \bar{\beta} h_{00}) \partial h_{00}}{h_{00}^2} \\
&= \frac{\partial' h_{01} h_{00} - h_{01} \partial' h_{00}}{h_{00}^2}
\end{aligned}$$

ensuring $\tilde{\mathcal{K}}_{12} = \mathcal{K}_{12}$ on \mathcal{Z} . Finally, it is clear that $\mathcal{K}_{11}(z) = \tilde{\mathcal{K}}_{11}(z)$, for $z \in \mathcal{Z}$ is equivalent to $\tilde{h}_{00} = |\alpha|^2 h_{00}$ for some holomorphic function α defined on \mathcal{Z} . \square

Proof of Theorem 2. We first prove the “if” part of the theorem. In this case, we have equality of all the coefficients of the two curvatures on the hypersurface \mathcal{Z} . This is equivalent to the

relationship given in the equations (5.13). We then find that

$$\begin{aligned} \begin{pmatrix} \alpha & 0 \\ \alpha\beta & \alpha \end{pmatrix} \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\alpha}\bar{\beta} \\ 0 & \bar{\alpha} \end{pmatrix} &= |\alpha|^2 \begin{pmatrix} h_{00} & h_{01} + \bar{\beta}h_{00} \\ h_{10} + \beta h_{00} & h_{11} + \bar{\beta}h_{10} + \beta h_{01} + h_{00}|\beta|^2 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{h}_{00} & \tilde{h}_{01} \\ \tilde{h}_{10} & \tilde{h}_{11} \end{pmatrix}. \end{aligned}$$

It follows that the bundle map $\Theta : J_1^{(2)}E|_{\mathcal{Z}} \rightarrow J_1^{(2)}\tilde{E}|_{\mathcal{Z}}$ defined by, $\Theta(z) = \begin{pmatrix} \alpha & 0 \\ \alpha\beta & \alpha \end{pmatrix}$, for $z \in \mathcal{Z}$ is holomorphic as well as isometric. Moreover, it intertwines the nilpotent action as well. Therefore, the quotient modules are isomorphic via this map.

For the proof of the only if part, we first observe that any unitary implementing the equivalence of the quotient modules must map the submodule \mathcal{M}_0 onto $\tilde{\mathcal{M}}_0$. This implies that the tangential curvatures must coincide. The matrix representation for the nilpotent action corresponding to the normal coordinate has the transverse curvature at the $(1, 2)$ position. So, if these nilpotent actions are equivalent, then the transverse curvature corresponding to them must coincide. Furthermore, any such intertwining unitary between the quotient modules must be of the form $\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$ for holomorphic functions a, b defined on the hypersurface \mathcal{Z} . We can assume, without loss of generality, that $b = ac$. Hence we must have

$$\begin{pmatrix} a & 0 \\ ac & a \end{pmatrix} \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{a}\bar{c} \\ 0 & \bar{a} \end{pmatrix} = \begin{pmatrix} \tilde{h}_{00} & \tilde{h}_{01} \\ \tilde{h}_{10} & \tilde{h}_{11} \end{pmatrix}.$$

It then follows that we must have that the relationship given by (5.13) holds, which completes the proof. \square

6. APPLICATIONS AND EXAMPLES

6.1. In this section, we give applications of our results to a familiar class of Hilbert modules over the bi-disc algebra. These applications involve homogeneity of the modules under the action of the Möbius group. The study of homogeneity for Hilbert modules over the algebra $\mathcal{A}(\Omega)$ for a bounded symmetric domain $\Omega \subseteq \mathbb{C}^m$ was initiated in [23] and was further studied in [2]. However, in these papers, it was assumed that Ω is irreducible. So, the question of considering the possibility of $\Omega = \mathbb{D}^2$ did not arise. Although, the theorem below is stated in this case, it is clear that the proof works just as well in the case of \mathbb{D}^m . The recent work of Ferugson and Rochberg [19] and [18] are very close to the discussion below – at least, in spirit. Similarly, the work of the second named author with Koranyi on homogeneous operators in the class $B_k(\mathbb{D})$ is closely related to what we report here.

For $\lambda > 0$, let $\mathcal{M}^{(\lambda)}$ be the Hilbert space which is determined by requiring that $\{e_n^{(\lambda)}(z) := c_n^{-1/2}z^n : n \geq 0\}$ is a complete orthonormal set in it, where c_n is the coefficient of x^n in the expansion of $(1-x)^{-\lambda}$ or c_n is the set of binomial coefficients: $\binom{-\lambda}{n} = \frac{\lambda(\lambda+1)\cdots(\lambda+n-1)}{n!}$. It follows

that $\mathcal{M}^{(\lambda)}$ possesses a reproducing kernel $K^{(\lambda)} : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$, which is given by the formula

$$\begin{aligned} K^{(\lambda)}(z, w) &= \sum_{n=0}^{\infty} e_n^{(\lambda)}(z) \overline{e_n^{(\lambda)}(w)} \\ &= (1 - z\bar{w})^{-\lambda}, \end{aligned}$$

where \mathbb{D} is the open unit disc. Thus $\mathcal{M}^{(\lambda)}$ consists of holomorphic functions on the open unit disc \mathbb{D} . For $\theta \in [0, 2\pi)$ and $\alpha \in \mathbb{D}$, let $\varphi_{\alpha, \theta}(z) = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha}z}$ for $z \in \mathbb{D}$. The group of bi-holomorphic automorphisms Möb of the unit disc is $\{\varphi_{\alpha, \theta} : \theta \in [0, 2\pi) \text{ and } \alpha \in \mathbb{D}\}$. We recall that for $\lambda > 0$, the natural action of the polynomial ring $\mathbb{C}[z]$ on each of the Hilbert spaces $\mathcal{M}^{(\lambda)}$, for $\lambda \geq 0$, makes it into a module. However, for each $\lambda > 1$, this action extends to the disc algebra $\mathcal{A}(\mathbb{D})$. The modules $\mathcal{M}^{(\lambda)}$, $\lambda \geq 0$, lie in the class $B_1(\mathbb{D})$. What is more, they are *Möb-homogeneous*, that is, the module $\varphi_* \mathcal{M}^{(\lambda)}$ defined by the action $(f, h) \mapsto (f \circ \varphi) \cdot h$ for $f \in \mathcal{A}(\mathbb{D})$, $h \in \mathcal{M}^{(\lambda)}$ is isomorphic to the module $\mathcal{M}^{(\lambda)}$ for all φ in Möb. It turns out these are the only homogeneous modules in the class $B_1(\mathbb{D})$. For a complete discussion, we refer the reader to the survey paper [3]. D. Wilkins [27] has obtained a classification of all homogeneous Hilbert modules over the disc algebra which are in the class $B_k(\mathbb{D})$ for $k > 1$. However, he was able to give an explicit description of these modules only for rank 2. In a recent pre-print, Ferguson and Rochberg have obtained a similar description of these modules, again only in the case of rank 2. A. Koranyi and the second named author have also obtained a model for these quotient modules [22] which works for an arbitrary $k \in \mathbb{N}$.

6.2. For $\lambda, \mu > 0$, there is a natural action of the group Möb \times Möb on the module $\mathcal{M}^{(\lambda, \mu)}$, which is just the tensor product $\mathcal{M}^{(\lambda)} \otimes \mathcal{M}^{(\mu)}$. The Hilbert space $\mathcal{M}^{(\lambda, \mu)}$ is then a space of holomorphic functions on the bi-disc via the identification of the elementary tensor $e_m^{(\lambda)} \otimes e_n^{(\mu)}$ with the function of two variables $z_1^m z_2^n$ on the bi-disc $\mathbb{D} \times \mathbb{D}$. It naturally possesses the reproducing kernel $K^{(\lambda, \mu)}(\mathbf{z}, \mathbf{w}) = (1 - z_1 \bar{w}_1)^{-\lambda} (1 - z_2 \bar{w}_2)^{-\mu}$, where $\mathbf{z} = (z_1, z_2)$ and $\mathbf{w} = (w_1, w_2)$ are both in $\mathbb{D} \times \mathbb{D}$. These modules are then Möb \times Möb-homogeneous, with respect to the obvious action of this group on $\mathcal{M}^{(\lambda, \mu)}$. We now show that these are the only Möb \times Möb-homogeneous modules which are in $B_1(\mathbb{D}^2)$.

THEOREM 3. *Let \mathcal{M} be a Hilbert module over the bi-disc algebra $\mathcal{A}(\mathbb{D}^2)$. Assume that \mathcal{M} is in $B_1(\mathbb{D}^2)$ and that it is homogeneous. Then \mathcal{M} is isomorphic to $\mathcal{M}^{(\lambda, \mu)}$ for some $\lambda, \mu > 0$.*

Proof. Let γ be a holomorphic section for the bundle E corresponding to \mathcal{M} . It then follows that $\gamma \circ \phi^{-1}$ is a holomorphic section for the module $\phi_* \mathcal{M}$, where $\phi = (\varphi_1, \varphi_2)$ is an arbitrary element of the group Möb \times Möb. These modules are then isomorphic if and only if the curvatures of the bundle E corresponding to \mathcal{M} and the bundle $\phi^* E$ corresponding to $\phi_* \mathcal{M}$ are equal. Let \mathcal{K}_E be the curvature of the line bundle E , that is,

$$\mathcal{K}_E(z) = \sum_{i,j=1}^2 \bar{\partial}_i \partial_j \log \|\gamma(z)\|^2 d\bar{z}_i \wedge dz_j.$$

It will be convenient to let \mathcal{K}_E also denote the coefficient matrix of the curvature of the line bundle E , namely

$$\mathcal{K}_E = D \log \|\gamma\|^2, \text{ where } D = (\bar{\partial}_i \partial_j)_{i,j=1,2}.$$

Using the chain rule, we find that the curvature of $\phi^* E$ can be related to the curvature of E as follows. For $\mathbf{z} \in \mathbb{D}^2$,

$$\begin{aligned} \mathcal{K}_{\phi^* E}(\mathbf{z}) &= D \log \|\gamma \circ \phi^{-1}(\mathbf{z})\|^2 \\ &= D \phi^{-1}(\mathbf{z})^* \mathcal{K}_E(\phi^{-1}(\mathbf{z})) D \phi^{-1}(\mathbf{z}) \\ &= \begin{pmatrix} e^{i\theta_1} \frac{1-|a_1|^2}{1+\bar{a}_1 z_1} & 0 \\ 0 & e^{i\theta_2} \frac{1-|a_2|^2}{1+\bar{a}_2 z_2} \end{pmatrix}^* \mathcal{K}_E(\phi^{-1}(\mathbf{z})) \begin{pmatrix} e^{i\theta_1} \frac{1-|a_1|^2}{1+\bar{a}_1 z_1} & 0 \\ 0 & e^{i\theta_2} \frac{1-|a_2|^2}{1+\bar{a}_2 z_2} \end{pmatrix}. \end{aligned} \quad (6.1)$$

The equality of the curvatures for E and $\phi^* E$ now amounts to

$$\mathcal{K}_E(\mathbf{z}) = \begin{pmatrix} e^{i\theta_1} \frac{1-|a_1|^2}{1+\bar{a}_1 z_1} & 0 \\ 0 & e^{i\theta_2} \frac{1-|a_2|^2}{1+\bar{a}_2 z_2} \end{pmatrix}^* \mathcal{K}_E(\phi^{-1}(\mathbf{z})) \begin{pmatrix} e^{i\theta_1} \frac{1-|a_1|^2}{1+\bar{a}_1 z_1} & 0 \\ 0 & e^{i\theta_2} \frac{1-|a_2|^2}{1+\bar{a}_2 z_2} \end{pmatrix}$$

for all ϕ in $\text{Möb} \times \text{Möb}$. By setting, $\mathbf{z} = 0$ in the equation relating the curvature \mathcal{K}_E at \mathbf{z} and at $\phi^{-1}(\mathbf{z})$, we see that

$$\mathcal{K}_E(a_1, a_2) = \begin{pmatrix} e^{-i\theta_1} \frac{1}{1-|a_1|^2} & 0 \\ 0 & e^{-i\theta_2} \frac{1}{1-|a_2|^2} \end{pmatrix}^* \mathcal{K}_E(0, 0) \begin{pmatrix} e^{-i\theta_1} \frac{1}{1-|a_1|^2} & 0 \\ 0 & e^{-i\theta_2} \frac{1}{1-|a_2|^2} \end{pmatrix}.$$

We can now put $a_1 = 0 = a_2$ to infer that $\mathcal{K}_E(0, 0)$ must be diagonal, with diagonals equal to λ, μ , say.

Finally, we can show, without loss of generality by setting $\theta_1 = 0 = \theta_2$, that the curvature has the form

$$\mathcal{K}_E(a_1, a_2) = \begin{pmatrix} \lambda(1-|a_1|^2)^{-2} & 0 \\ 0 & \mu(1-|a_2|^2)^{-2} \end{pmatrix},$$

for $(a_1, a_2) \in \mathbb{D}^2$. However, the curvature of the module $\mathcal{M}^{(\lambda, \mu)}$ has exactly this form. So, we conclude that the homogeneous module \mathcal{M} is isomorphic to $\mathcal{M}^{(\lambda, \mu)}$. \square

The notion of homogeneity can be adapted easily to *quotient modules* over the bi-disc algebra. Let \mathcal{M} be a module over the bi-disc algebra which is in the class $B_2(\mathbb{D}^2)$. Let us define the module $\phi_* \mathcal{M}$ to be the module which as a Hilbert space is the same as \mathcal{M} . However, the algebra $\mathcal{A}(\mathbb{D}^2)$ now acts via the map $(f, h) \mapsto (f \circ \phi) \cdot h$, where $\phi = (\varphi, \varphi)$ with φ in Möb . Let \mathcal{M}_0 be the submodule of functions vanishing to order k on the diagonal set $\{(z, z) : z \in \mathbb{D}\} \subseteq \mathbb{D}^2$. Then the action $(f, h) \mapsto (f \circ \phi) \cdot h$ of the algebra $\mathcal{A}(\mathbb{D}^2)$ on \mathcal{M} leaves the submodule \mathcal{M}_0 invariant. The Hilbert space \mathcal{M} equipped with this action of the bi-disc algebra is denoted by $\varphi_* \mathcal{M}$.

We note that the submodule $\mathcal{M}_0^{(\lambda, \mu)} \subseteq \mathcal{M}^{(\lambda, \mu)}$ of functions vanishing on the diagonal $\{(z, z) : z \in \mathbb{D}\} \subseteq \mathbb{D}^2$ is invariant under this action. Therefore, $\varphi_* \mathcal{M}_0^{(\lambda, \mu)}$ is a submodule of $\varphi_* \mathcal{M}^{(\lambda, \mu)}$. Consequently, we may form the quotient module $\varphi_* \mathcal{Q}^{(\lambda, \mu)} = \varphi_* \mathcal{M}^{(\lambda, \mu)} / \varphi_* \mathcal{M}_0^{(\lambda, \mu)}$. We clearly have, in view of Corollary 6.1, $\varphi_* \mathcal{Q}$ isomorphic to $\varphi_* \mathcal{Q}^{(\lambda, \mu)}$ for φ in Möb . This prompts the following Definition.

DEFINITION 6.1. Let $0 \leftarrow \mathcal{Q} \leftarrow \mathcal{M} \leftarrow \mathcal{M}_0 \leftarrow 0$ be a short exact sequence of Hilbert modules over the bi-disc algebra with the property that the natural action of the group Möb leaves the submodule \mathcal{M}_0 invariant. The quotient module \mathcal{Q} is said to be *homogeneous* if $\varphi_*\mathcal{Q} := \phi_*\mathcal{M}/\phi_*\mathcal{M}_0$ is isomorphic to \mathcal{Q} for all φ in the Möbius group.

COROLLARY 6.1. Let $\mathcal{M}_0^{(\lambda,\mu)}$ be the submodule of $\mathcal{M}^{(\lambda,\mu)}$ which consist of functions vanishing to order 2 on the diagonal set $\Delta = \{(z, z) : z \in \mathbb{D}\}$. Then the quotient module $\mathcal{Q}^{(\lambda,\mu)} = \mathcal{M}^{(\lambda,\mu)}/\mathcal{M}_0^{(\lambda,\mu)}$ is homogeneous.

The proof of this Corollary is a straightforward application of the Theorem 1 which in the case of rank 2, as we have pointed out, says that the restriction of the curvature to the zero set is a complete set of invariants for the quotient modules. An explicit description of these quotient modules follows.

6.3. Let $\mathcal{M}_0^{(\lambda,\mu)}$ be the subspace of all functions in $\mathcal{M}^{(\lambda,\mu)}$ which vanish to order k on the diagonal $\{(z, z) : z \in \mathbb{D}\} \subseteq \mathbb{D} \times \mathbb{D}$. To describe the quotient $\mathcal{M}^{(\lambda,\mu)}/\mathcal{M}_0^{(\lambda,\mu)}$, it will be useful to consider the ascending chain

$$(6.2) \quad \{0\} = V_0(p) \subseteq V_1(p) \subseteq V_2(p) \cdots \subseteq V_{p+1}(p) = \text{Hom}(p),$$

where $\text{Hom}(p)$ is the space of homogeneous polynomials of degree p and $V_k(p)$ is the subspace of $\text{Hom}(p)$ which is orthogonal to the submodule $\mathcal{M}_0^{(\lambda,\mu)}$. The second named author and B. Bagchi have developed methods to calculate $f_p^{(k)} \in V_k(p) \ominus V_{k-1}(p)$ for $1 \leq k \leq p+1$. These calculations are also related to the recent work of Ferguson and Rochberg on higher order Hankel forms [19]. Also, in a recent paper, Peng and Zhang [24] have shown how to carry out such calculations in the context of much more general domains. However, for our purposes, we will give the details of these calculations for the case of $k = 2$ only.

First, we compute an orthonormal basis for the quotient module $\mathcal{Q} = \mathcal{M}^{(\lambda,\mu)}/\mathcal{M}_0^{(\lambda,\mu)}$. We then describe the compression of the two operators, $M_1 : f \mapsto z_1 f$ and $M_2 : f \mapsto z_2 f$ for $f \in \mathcal{M}^{(\lambda,\mu)}$, on the quotient module \mathcal{Q} , as a block weighted shift operator with respect to the orthonormal basis we have computed. These are homogeneous operators in the class $B_2(\mathbb{D})$ which were first discovered by Wilkins [27].

It is easily seen that

$$\begin{aligned} g_p^{(1)} &= \sum_{\ell=0}^p \frac{z_1^{p-\ell} z_2^\ell}{\|z_1^{p-\ell}\|^2 \|z_2^\ell\|^2} \\ g_p^{(2)} &= \sum_{\ell=0}^p \frac{\ell z_1^{p-\ell} z_2^\ell}{\|z_1^{p-\ell}\|^2 \|z_2^\ell\|^2} \end{aligned}$$

are in $V_1(p)$ and $V_2(p)$ respectively. We set $f_p^{(1)} = g_p^{(1)}$. To find $f_p^{(2)}$, all we have to do is to find constants a_p, b_p such that

$$\sum_{\ell=0}^p \frac{a_p \ell + b_p}{\|z_1^{p-\ell}\|^2 \|z_2^\ell\|^2} = 0.$$

This will ensure that $f_p^{(2)} = b_p g_p^{(1)} + a_p g_p^{(2)}$ vanishes on the set $\{(z, z) : z \in \mathbb{D}\}$. Hence it must be orthogonal to V_1 . It is clear that $a_p = -\sum_{\ell=0}^p \frac{1}{\|z_1^{p-\ell}\|^2 \|z_2^\ell\|^2}$ and $b_p = \sum_{\ell=0}^p \frac{\ell}{\|z_1^{p-\ell}\|^2 \|z_2^\ell\|^2}$ meet the requirement. Therefore,

$$\left\{ e_p^{(1)} = \frac{f_p^{(1)}}{\|f_p^{(1)}\|}, e_p^{(2)} \stackrel{\text{def}}{=} \frac{f_p^{(2)}}{\|f_p^{(2)}\|} \right\}_{p=0}^\infty$$

forms an orthonormal set of vectors in the quotient module $\mathcal{M}/\mathcal{M}_2^{(\lambda, \mu)}$. To calculate the module action, we first note that

$$\begin{aligned} (1 - |z_1|^2)^{-(\lambda+\mu)} &= (1 - |z_1|^2)^{-\lambda} (1 - |z_2|^2)^{-\mu} \Big|_{z_1=z_2} \\ &= \sum_{p=0}^\infty \sum_{\ell=0}^p \frac{|z_1|^{2(p-\ell)} |z_2|^{2\ell}}{\|z_1^{p-\ell}\|^2 \|z_2^\ell\|^2} \Big|_{z_1=z_2} \\ &= \sum_{p=0}^\infty |z_1|^{2p} \sum_{\ell=0}^p \|z_1^{p-\ell}\|^{-2} \|z_2^\ell\|^{-2}. \end{aligned}$$

It follows that $-a_p = \|f_p^{(1)}\|^2$ is the coefficient of z^p in the expansion of $(1 - |z_1|^2)^{-(\lambda+\mu)}$ which is $\binom{-(\lambda+\mu)}{p}$. Similarly,

$$\begin{aligned} \mu(1 - |z_1|^2)^{-(\lambda+\mu+1)} &= (1 - |z_1|^2)^{-\lambda} \frac{d}{d|z_2|^2} (1 - |z_2|^2)^{-\mu} \Big|_{z_1=z_2} \\ &= \sum_{p=0}^\infty \sum_{\ell=0}^p \frac{|z_1|^{2(p-\ell)} \ell |z_2|^{2(\ell-1)}}{\|z_1^{p-\ell}\|^2 \|z_2^\ell\|^2} \Big|_{z_1=z_2} \\ &= \sum_{p=0}^\infty |z_1|^{2(p-1)} \sum_{\ell=0}^p \ell \|z_1^{p-\ell}\|^{-2} \|z_2^\ell\|^{-2}. \end{aligned}$$

Therefore, we see that $b_p = \langle g_p^{(1)}, g_p^{(2)} \rangle$ is the coefficient of z^{p-1} in the expansion of $\mu(1 - |z_1|^2)^{-(\lambda+\mu+1)}$ which is $\mu \binom{-(\lambda+\mu+1)}{p-1}$.

Further,

$$\begin{aligned} \mu(1 + \mu|z_1|^2)(1 - |z_1|^2)^{-(\lambda+\mu+2)} &= (1 - |z_1|^2)^{-\lambda} \frac{d}{d|z_2|^2} (|z_2|^2 \frac{d}{d|z_2|^2} (1 - |z_2|^2)^{-\mu}) \Big|_{z_1=z_2} \\ &= \sum_{p=0}^\infty \sum_{\ell=0}^p \frac{|z_1|^{2(p-\ell)} \ell^2 |z_2|^{2(\ell-1)}}{\|z_1^{p-\ell}\|^2 \|z_2^\ell\|^2} \Big|_{z_1=z_2} \\ &= \sum_{p=0}^\infty |z_1|^{2(p-1)} \sum_{\ell=0}^p \ell^2 \|z_1^{p-\ell}\|^{-2} \|z_2^\ell\|^{-2}. \end{aligned}$$

Consequently, if we set $c_p = \|g_p^{(2)}\|^2$, then c_p is the coefficient of z^{p-1} in the expansion of $\mu(1 + \mu|z_1|^2)(1 - |z_1|^2)^{-(\lambda+\mu+2)}$ which is $\mu \binom{-(\lambda+\mu+2)}{p-1} + \mu \binom{-(\lambda+\mu+2)}{p-2}$. We find that

$$(6.3) \quad \|g_p^{(1)}\|^2 \|g_p^{(2)}\|^2 - \langle g_p^{(1)}, g_p^{(2)} \rangle^2 = \frac{\lambda\mu}{\lambda + \mu} \binom{-(\lambda + \mu)}{p} \binom{-(\lambda + \mu + 2)}{p-1}.$$

It is now easy to compute the norm of $f_p^{(2)}$:

$$\begin{aligned}
 \|f_p^{(2)}\|^2 &= \left\| \langle g_p^{(1)}, g_p^{(2)} \rangle g_p^{(1)} - \|g_p^{(1)}\|^2 g_p^{(2)} \right\|^2 \\
 &= \|g_p^{(1)}\|^2 (\|g_p^{(1)}\|^2 \|g_p^{(2)}\|^2 - \langle g_p^{(1)}, g_p^{(2)} \rangle^2) \\
 (6.4) \quad &= \frac{\lambda\mu}{\lambda+\mu} \binom{-(\lambda+\mu)}{p}^2 \binom{-(\lambda+\mu+2)}{p-1}.
 \end{aligned}$$

Now, we have all the ingredients to compute the module action. Let us first compute the matrix $M_p^{(1)} = \begin{pmatrix} \alpha_p^{(1)} & 0 \\ \beta_p^{(1)} & \eta_p^{(1)} \end{pmatrix}$ for multiplication by z_1 with respect to the orthonormal basis $\{e_p^{(1)}, e_p^{(2)}\}_{p=0}^\infty$. It is clear that

$$\begin{aligned}
 \alpha_p^{(1)} &= \langle z_1 e_p^{(1)}, e_{p+1}^{(1)} \rangle \\
 &= \frac{1}{\|g_{p+1}^{(1)}\| \|g_p^{(1)}\|} \left\langle \sum_{\ell=0}^p \frac{z_1^{p+1-\ell}}{\|z_1^{p-\ell}\|^2} \frac{z_2^\ell}{\|z_2^\ell\|^2}, \sum_{\ell=0}^{p+1} \frac{z_1^{p+1-\ell}}{\|z_1^{p+1-\ell}\|^2} \frac{z_2^\ell}{\|z_2^\ell\|^2} \right\rangle \\
 &= \frac{1}{\|g_{p+1}^{(1)}\| \|g_p^{(1)}\|} \sum_{\ell=0}^p \|z_1^{p-\ell}\|^{-2} \|z_2^\ell\|^{-2} \\
 &= \frac{\|g_p^{(1)}\|}{\|g_{p+1}^{(1)}\|} \\
 &= \frac{\binom{-(\lambda+\mu)}{p}^{1/2}}{\binom{-(\lambda+\mu)}{p+1}^{1/2}}.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \beta_p^{(1)} &= \langle z_1 e_p^{(1)}, e_{p+1}^{(2)} \rangle \\
 &= \frac{1}{\|g_p^{(1)}\| \|f_p^{(2)}\|} \langle g_p^{(1)}, f_p^{(2)} \rangle \\
 &= \frac{1}{\|g_p^{(1)}\| \|f_p^{(2)}\|} \langle g_{p+1}^{(1)}, g_{p+1}^{(2)} \rangle \|g_p^{(1)}\|^2 - \langle g_p^{(1)}, g_p^{(2)} \rangle \|g_{p+1}^{(1)}\|^2 \\
 &= \left(\frac{\mu}{\lambda}\right)^{1/2} (\lambda + \mu + 1)^{1/2} \left((\lambda + \mu + p)(\lambda + \mu + p + 1) \right)^{-1/2}
 \end{aligned}$$

Finally, we have

$$\begin{aligned}
\eta_p^{(1)} &= \langle z_1 e_p^{(2)}, e_{p+1}^{(2)} \rangle \\
&= \frac{1}{\|f_p^{(2)}\| \|f_{p+1}^{(2)}\|} \langle z_1 f_p^{(2)}, f_{p+1}^{(2)} \rangle \\
&= \frac{1}{\|f_p^{(2)}\| \|f_{p+1}^{(2)}\|} \|g_{p+1}^{(1)}\|^2 (\|g_p^{(1)}\|^2 \|g_p^{(2)}\|^2 - \langle g_p^{(1)}, g_p^{(2)} \rangle^2) \\
&= \frac{\left(\frac{-(\lambda+\mu+2)}{p-1}\right)^{1/2}}{\left(\frac{-(\lambda+\mu+2)}{p}\right)^{1/2}}
\end{aligned}$$

Since $e_p^{(2)} = 0$ on the set $\{(z_1, z_2) : z_1 = z_2\}$, it follows that $z_2 e_p^{(2)} = 0$ on this set as well. Hence the projection of $z_2 e_p^{(2)}$ to the subspace $V_1(p)$ is 0. Consequently, $M_p^{(1)}(1, 2) = 0$. Similarly, we can compute the matrix $M_p^{(2)} = \begin{pmatrix} \alpha_p^{(2)} & 0 \\ \beta_p^{(2)} & \eta_p^{(2)} \end{pmatrix}$ for multiplication by z_2 with respect to the same orthonormal basis $\{e_p^{(1)}, e_p^{(2)}\}_{p=0}^\infty$ as before. Calculations similar to the ones described above show that $\alpha_p^{(1)} = \alpha_p^{(2)}$ and $\beta_p^{(1)} = \beta_p^{(2)}$. However, $\eta_p^{(2)} = -\frac{\lambda}{\mu} \eta_p^{(1)}$.

Summarizing, the matrix

$$M_p^{(1)} = \begin{pmatrix} \frac{\left(\frac{-(\lambda+\mu)}{p}\right)^{1/2}}{\left(\frac{-(\lambda+\mu)}{p+1}\right)^{1/2}} & 0 \\ \left(\frac{\mu}{\lambda}\right)^{1/2} \frac{(\lambda+\mu+1)^{1/2}}{((\lambda+\mu+p)(\lambda+\mu+p+1))^{1/2}} & \frac{\left(\frac{-(\lambda+\mu+2)}{p-1}\right)^{1/2}}{\left(\frac{-(\lambda+\mu+2)}{p}\right)^{1/2}} \end{pmatrix}$$

represents the operator M_1 which is multiplication by z_1 with respect to the orthonormal basis $\{e_p^{(1)}, e_p^{(2)}\}_{p=0}^\infty$. Similarly,

$$M_p^{(2)} = \begin{pmatrix} \frac{\left(\frac{-(\lambda+\mu)}{p}\right)^{1/2}}{\left(\frac{-(\lambda+\mu)}{p+1}\right)^{1/2}} & 0 \\ -\left(\frac{\lambda}{\mu}\right)^{1/2} \frac{(\lambda+\mu+1)^{1/2}}{((\lambda+\mu+p)(\lambda+\mu+p+1))^{1/2}} & \frac{\left(\frac{-(\lambda+\mu+2)}{p-1}\right)^{1/2}}{\left(\frac{-(\lambda+\mu+2)}{p}\right)^{1/2}} \end{pmatrix}$$

represents the operator M_2 which is multiplication by z_2 with respect to the orthonormal basis $\{e_p^{(1)}, e_p^{(2)}\}_{p=0}^\infty$. Therefore, we see that $Q_1^{(p)} = \frac{1}{2}(M_1^{(p)} - M_2^{(p)})$ is a nilpotent matrix of index 2 while $Q_2^{(p)} = \frac{1}{2}(M_1^{(p)} + M_2^{(p)})$ is a diagonal matrix in case $\mu = \lambda$. These definitions naturally give a pair of operators Q_1 and Q_2 on the quotient module $\mathcal{Q}^{(\lambda, \mu)}$. Let f be a function in the bi-disc algebra $\mathcal{A}(\mathbb{D}^2)$ and

$$f(u_1, u_2) = f_0(u_1) + f_1(u_1)u_2 + f_2(u_1)u_2^2 + \cdots$$

be the Taylor expansion of the function f with respect to the coordinates $u_1 = \frac{z_1+z_2}{2}$ and $u_2 = \frac{z_1-z_2}{2}$. Now the module action for $f \in \mathcal{A}(\mathbb{D}^2)$ in the quotient module $\mathcal{Q}^{(\lambda, \mu)}$ is then given

by

$$\begin{aligned}
f \cdot h &= f(Q_1, Q_2) \cdot h \\
&= f_0(Q_1) \cdot h + f_1(Q_1)Q_2 \cdot h \\
&\stackrel{\text{def}}{=} \begin{pmatrix} f_0 & 0 \\ f_1 & f_0 \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix},
\end{aligned}$$

where $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathcal{Q}^{(\lambda, \mu)}$ is the unique decomposition obtained from realizing the quotient module as the direct sum $\mathcal{Q}^{(\lambda, \mu)} = (\mathcal{M}^{(\lambda, \mu)} \ominus \mathcal{M}_1^{(\lambda, \mu)}) \oplus (\mathcal{M}_1^{(\lambda, \mu)} \ominus \mathcal{M}_2^{(\lambda, \mu)})$, where $\mathcal{M}_i^{(\lambda, \mu)}$, $i = 1, 2$, are the submodules in $\mathcal{M}^{(\lambda, \mu)}$ consisting of all functions vanishing on \mathcal{Z} to order 1 and 2 respectively.

We now calculate the curvature $\mathcal{K}^{(\lambda, \mu)}$ for the bundle $E^{(\lambda, \mu)}$ corresponding to the metric $K^{(\lambda, \mu)}(\mathbf{u}, \mathbf{u})$, where $\mathbf{u} = (u_1, u_2) \in \mathbb{D}^2$. The curvature $\mathcal{K}^{(\lambda, \mu)}$ is easy to compute:

$$\mathcal{K}^{(\lambda, \mu)}(u_1, u_2) = (1 - |u_1 + u_2|^2)^{-2} \begin{pmatrix} \lambda & \lambda \\ \lambda & \lambda \end{pmatrix} + (1 - |u_1 - u_2|^2)^{-2} \begin{pmatrix} \mu & -\mu \\ -\mu & \mu \end{pmatrix}.$$

The restriction of the curvature to the hyper-surface $\{u_2 = 0\}$ is

$$\mathcal{K}^{(\lambda, \mu)}(u_1, u_2)|_{u_2=0} = (1 - |u_1|^2)^{-2} \begin{pmatrix} \lambda + \mu & \lambda - \mu \\ \lambda - \mu & \lambda + \mu \end{pmatrix},$$

where $u_1 \in \mathbb{D}$. Thus we find that if $\lambda = \mu$, then the curvature is of the form $2\lambda(1 - |u_1|^2)^{-2}I_2$.

REMARK 6.1. Let us now compare the two jet bundles, corresponding to $\lambda = \mu$ and $\lambda_1 \neq \mu_1$ such that $\lambda_1 + \mu_1 = 2\lambda$. We see that the tangential and the transverse curvatures of these line bundles restricted to the hyper-surface $\{u_2 = 0\}$ are then equal. However, the jet bundles in these two cases are not equivalent (which is the same as saying that the quotient modules are not equivalent). The second fundamental form, which is "essentially" the off diagonal entry in the restriction of the curvature, distinguishes them. In the first case it is 0 and in the second case it is not!

We now describe the unitary map which is basic to the construction of the quotient module, namely,

$$h \mapsto \sum_{\ell=0}^{k-1} \partial_1^\ell h \otimes \varepsilon_{\ell+1} \quad |_{z_1=z_2}$$

for $h \in \mathcal{M}^{(\lambda, \mu)}$. For $k = 2$, it is enough to describe this map just for the orthonormal basis $\{e_p^{(1)}, e_p^{(2)} : p \geq 0\}$. A simple calculation shows that

$$\begin{aligned}
e_p^{(1)}(z_1, z_2) &\mapsto \begin{pmatrix} \binom{-(\lambda+\mu)}{p}^{1/2} z_1^p \\ \mu \sqrt{\frac{p}{\lambda+\mu}} \binom{-(\lambda+\mu+1)}{p-1}^{1/2} z_1^{p-1} \end{pmatrix} \\
e_p^{(2)}(z_1, z_2) &\mapsto \begin{pmatrix} 0 \\ \sqrt{\frac{\lambda\mu}{\lambda+\mu}} \binom{-(\lambda+\mu+2)}{p-1}^{1/2} z_1^{p-1} \end{pmatrix}.
\end{aligned} \tag{6.5}$$

This allows us to compute the 2×2 matrix-valued kernel function

$$K_Q(\mathbf{z}, \mathbf{w}) = \sum_{p=0}^{\infty} e_p^{(1)}(\mathbf{z}) e_p^{(1)}(\mathbf{w})^* + \sum_{p=0}^{\infty} e_p^{(2)}(\mathbf{z}) e_p^{(2)}(\mathbf{w})^*, \quad \mathbf{z}, \mathbf{w} \in \mathbb{D}^2$$

corresponding to the quotient module. Indeed, a straight forward computation shows that

$$\begin{aligned} & K_Q(\mathbf{z}, \mathbf{w})|_{\text{res } \Delta} \\ &= \begin{pmatrix} (1 - |z|^2)^{-(\lambda+\mu)} & \mu z (1 - |z|^2)^{-(\lambda+\mu+1)} \\ \mu \bar{z} (1 - |z|^2)^{-(\lambda+\mu+1)} & \frac{\mu^2}{\lambda+\mu} \frac{d}{d|z|^2} (|z|^2 (1 - |z|^2)^{-(\lambda+\mu+1)}) + \frac{\mu\lambda}{\lambda+\mu} (1 - |z|^2)^{-(\lambda+\mu+2)} \end{pmatrix} \\ &= \left((1 - |z_1|^2)^{-\lambda} \partial^i \bar{\partial}^j (1 - |z_2|^2)^{-\mu} |_{\text{res } \Delta} \right)_{i,j=0,1} \\ &= (JK)(\mathbf{z}, \mathbf{w})|_{\text{res } \Delta}, \quad \mathbf{z}, \mathbf{w} \in \mathbb{D}^2, \end{aligned}$$

where $\Delta = \{(z, z) \in \mathbb{D}^2 : z \in \mathbb{D}\}$. These calculations give an explicit illustration of one of the main theorems on quotient modules from [14, Theorem 3.4].

6.4. Let E be a holomorphic hermitian line bundle defined on the bi-disc and $J^{(k)}E$ be the jet bundle of order k associated to E . The Möbius group acts on the holomorphic sections of the jet bundle $J^{(k)}E$ via the module map $s \mapsto \mathcal{J}\phi \cdot s$, where $\phi = (\varphi, \varphi)$ for φ in Möb. The jet bundle along with this action of the group Möb on its sections will be denoted by $(\mathcal{J}\varphi)^*(J^{(k)}E)$. The bundle E is said to be *Möb – homogeneous of rank k* if the jet bundle $J^{(k)}E$ of E is equivalent to $(\mathcal{J}\varphi)^*(J^{(k)}E)$ on the set $\Delta = \{(z, z) : z \in \mathbb{D}\} \subseteq \mathbb{D}^2$ for all φ in the Möbius group.

It is then natural to ask which quotient modules over the bi-disc algebra are Möb – homogeneous. In the case of rank $k = 2$, we have shown that the modules $\mathcal{M}^{(\lambda, \mu)}$ are Möb \times Möb – homogeneous. Therefore, these are Möb – homogeneous as well. Are there any others? We first consider this question for bundles E over the bi-disc.

Let $\pi : E_{\alpha, \delta}^{\beta} \rightarrow \mathbb{D}^2$ be a hermitian (trivial) holomorphic line bundle determined by the holomorphic frame

$$\gamma(\mathbf{w})(\mathbf{z}) = (1 - z_1 \bar{w}_2)^{\beta} (1 - z_2 \bar{w}_1)^{\beta} (1 - z_1 \bar{w}_1)^{-\alpha} (1 - z_2 \bar{w}_2)^{-\delta}$$

at $\mathbf{w} \in \mathbb{D}^2$. Let $\|\gamma(\mathbf{w})\|^2 = |(1 - w_1 \bar{w}_2)|^{2\beta} (1 - |w_1|^2)^{-\alpha} (1 - |w_2|^2)^{-\delta}$. We note that the metric for the jet bundle $J^{(2)}E_{\alpha, \delta}^{\beta}$ is then given by $(\partial_1^i \bar{\partial}_1^j \|\gamma(\mathbf{w})\|^2)_{i,j=0,1}$. But for this to be positive definite at $\mathbf{w} = (w, w)$, $w \in \mathbb{D}$, we must have the conditions: $\alpha, \delta > 0$ and $\alpha\delta - |\beta|^2 > 0$.

THEOREM 4. *A holomorphic hermitian line bundle E over the bi-disc is Möb – homogeneous of rank 2 if and only if E is isomorphic to $E_{\alpha, \delta}^{\beta}$ for $\alpha, \delta > 0$ and some real number β satisfying $\alpha\delta - |\beta|^2 > 0$.*

Proof. To prove the “if” part, we compute the curvatures of $E_{\alpha, \delta}^{\beta}$ as well as that of $\varphi^*(E_{\alpha, \delta}^{\beta})$, using the chain rule (6.1), and verify that the restrictions of these to the set Δ are equal.

For the “only if” part, let E be a line bundle which is Möb – homogeneous of rank 2. Let $\mathcal{K}(\mathbf{z}) = \sum_{i,j=1}^2 \mathcal{K}_{ij}(\mathbf{z}) dz_i \wedge d\bar{z}_j$ be the $(1, 1)$ form valued curvature of the line bundle E . Then the coefficients $\mathcal{K}_{ij}|_{\text{res } \Delta}$ form a complete set of invariants for $J^{(2)}E|_{\text{res } \Delta}$.

On the other hand, it is easy to see using the chain rule (6.1) that the curvature \mathcal{K}_{φ^*E} restricted to the set Δ is given by the formula

$$\mathcal{K}_{\varphi^*E}(z, z) = \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} (\mathcal{K}_E \circ \phi^{-1})(z, z),$$

where $\phi^{-1} = (\varphi^{-1}, \varphi^{-1})$ and $\varphi(z) = \frac{z-a}{1-\bar{a}z}$ for $a \in \mathbb{D}$. Now, if $\varphi^*(J^{(2)}E)$ is unitarily equivalent to $J^{(2)}E$ on $\Delta \subseteq \mathbb{D}^2$, then

$$(\mathcal{K}_E)_{ij}(z, z) = \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} ((\mathcal{K}_E)_{ij} \circ \varphi^{-1})(z, z)$$

for all $a \in \mathbb{D}$. Putting $z = 0$, we obtain

$$(6.6) \quad (\mathcal{K}_E)_{ij}(0, 0)(1 - |a|^2)^{-2} = (\mathcal{K}_E)_{ij}(a, a), \quad (\mathcal{K}_E)_{ij}(0, 0) = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \delta \end{pmatrix}.$$

We assume that the metric h for E is normalized at 0. The curvature of E at 0 for a normalized metric is $\sum_{i,j=1}^2 (\bar{\partial}_i \partial_j h)(0) d\bar{z}_i \wedge dz_j$. However, the metric for the jet bundle $J^{(2)}E$ at 0 is $(\bar{\partial}_i \partial_j h)(0)_{i,j=1}^2$. This metric must be positive definite which is equivalent to the condition $\alpha\delta - |\beta|^2 > 0$.

For the rest of the proof, it will be convenient to work with the coordinates $u_1 = (z_1 + z_2)/2$ and $u_2 = (z_1 - z_2)/2$. The curvature of the bundle E with respect to these new coordinates is then easily seen to be of the form

$$(6.7) \quad \mathcal{K}_E(u_1, u_2)|_{u_2=0} = \begin{pmatrix} \alpha + \delta + \beta + \bar{\beta} & \alpha - \delta + \beta - \bar{\beta} \\ \alpha - \delta + \beta - \bar{\beta} & \alpha + \delta - (\beta + \bar{\beta}) \end{pmatrix} (1 - |u_1|^2)^{-2}.$$

Let us set $a = \alpha + \delta + \beta + \bar{\beta}$, $b = \alpha - \delta + \beta - \bar{\beta}$, and $c = \alpha + \delta - (\beta + \bar{\beta})$. Let $\gamma(u_1, u_2) = \sum_{m,n=0}^{\infty} a_{mn}(u_1, \bar{u}_1) u_2^m \bar{u}_2^n$ be a positive real analytic function on \mathbb{D}^2 . We will try to find the coefficients a_{mn} so as to ensure that the curvature of γ restricted to the set $u_2 = 0$ satisfies the equation (6.7). We will let ∂_i denote differentiation with respect to u_1 or u_2 depending on whether $i = 1$ or $i = 2$. It is clear that the equation (6.7) forces

$$(6.8) \quad \begin{aligned} \left(\frac{\partial^2}{\partial_1 \bar{\partial}_1} \log \|\gamma\|^2 \right)_{|u_2=0} &= \frac{\partial^2}{\partial_1 \bar{\partial}_1} \log a_{00} \\ &= a(1 - |u_1|^2)^{-2}. \end{aligned}$$

It then follows that $a_{00} = (1 - |u_1|^2)^{-a}$. Similar calculations show that

$$(6.9) \quad \begin{aligned} \left(\frac{\partial^2}{\partial_1 \bar{\partial}_2} \log \|\gamma\|^2 \right)_{|u_2=0} &= a_{00}^{-2} (a_{00} \bar{\partial}_1 a_{10} - a_{10} \bar{\partial}_1 a_{00}) \\ &= b(1 - |u_1|^2)^{-2}. \end{aligned}$$

Choosing $a_{10} = (b/a) \partial_1 a_{00} = b \bar{u}_1 (1 - |u_1|^2)^{-a-1} = b a_{00} \bar{u}_1 (1 - |u_1|^2)^{-1}$, we verify the equation (6.9). Finally, we have

$$(6.10) \quad \begin{aligned} \left(\frac{\partial^2}{\partial_2 \bar{\partial}_2} \log \|\gamma\|^2 \right)_{|u_2=0} &= a_{00}^{-2} (a_{11} a_{00} - |a_{10}|^2) \\ &= c(1 - |u_1|^2)^{-2}. \end{aligned}$$

We can now solve for

$$\begin{aligned}
 (6.11) \quad a_{11} &= a_{00}^{-1} (c(1 - |u_1|^2)^{-2} a_{00}^2 + b^2 |u_1|^2 a_{00}^2 (1 - |u_1|^2)^{-2}) \\
 &= a_{00} (1 - |u_1|^2)^{-2} (c + b^2 |u_1|^2).
 \end{aligned}$$

Recall that the restriction of the curvature determines the coefficients a_{00}, a_{10} and a_{11} in the metric γ modulo unitary equivalence of the quotient modules. Therefore, the positive definite matrix-valued function $\Gamma = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$ describes all possible *homogeneous* bundles of rank 2 on the bi-disc. We see that the jet bundle $J^{(2)} E_{\alpha, \delta}^\beta$ on Δ may be obtained from the line bundle $E_{\alpha, \delta}^\beta$ and that the curvature of this line bundle, computed with respect to the variables u_1, u_2 at $(u_1, 0)$, $u_1 \in \mathbb{D}$, is exactly what is prescribed in (6.7). This completes the proof. \square

Whether the holomorphic hermitian line bundles $E_{\alpha, \delta}^\beta$, $\alpha\delta - |\beta|^2 > 0$, correspond to a Hilbert module \mathcal{M} over the algebra $\mathcal{A}(\Omega)$ depends on the question of positive definiteness of the function $\gamma(\mathbf{w})(\mathbf{z})$ for $\mathbf{z}, \mathbf{w} \in \mathbb{D}^2$.

7. SOME CLOSING REMARKS

As is true in many cases, the current paper probably raises as many questions as it answers. While our hope is to investigate many of the directions suggested in the future, we want to point them out here. Also, other thoughts seem to be of a more intuitive, preliminary nature but promise tantalizing connections with other topics. We will attempt to record these possibilities as well.

7.1. We begin with a succinct conceptual recollection of the original connection of operator theory with complex geometry couched in the context studied in this paper.

As mentioned in Section 1, the kernel function $K_{\mathcal{M}}$ defined for a finite-rank k quasi-free Hilbert module \mathcal{M} over a domain Ω can be used to define a hermitian holomorphic rank k vector bundle $E_{\mathcal{M}}$ which is a pullback of a holomorphic map from Ω to the Grassmanian of k -dimensional subspaces of \mathcal{M} . Moreover, this bundle determines the module up to unitary equivalence. Since for U an open subset of Ω , one can show that the span of the fibers of $E_{\mathcal{M}}$ over U equals \mathcal{M} , the restriction $(E_{\mathcal{M}})|_U$ of $E_{\mathcal{M}}$ to U also determines \mathcal{M} . Hence there is no compelling reason to consider the bundle over the largest open set possible. However, the fibers of $E_{\mathcal{M}}$ over any point of Ω can still be seen in terms of \mathcal{M} .

In particular, the fiber of $E_{\mathcal{M}}$ at $w \in \Omega$ can be identified naturally with the quotient $\mathcal{M}/[\mathcal{A}(\Omega)_w \mathcal{M}]$, where $[\mathcal{A}(\Omega)_w \mathcal{M}]$ denotes the closure of the linear span of the products of $\mathcal{A}(\Omega)_w$ with the functions in \mathcal{M} , and $\mathcal{A}(\Omega)_w$ is the maximal ideal of functions in $\mathcal{A}(\Omega)$ that vanish at w . It is shown in [14] that the disjoint union of these fibers can be identified with $E_{\mathcal{M}}$. Moreover, for f a function in $\mathcal{A}(\Omega)$, the module action defines a holomorphic bundle map on $E_{\mathcal{M}}$ which is multiplication by the scalar $f(w)$. We complete this brief summary by stating the three basic parts of the theory in the form of a theorem.

THEOREM 7.1. *Let \mathcal{M} be a Hilbert module in the class $B_k(\Omega)$ with associated bundle $E_{\mathcal{M}}$. Then*

- (a) *a complete set of “geometric invariants” for a hermitian holomorphic vector bundle E , which determines the bundle up to equivalence, consists of its curvature and sufficiently many partial derivatives of the curvature;*
- (b) *a complete set of “operator invariants” which determines the Hilbert module \mathcal{M} up to unitary equivalence, consists of the m -tuples of commuting nilpotent matrices obtained by restricting the coordinate multiplication operators to the common generalized eigenspaces to high enough order; and*
- (c) *the “geometric invariants” of (a) determines the “operator invariants” of (b) and vice versa.*

We refer the reader to the earlier papers [6, 7, 8, 5] for complete details.

7.2. Now we want to consider the same set of questions for the quotient Hilbert modules considered in this paper.

We begin with a few comments on the notion of an analytic hypersurface. In general, a subset \mathcal{Z} of Ω defined as the zero set of a holomorphic function possesses singularities of various kinds. Even so, the set of smooth manifold points forms a dense open subset \mathcal{Z}' of \mathcal{Z} . Although one can restrict attention to \mathcal{Z}' contained in a smaller open subset of Ω , as we have done, a function in \mathcal{M} that vanishes on \mathcal{Z}' will actually vanish on all of \mathcal{Z} . Moreover, the quotient Hilbert module will yield a kind of spectral sheaf defined over all of \mathcal{Z} with the fibers over singular points also having an operator theoretic meaning. But this phenomenon is a topic for a later investigation. Thus we will assume, as we have done in the paper, that \mathcal{Z} is a smooth manifold.

In Section 3 we showed how to construct the jet bundle $J^{(k)}E_{\mathcal{M}}$ over an open subset U of Ω on which there is a “good defining function” φ and determined the change in this construction corresponding to a change in defining function. An obvious question which presents itself at this point is whether or not the jet bundle can be defined over a neighborhood of \mathcal{Z} or even on all of \mathcal{Z} ? As in the previous section, one can use the fact that the $J^{(k)}E_{\mathcal{M}}$ constructed on an open set U is defined as a pullback bundle from the Grassmanian of k -dimensional subspaces of the quotient Hilbert space \mathcal{Q} to identify it and its fibers concretely, at least over points of \mathcal{Z} . Analogous to the earlier case, such a fiber can be identified with $\mathcal{Q}/[\mathcal{A}^{(k)}(\Omega)_w\mathcal{Q}]$ for w in \mathcal{Z} . Thus, one can show that $J^{(k)}E_{\mathcal{M}}$ is a well-defined hermitian holomorphic vector bundle over \mathcal{Z} . Actually, there is a simpler expression for these fibers. For $w \in \mathcal{Z}$ and v a vector normal to \mathcal{Z} at w , let $\mathcal{A}(\Omega)_{w,v}$ denote the functions in $\mathcal{A}(\Omega)$ for which both the function and the partial derivative in the v -direction vanish at w to order k . Then one can show that $\mathcal{Q}/[\mathcal{A}^{(k)}(\Omega)_{w,v}\mathcal{Q}]$ is naturally isomorphic to $\mathcal{M}/[\mathcal{A}^{(k)}(\Omega)_{w,v}\mathcal{M}]$. (Here the exponent again refers to the linear span of k -fold products.) In this context, even more is true.

The identification of $\mathcal{Q}/[\mathcal{A}^{(k)}(\Omega)_{w,v}\mathcal{Q}]$ with the fiber over w preserves $\mathcal{Q}/[\mathcal{A}^{(i)}(\Omega)_{w,v}\mathcal{Q}]$ for $1 \leq i \leq k$, and hence the flag structure of $J^{(k)}E_{\mathcal{M}}$ is also well-defined over \mathcal{Z} . To make this more precise, one needs to recall the special frame for the jet bundle constructed over an open set U in Section 4. Now the metric on $J^{(k)}E_{\mathcal{M}}$ defined in Section 3 is the same as the one

inherited from the Grassmanian or the quotient norm on $\mathcal{M}/[\mathcal{A}^{(k)}(\Omega)_{w,v}\mathcal{M}]$. But there is even more structure present.

For ψ a function in $\mathcal{A}(\Omega)$, a bounded operator is defined on \mathcal{Q} and hence also on each fiber $\mathcal{Q}/[\mathcal{A}^{(k)}(\Omega)_{w,v}\mathcal{Q}]$. Relative to the special basis chosen in Section 4, the operator at each point w is a Toeplitz-like matrix. In particular, the matrix for a defining function for \mathcal{Z} at w in \mathcal{Z} is a nilpotent matrix of order k . It is the unitary equivalence class of this nilpotent at the points w in \mathcal{Z} that corresponds to the operator invariants for this case. We summarize these results in the following theorem:

THEOREM 7.2. *Let \mathcal{M} be a rank one quasi-free Hilbert module over $\mathcal{A}(\Omega)$ and \mathcal{Z} be an analytic hypersurface contained in Ω . Then the jet bundle $J^{(k)}E_{\mathcal{M}}$ over \mathcal{Z} can be identified with the union of the fibers $\mathcal{M}/[\mathcal{A}^{(k)}(\Omega)_{w,v}\mathcal{M}]$. Moreover, the module action induces by restriction to each fiber an algebra isomorphic to the lower triangular Toeplitz matrices. Finally, the quotient module determines these fiber operators up to unitary equivalence.*

Unfortunately, at this point we don't understand what constitutes a complete set of “operator-theoretic invariants”, although, in analogy with the results described in Section 7.1, we might expect it to be the commuting m - tuple of nilpotents obtained from the restriction of the coordinate multipliers to higher order generalized eigenspaces. These invariants can be viewed as analogues to “geometric invariants” but except for the case $k = 2$, a better description should be possible. We will say more about this matter below. We were, however, able to obtain a complete set of invariants in terms of the operator D_k which is the result we presented in Section 4.

7.3. In this part we begin by reviewing what it means for bundles to be equivalent in terms of frames, in both the contexts of sections 7.1 and 7.2. With that information in hand, we will see that characterizing equivalence can be divided into two parts, equivalence at a point and equivalence in a neighborhood of the point. After that, we will attempt to use this framework to interpret the invariants we have obtained earlier in the paper.

In section 7.1 the bundle $E_{\mathcal{M}}$ in question has rank k and a hermitian holomorphic structure and is defined as a pullback from the Grassmanian. Moreover, at least locally on an open set U of Ω , one can find a holomorphic frame $\{s_1(w), \dots, s_k(w)\}$ which we can take to be holomorphic \mathcal{M} - valued functions on U , where \mathcal{M} is a Hilbert module over $\mathcal{A}(\Omega)$.

Now suppose $\tilde{\mathcal{M}}$ is another Hilbert module over $\mathcal{A}(\Omega)$ which defines a rank k hermitian holomorphic bundle $E_{\tilde{\mathcal{M}}}$ with a holomorphic frame $\{\tilde{s}_1(w), \dots, \tilde{s}_k(w)\}$ also over U . What does it mean to say that $E_{\mathcal{M}}$ and $E_{\tilde{\mathcal{M}}}$ are equivalent over U ?

Essentially, there must exist a $k \times k$ matrix of holomorphic functions $(\psi_{i,j})$ on U such that

- (1) $\tilde{s}_p(w) = \sum_{j=1}^k \psi_{p,j}(w)s_j(w)$ for $p = 1, \dots, k$; and
- (2) the matrix $(\psi_{i,j}(w))$ defines a unitary map between the corresponding fibers of $E_{\mathcal{M}}$ and $E_{\tilde{\mathcal{M}}}$ for $w \in U$.

Now an obvious necessary condition for the existence of such a matrix of functions is that such a matrix must exist at each point w . This is the pointwise condition mentioned above. However, here that condition is vacuously satisfied.

By hypothesis, the set of values at w of a frame over U for $E_{\mathcal{M}}$ forms a basis for the k -dimensional fiber as does the set of values at w of a frame over U for $E_{\tilde{\mathcal{M}}}$. Now both fibers have an inner product and we can find a matrix taking one basis to the other and acting as a unitary. (Note this is not the same thing as saying that the matrix is a unitary matrix since the inner products on the domain and range are different.) However, note that such a matrix is far from being unique since we can both pre- and post-multiply it by a unitary matrix. In case $k = 1$, or the bundles are line bundles, the non-uniqueness is a scalar of modulus one.

In the general case, one can choose a matrix of functions which accomplishes both (1) and (2) but the question is whether or not those functions can be chosen to be holomorphic. That is the question answered in [6], [7], and [8] with the answer involving the curvature and partial derivatives of the curvature. We will not proceed any further with a descriptive analysis in this case.

7.4. Now we want to treat the bundle discussed in 7.2 which arises from the quotient Hilbert module in the same fashion as we did for $E_{\mathcal{M}}$.

In particular, we have the jet bundles for the two Hilbert modules \mathcal{M} and $\tilde{\mathcal{M}}$. Each has rank k and there is a canonical frame $s(w)$ over an open set U for each once one fixes sections $s(w)$ and $\tilde{s}(w)$. The other elements of the frame are obtained by differentiating the given section in the direction normal to the hypersurface \mathcal{Z} using the same good defining function for each. Again, we ask when these two bundles are equivalent but now we want more, not just equivalence of the two bundles but a bundle map effecting that equivalence which is also a module map. Before discussing just what that entails, let us point out that although we didn't mention it in 7.3, the bundle maps discussed there were module maps because the action induced by a multiplier ψ in $\mathcal{A}(\Omega)$ on the fiber over w is just multiplication by the scalar $\psi(w)$.

Now a bundle map effecting an equivalence between $J^{(k)}E_{\mathcal{M}}$ and $J^{(k)}E_{\tilde{\mathcal{M}}}$ must again be a matrix of holomorphic functions satisfying (1) and (2) but now there is also a condition:

- (3) the matrix for the value of $(\psi_{i,j})$ at w Toeplitz-like matrices, that is, it is lower triangular and the entries on a diagonal are predetermined multiples of each other. Moreover, the matrix corresponding to a defining function for \mathcal{Z} at w is a nilpotent of order k which is a single Jordan block.

This latter condition places strong restrictions on the matrix function Ψ , particularly in view of (2) which means it must define a unitary map. And, whereas in the case of 7.3 there is no pointwise obstruction, now there is. This issue can be approached as follows.

Consider a separable, infinite dimensional Hilbert space \mathcal{H} and the collection $N_k(\mathcal{H})$ of ordered, linearly independent k -tuples $X = \{x_1, \dots, x_k\}$ in \mathcal{H} . For a given X in $N_k(\mathcal{H})$, there is a unique element $\text{Gr}(X)$ of the Grassmanian, $\text{Gr}_k(H)$, of k - dimensional subspaces of \mathcal{H} which it determines. There is also an element $\text{St}(X)$ in the complex Stieffel manifold of linearly

independent subsets with k elements. Finally, let us consider the order k nilpotent operator $\text{Nil}(X)$ defined on the span of the vectors in X by the simple shift, that is, the operator which takes x_i to x_{i+1} for $0 \leq i < k$.

We can define several notions of equivalence on $N_k(\mathcal{H})$ as follows. First, we can identify X and X' if the subspaces they span are equal or equivalently, if $\text{Gr}(X) = \text{Gr}(X')$. Second, we can identify them if the two Stieffel elements, $\text{St}(X)$ and $\text{St}(X')$, are unitarily equivalent. Finally, we can identify them if the nilpotent operators $\text{Nil}(X)$ and $\text{Nil}(X')$ are unitarily equivalent. One can easily see that equivalence of the nilpotents implies equivalence of the Stieffel elements implies equivalence of the Grassmanians, and none of the equivalences are the same. Moreover, one can easily determine the Lie group of operators that respect each of the equivalences, in case $X = X'$.

Now let us return to the question of a pointwise obstruction to the existence of a $k \times k$ matrix of holomorphic functions satisfying (1), (2) and (3).

THEOREM 7.3. *Let $s(w)$ and $s'(w)$ be the canonical frames over U for two jet bundles determined by the same defining function and consider the elements $S(w)$ and $S'(w)$ of $N_k(\mathcal{M})$ and $N_k(\mathcal{M}')$, respectively, that they determine. Then a necessary condition for the jet bundles to be equivalent is that $N_k(\mathcal{M})$ and $N_k(\mathcal{M}')$ are equivalent.*

The proof is straightforward since (1), (2), and (3) imply equivalence of the nilpotents.

If $s(w)$ and $s'(w)$ are the canonical frames over U for the two jet bundles determined by the same defining function, then for each $w \in U$ they yield elements in $N_k(\mathcal{M})$ and $N_k(\mathcal{M}')$, respectively, by evaluating the ordered frames at w . Conditions (1), (2) and (3) imply that the corresponding nilpotent operators are unitarily equivalent. Hence for each w , a necessary condition for the jet bundles to be equivalent is that the elements in N_k are equivalent, and this does not always happen. The relationship of this condition to the unitary invariants obtained in this paper will be considered in subsequent work.

7.5. We conclude with a number of comments suggesting additional connections or further lines of investigation of the results of this paper.

The “nilpotent invariants” identified in the previous subsection refine the Stieffel invariants studied earlier and would seem to be related to the “moving frames” of Cartan [4]. Further, one should be able to use the Lie algebra structure relative to the Toeplitz Lie group to define characteristic forms which capture these invariants. Moreover, if one assumes that those invariants are the same for the jet bundles for two line bundles, then the remaining degrees of freedom in choosing the bundle map to be holomorphic essentially amounts to a phase which in this case is a unitary-valued function. The existence question for such a phase would seem to be related to the existence of complex structure and thus to Chern-Moser invariants.

If one considers quotient modules for submodules of functions that vanish to increasing order, then they form a natural inverse limit of Hilbert modules whose limit will be \mathcal{M} . In a dual manner, one should be able to show that the direct limit of the jet bundles constructed for

these quotient modules has a limit equal to $\Omega \times \mathcal{M}$. One way of viewing these constructions would be as expanding \mathcal{M} as a “Taylor series” of modules over \mathcal{Z} .

Finally, assume that there is a global defining function ϕ for \mathcal{Z} in $\mathcal{A}(\Omega)$ and consider the operator it defines on the quotient module defined by the functions which vanish to order k in the direction of \mathcal{Z} . Then ϕ defines a bundle map on the $k \times k$ matrix-valued kernel Hilbert space for the quotient which can be written as the scalar multiplier ϕI_k plus a nilpotent matrix-valued multiplier. Such an operator can be seen to be analogous to the spectral operators of Dunford [17]. That is the case if one replaces a normal operator by a multiplication operator on a space of holomorphic functions. An abstract characterization of operators having such a representation as well as a study of their properties would seem to be of interest.

It is clear that the ideas and techniques of this paper raise many questions that warrant additional study.

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