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Abstract. In this note we settle some technical questions concerning finite rank quasi-free Hilbert modules and develop some useful machinery. In particular, we provide a method for determining when two such modules are unitarily equivalent. Along the way we obtain representations for module maps and study how to determine the underlying holomorphic structure on such modules.

0. Introduction

One approach to multivariate operator theory is via the study of Hilbert modules, which are Hilbert spaces that are acted upon by a natural algebra of functions holomorphic on some bounded domain in complex n-space $\mathbb{C}^n$, (cf. [13], [5]). In this setting, concepts and techniques from commutative algebra as well as from algebraic and complex geometry can be used. In particular, general Hilbert modules can be studied using resolutions by simpler or more basic Hilbert modules. Such an approach generalizes the dilation theory studied in the one variable or single operator setting (cf. [13]). In [11] the existence of resolutions for a large class of Hilbert modules was established with the class of quasi-free Hilbert modules forming the building blocks. Such modules are defined as the Hilbert space completion of a space of vector-valued holomorphic functions that possesses a kernel function. It then follows that a natural Hermitian holomorphic bundle is determined by such a module. However, for a given algebra there are many distinct, inequivalent Hilbert space completions, which raises the question of determining the relation between two such modules.

In this note, we consider this question by examining more carefully the bundle associated with a quasi-free module and introduce a non-negative matrix-valued modulus function for any pair of finite rank quasi-free Hilbert modules. We show that a necessary condition for the modules to be unitarily equivalent is for the modulus to be the absolute value of a holomorphic matrix-valued function. Moreover, if the domain is starlike or bounded, strongly pseudo-convex, we show that this condition is also sufficient. The Hermitian holomorphic vector bundle over Ω associated with a quasi-free Hilbert module possesses a natural connection and curvature. To prove our results we rely upon the localization characterization of unitary equivalence obtained in [13]. In the rank one case, we have line bundles and we show that the difference of the two

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curvatures is equal to the complex two-form-valued Laplacian of the logarithm of the modulus function. This identity enables one to reduce the question of unitary equivalence of two rank one quasi-free Hilbert modules to showing that the latter function vanishes identically.

Along the way we examine closely how one obtains the holomorphic structure on the vector bundle defined by a quasi-free Hilbert module. To accomplish this we introduce the notion of kernel functions dual to a generating set and study concrete representations for module maps between two quasi-free Hilbert modules. These dual kernel functions are closely related to the usual two-variable kernel function. We also raise some related questions for more general Hilbert modules.

In our earlier work, we have assumed the algebra of functions is complete in the supremum norm and hence that it is a commutative Banach algebra. While we continue to make that assumption in this note, we will point out along the way that much weaker assumptions are sufficient for many of the results. In particular, when the domain is the unit ball, it is enough for the polynomial algebra to act on the Hilbert space so that the coordinate functions define contraction operators.

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1. The Modulus for Quasi-Free Hilbert Modules

We use kernel Hilbert spaces over bounded domains in $\mathbb{C}^n$, which are also contractive Hilbert modules for the natural function algebra over the domain. More precisely, we use the kind of Hilbert module introduced in [11] for the study of module resolutions. We first recall the necessary terminology.

For Ω a bounded domain in $\mathbb{C}^n$, let $A(\Omega)$ be the function algebra obtained as the completion of the set of functions that are holomorphic in some neighborhood of the closure of Ω. For Ω the unit ball $\mathbb{B}^n$ or the polydisk $\mathbb{D}^n$ in $\mathbb{C}^n$, we obtain the familiar ball and polydisk algebras, $A(\mathbb{B}^n)$ and $A(\mathbb{D}^n)$, respectively. The Hilbert space $\mathcal{M}$ is said to be a *contractive Hilbert module over* $A(\Omega)$ if $\mathcal{M}$ is a unital module over $A(\Omega)$ with module map $A(\Omega) \times \mathcal{M} \to \mathcal{M}$ such that

$$\|\varphi f\|_{\mathcal{M}} \leq \|\varphi\|_{A(\Omega)} \|f\|_{\mathcal{M}} \text{ for } \varphi \text{ in } A(\Omega) \text{ and } f \text{ in } \mathcal{M}.$$

The space $\mathcal{R}$ is said to be a *quasi-free Hilbert module of rank* m *over* $A(\Omega)$, $1 \leq m \leq \infty$, if it is obtained as the completion of the algebraic tensor product $A(\Omega) \otimes \ell^2_m$ relative to an inner product such that

1) $eval_{\boldsymbol{z}}\colon\ A(\Omega) \otimes \ell^2_m \to \ell^2_m$ is bounded for $\boldsymbol{z}$ in Ω and locally uniformly bounded on Ω;
2) $\|\varphi(\Sigma\theta_i \otimes x_i)\| = \|\Sigma\varphi\theta_i \otimes x_i\|_{\mathcal{R}} \leq \|\varphi\|_{A(\Omega)} \|\Sigma\theta_i \otimes x_i\|_{\mathcal{R}}$ for φ, $\{\theta_i\}$ in $A(\Omega)$ and $\{x_i\}$ in ℓ^2_m; and
3) for $\{F_i\}$ a sequence in $A(\Omega) \otimes \ell^2_m$ that is Cauchy in the $\mathcal{R}$-norm, it follows that $eval_{\boldsymbol{z}}(F_i) \to 0$ for all $\boldsymbol{z}$ in Ω iff $\|F_i\|_{\mathcal{R}} \to 0$.

Here, ℓ^2_m is the m-dimensional Hilbert space.

Actually, condition 2) can be replaced in this paper by:

2′) $\|\varphi(\Sigma\theta_i \otimes x_i)\| \leq K\|\varphi\|_{A(\Omega)}\|\Sigma\theta_i \otimes x_i\|_R$ for $\varphi, \{\theta_i\}$ in $A(\Omega)$ and $\{x_i\}$ in ℓ^2_m for some $K > 0$.

Also, note that condition 3) already occurs in the fundamental paper of Aronszajn [2] in which it is used to conclude that the abstract completion of a space of functions on some domain is again a space of functions.

There is another equivalent definition of quasi-free Hilbert module in terms of a generating set. The contractive Hilbert module $\mathcal{R}$ over $A(\Omega)$ is said to be quasi-free relative to the vectors $\{f_1, \ldots, f_m\}$ if the set generates $\mathcal{R}$ and $\{f_i \otimes_A 1_z\}_{i=1}^m$ forms a basis for $\mathcal{R} \otimes_A \mathbb{C}_z$ for $\boldsymbol{z}$ in Ω. The set of vectors $\{f_i\}$ is called a *generating set* for $\mathcal{R}$. One must also assume that the evaluation functions obtained are locally uniformly bounded and that property 3) holds. In [11], this characterization and other properties of quasi-free Hilbert modules are given. This concept is closely related to the notions of sharp and generalized Bergman kernels studied by Curto and Salinas [7], Agrawal and Salinas [1], and Salinas [19]. In fact, a matrix-valued kernel function $K(\boldsymbol{z},\boldsymbol{\omega})$ on Ω defines a finite rank quasi-free Hilbert module over $A(\Omega)$ if we assume that $K(\boldsymbol{z},\boldsymbol{z})$ is positive definite for $\boldsymbol{z}$ in Ω and the corresponding Hilbert space of vector-valued holomorphic functions on Ω is a contractive Hilbert module over $A(\Omega)$. The proof used the uniform boundedness principle and arguments in [12, p. 286]. We'll say more about this relationship later.

Note that there is a significant difference between the notion of quasi-freeness and membership in the class $\mathcal{B}_n(\Omega)$ introduced in [6] and [7]. For example, let $\mathcal{M}$ be the contractive Hilbert module over $A(\Gamma)$ defined by the analytic Toeplitz operator T_p on the Hardy space $H^2(\mathbb{D})$ for some polynomial $p(z)$, where the closure of $p(\mathbb{D})$ equals the closure of Γ. Then $\mathcal{M}$ is in $\mathcal{B}_k(\Gamma')$ for Γ' any domain in Γ disjoint from $p(\mathbb{T})$, where k is the winding number of the curve $p(\mathbb{T})$ around Γ'. However, $\mathcal{M}$ is a rank k quasi-free Hilbert module relative to an algebra $A(\Gamma')$ iff $p(\mathbb{T})$ equals the boundary of Γ, in which case $\Gamma' = \Gamma$ and k is again the winding number.

We should mention that other authors have investigated the proper notion of freeness for topological modules over Frechet algebras (cf. pp. 76, 123 [14]). Since one allows modules that are the direct sum of finitely many copies of the algebra or the topological tensor product of the algebra with a Frechet space, there can be a closer parallel with what is done in algebra.

Let $\mathcal{R}$ and $\mathcal{R}'$ each be a rank m $(1 \leq m < \infty)$ quasi-free Hilbert module over $A(\Omega)$ for the generating sets of vectors $\{f_i\}$ and $\{g_i\}$, respectively. Then $\{f_i(\boldsymbol{z})\}$ and $\{g_i(\boldsymbol{z})\}$ each forms a basis for ℓ^2_m for $\boldsymbol{z}$ on Ω and $\mathcal{R}$ is the closure of the span of $\{\varphi f_i \mid \varphi \in A(\Omega), 1 \leq i \leq m\}$ while $\mathcal{R}'$ is the closure of the span of $\{\varphi g_i \mid \varphi \in A(\Omega), 1 \leq i \leq m\}$. Consider the subspace Δ of $\mathcal{R} \oplus \mathcal{R}'$ which is the closure of the linear span of $\{\varphi f_i \oplus \varphi g_i \mid \varphi \in A(\Omega), 1 \leq i \leq m\}$ in $\mathcal{R} \oplus \mathcal{R}'$. Let $Hol_m(\Omega)$ be the space of all holomorphic $\mathfrak{L}(\ell^2_m)$-valued functions on Ω.

Lemma 1. *The subspace Δ is the graph of a closed, densely defined, one-to-one transformation $\delta = \delta(\mathcal{R}, \mathcal{R}')$ having dense range. Moreover, the domain and range of δ are invariant under the module action and δ is a module transformation.*

Proof. Since Δ is closed and the domain and range of δ, if it is well-defined, will contain the linear spans of $\{\varphi f_i \mid \varphi \in A(\Omega), 1 \leq i \leq m\}$ and $\{\varphi g_i \mid \varphi \in A(\Omega), 1 \leq i \leq m\}$, respectively, the only thing needing proof is that $h \oplus 0$ or $0 \oplus k$ in Δ implies $h = 0$ and $k = 0$. For $0 \oplus k$

in Δ we have sequences $\{\varphi_i^{(n)}\}$, $1 \leq i \leq m$, such that $\Sigma\varphi_i^{(n)} f_i \to 0$, while $\Sigma\varphi_i^{(n)} g_i \to k$. Since evaluation at $\boldsymbol{z}$ in Ω is continuous in the norm of $\mathcal{R}$, we have that $\Sigma\varphi_i^{(n)}(\boldsymbol{z}) f_i(\boldsymbol{z}) \to 0$ for $\boldsymbol{z}$ in Ω. Since $\{f_i(\boldsymbol{z})\}$ is a fixed basis for ℓ^2_m, it follows that $\varphi_i^{(n)}(\boldsymbol{z}) \to 0$ for $1 \leq i \leq m$. Hence, it follows that $k(\boldsymbol{z}) = \lim_n \Sigma\varphi_i^{(n)}(\boldsymbol{z}) g_i(\boldsymbol{z}) = 0$ and since $k(\boldsymbol{z}) = 0$ for $\boldsymbol{z}$ in Ω, we have $k = 0$ by 3). The same argument works to show $h \oplus 0$ in Δ implies that $h = 0$. □

Although the definition of δ is given in terms of its graph for technical reasons, one should note that δ merely takes the given generating set for $\mathcal{R}$ to the given generating set for $\mathcal{R}'$.

To consider the infinite rank case, we would need to know more about the relationship as bases between the sets of values of the generating sets $\{f_i(\boldsymbol{z})\}$ and $\{g_i(\boldsymbol{z})\}$ in ℓ^2_m for the preceding argument to succeed (cf. [11]).

Note that the graph Δ can also be interpreted as a rank m quasi-free Hilbert module over $A(\Omega)$ relative to the generating set $\{f_i \oplus g_i\}$. Moreover, if we repeat the above construction relative to the pairs $\{\Delta, \mathcal{R}\}$ and $\{\Delta, \mathcal{R}'\}$, the transformations $\delta(\Delta, \mathcal{R})$ and $\delta(\Delta, \mathcal{R}')$ are bounded. Finally, since $\delta(\mathcal{R}, \mathcal{R}') = \delta(\Delta, \mathcal{R}')^{-1}\delta(\Delta, \mathcal{R})$, many calculations for $\delta(\mathcal{R}, \mathcal{R}')$ can be reduced to the analogous calculations for a bounded module map composed with the inverse of a bounded module map.

If evaluation on $\mathcal{R}$ and $\mathcal{R}'$ are both continuous, the lemma holds if we replace $A(\Omega)$ by any algebra of holomorphic functions A so long as A is norm dense in $A(\Omega)$. For example, if Ω is the unit ball $\mathbb{B}^n$ or the polydisk $\mathbb{D}^n$, one could take A to be the algebra of all polynomials $\mathbb{C}[\boldsymbol{z}]$ or the algebra of functions holomorphic on some fixed neighborhood of the closure of Ω.

Now recall that for $\boldsymbol{z}$ in Ω, one defines the module $\mathbb{C}_{\boldsymbol{z}}$ over $A(\Omega)$, where $\mathbb{C}_{\boldsymbol{z}}$ is the one-dimensional Hilbert space $\mathbb{C}$, such that $\varphi \times \lambda = \varphi(\boldsymbol{z})\lambda$ for φ in $A(\Omega)$ and λ in $\mathbb{C}_{\boldsymbol{z}}$. Note that $\mathcal{R} \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{z}} \cong \mathbb{C}_{\boldsymbol{z}} \otimes \ell^2_m$ for $\mathcal{R}$ any rank m quasi-free Hilbert module. Localization of a Hilbert module $\mathcal{M}$ at $\boldsymbol{z}$ in Ω is defined to be the module tensor product $\mathcal{M} \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{z}}$ (cf. [13]), which is canonically isomorphic to the quotient module $\mathcal{M}/\mathcal{M}_{\boldsymbol{z}}$, where $\mathcal{M}_{\boldsymbol{z}}$ is the closure of $A(\Omega)_{\boldsymbol{z}}\mathcal{M}$ and $A(\Omega)_{\boldsymbol{z}} = \{\varphi \in A(\Omega) \mid \varphi(\boldsymbol{z}) = 0\}$. (Again, we can define this construction for an algebra A, as above, so long as the set of functions in A that vanish at a fixed point $\boldsymbol{z}$ in Ω is dense in $A(\Omega)_{\boldsymbol{z}}$.)

In addition to localizing Hilbert modules, one can localize module maps. While localization of bounded module maps is straightforward, here we need to localize δ which is possibly unbounded and hence we must be somewhat careful.

Lemma 2. *For $\boldsymbol{z}$ in Ω, the map $\delta \otimes_{A(\Omega)} 1_{\boldsymbol{z}}\colon\ \mathcal{R} \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{z}} \longrightarrow \mathcal{R}' \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{z}}$ is well-defined. Moreover, $\delta \otimes_{A(\Omega)} 1_{\boldsymbol{z}}$ is an invertible operator on the m-dimensional Hilbert space $\mathbb{C}_{\boldsymbol{z}} \otimes \ell^2_m$.*

Proof. Since for $\boldsymbol{z}$ in Ω, $A(\Omega)_{\boldsymbol{z}} f_i$ is contained in the domain of δ for $1 \leq i \leq m$ and $\delta(A(\Omega)_{\boldsymbol{z}} f_i)$ is contained in the linear span of $\{A(\Omega)_{\boldsymbol{z}} g_i\}$, $1 \leq i \leq m$, we see that one can define δ from $\mathcal{R}/\mathcal{R}_{\boldsymbol{z}}$ to $\mathcal{R}'/\mathcal{R}'_{\boldsymbol{z}}$ as a densely defined, module transformation having dense range. Both $\mathcal{R}/\mathcal{R}_{\boldsymbol{z}}$ and $\mathcal{R}'/\mathcal{R}'_{\boldsymbol{z}}$ are m-dimensional since they are isomorphic to $\mathcal{R} \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{z}}$ and $\mathcal{R}' \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{z}}$, respectively. Since δ has dense range, it follows that $\delta \otimes_{A(\Omega)} 1_{\boldsymbol{z}}$ is onto and thus invertible. Therefore, the final statement holds. □

Localization as defined above is used implicitly in the work of Arveson and others. Consider, for example, the recent paper [3] involving free covers. Since the defect space is simply $F \otimes_{\mathbb{C}[z]} \mathbb{C}_0$, the assumption in Definition 2.2 of [3] is that the localization map $A \otimes_{\mathbb{C}[z]} I_z = \dot{A}$ is unitary. While this observation doesn't add anything per se, it does raise the question about the meaning of localization at other $\boldsymbol{z}$, not just at the origin. We'll say more about this matter later in this note. A similar question can be raised in the work of Davidson [8] who uses the trace which is just the localization map from a module $\mathcal{M}$ to $\mathcal{M} \otimes_A \mathbb{C}_0$. Does consideration of localization at other $\boldsymbol{z}$ add anything? Since the algebra in this case is non-commutative, this question would likely take one into the realm of non-commutative algebraic geometry such as considered by Kontsevich and Rosenberg [18].

The modulus $\mu = \mu(\mathcal{R}, \mathcal{R}')$ of $\mathcal{R}$ and $\mathcal{R}'$ is defined to be the absolute value of $\delta \otimes_{A(\Omega)} 1_{\boldsymbol{z}}$. For $m > 1$, there are two possibilities: the square root of $(\delta \otimes_{A(\Omega)} 1_{\boldsymbol{z}})^*(\delta \otimes_{A(\Omega)} 1_{\boldsymbol{z}})$ and the square root of $(\delta \otimes_{A(\Omega)} 1_{\boldsymbol{z}})(\delta \otimes_{A(\Omega)} 1_{\boldsymbol{z}})^*$. The first operator, which we'll denote by $\mu(\mathcal{R}, \mathcal{R}')$, is defined on $\mathcal{R} \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{z}}$ while the second one, which corresponds to $\mu'(\mathcal{R}, \mathcal{R}')$, is defined on $\mathcal{R}' \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{z}}$. In either case, μ is an invertible positive $m \times m$ matrix function which is distinct from the absolute value of $\delta(\mathcal{R}', \mathcal{R}) = \delta(\mathcal{R}, \mathcal{R}')^{-1}$.

Next we need to know more about the adjoint transformation $\delta^* \colon \mathcal{R}' \to \mathcal{R}$. Recall we know from von Neumann's fundamental results [20], that δ^* exists and its graph is given by the orthogonal complement of Δ, the graph of δ, in $\mathcal{R} \oplus \mathcal{R}'$ after reversing the roles of $\mathcal{R}$ and $\mathcal{R}'$ and introducing a minus sign. In particular, the graph Δ^* of δ^* is equal to $\{h \oplus k \in \mathcal{R}' \oplus \mathcal{R} \mid -k \oplus h \perp \Delta\}$.

For $\boldsymbol{z}$ in Ω, let $\{k^i_{\boldsymbol{z}}\}$ and $\{k'^i_{\boldsymbol{z}}\}$ be elements in $\mathcal{R}$ and $\mathcal{R}'$, respectively, such that $\langle h(\boldsymbol{z}), g_i(\boldsymbol{z})\rangle_{\ell^2_m} = \langle h, k'^i_{\boldsymbol{z}}\rangle_{\mathcal{R}'}$ and $\langle k(\boldsymbol{z}), f_i(\boldsymbol{z})\rangle_{\ell^2_m} = \langle k, k^i_{\boldsymbol{z}}\rangle_{\mathcal{R}}$ for h and k in $\mathcal{R}'$ and $\mathcal{R}$, respectively. Note that the sets $\{k^i_{\boldsymbol{z}}\}$ and $\{k'^i_{\boldsymbol{z}}\}$ span the orthogonal complements of $\mathcal{R}_{\boldsymbol{z}}$ and $\mathcal{R}'_{\boldsymbol{z}}$, respectively. We will refer to the sets $\{k^i_{\boldsymbol{z}}\}$ and $\{k'^i_{\boldsymbol{z}}\}$, as the *dual sets of kernel functions* for the generating sets $\{f_i\}$ for $\mathcal{R}$ and $\{g_i\}$ for $\mathcal{R}'$, respectively. Finally, for $\boldsymbol{z}$ in Ω let $X_{ij}(\boldsymbol{z})$ be the matrix in $\mathfrak{L}(\ell^2_m)$ that satisfies

$$\left\langle \sum_j X_{ij}(\boldsymbol{z}) f_j(\boldsymbol{z}), f_\ell(\boldsymbol{z}) \right\rangle_{\ell^2_m} = \langle g_i(\boldsymbol{z}), g_\ell(\boldsymbol{z})\rangle_{\ell^2_m} \text{ for } 1 \le i, \ell \le m.$$

In other words, $\{X_{ij}\}$ effects the change of basis from $\{f_i\}$ for $\mathcal{R}$ to $\{g_i\}$ for $\mathcal{R}'$. If we define $Y(\boldsymbol{z}) \colon \ell^2_m \to \ell^2_m$ so that $Y(\boldsymbol{z}) f_i(\boldsymbol{z}) = g_i(\boldsymbol{z})$ for $1 \le i \le m$, then $Y(\boldsymbol{z})$ is invertible and $\{X_{ij}(\boldsymbol{z})\}$ is the matrix defining the operator $Y(\boldsymbol{z})^* Y(\boldsymbol{z})$ on ℓ^2_m. Moreover, since the generating sets $\{f_i(\boldsymbol{z})\}$ and $\{g_i(\boldsymbol{z})\}$ are holomorphic, the matrix-function $X_{ij}(\boldsymbol{z})$ is real-analytic.

Lemma 3. *The domain of δ^* contains the finite linear span of $\{k'^i_{\boldsymbol{z}} \mid \boldsymbol{z} \in \Omega, 1 \le i \le m\}$. Moreover,*

$$\delta^* k'^i_{\boldsymbol{z}} = \sum_j X_{ij}(\boldsymbol{z}) k^j_{\boldsymbol{z}}.$$

Proof. Since the span of $\{\varphi f_i \oplus \varphi g_i \mid \varphi \in A(\Omega), 1 \le i \le m\}$ is dense in Δ, it is enough to show that

$$\left\langle \left(-\sum_j X_{ij}(\boldsymbol{z}) k^j_{\boldsymbol{z}} \right) \oplus k'^i_{\boldsymbol{z}}, \varphi f_\ell \oplus \varphi g_\ell \right\rangle = 0$$

for φ in $A(\Omega)$ and $1 \leq \ell \leq m$. But

$$\left\langle \left(-\sum_j X_{ij}(\boldsymbol{z}) k^j_{\boldsymbol{z}} \right) \oplus {k'}^i_{\boldsymbol{z}}, \varphi f_\ell \oplus \varphi g_\ell \right\rangle_{\mathcal{R}\oplus\mathcal{R}'} = \left\langle -\sum_j X_{ij}(\boldsymbol{z}) k^j_{\boldsymbol{z}}, \varphi f_\ell \right\rangle_{\mathcal{R}} + \langle {k'}^i_{\boldsymbol{z}}, \varphi g_\ell \rangle_{\mathcal{R}'}$$

$$= -\sum_j X_{ij}(\boldsymbol{z}) \overline{\varphi(\boldsymbol{z})} \langle k^j_{\boldsymbol{z}}, f_\ell \rangle_{\mathcal{R}} + \overline{\varphi(\boldsymbol{z})} \langle {k'}^i_{\boldsymbol{z}}, g_\ell \rangle_{\mathcal{R}'}$$

$$= \overline{\varphi(\boldsymbol{z})} \left(\left\langle -\sum_j X_{ij}(\boldsymbol{z}) f_j(\boldsymbol{z}), f_\ell(\boldsymbol{z}) \right\rangle_{\ell^2_m} + \langle g_i(\boldsymbol{z}), g_\ell(\boldsymbol{z}) \rangle_{\ell^2_m} \right) = 0$$

by the definition of $\{X_{ij}(\boldsymbol{z})\}$ and thus the result is proved. □

Before we proceed, the notion of the dual set of kernel functions can be used to establish the first notion of holomorphicity, or in fact in this case, anti-holomorphicity, of a quasi-free Hilbert module.

Suppose $\mathcal{R}$ is the completion of $A(\Omega) \otimes_{alg} \ell^2_m$ and we consider the generating set $\{1 \otimes e_i\}$ for $\mathcal{R}$ with the dual set of kernel functions $\{k^i_{\boldsymbol{z}}\}$. As we pointed out above, $\{k^i_{\boldsymbol{z}}\}_{i=1}^m$ spans the orthonormal complement of $\mathcal{R}_{\boldsymbol{z}}$ in $\mathcal{R}$ for $\boldsymbol{z}$ in Ω. For h in $\mathcal{R}$ we have $\langle k^i_{\boldsymbol{z}}, h\rangle_{\mathcal{R}} = \overline{\langle h(\boldsymbol{z}), e_i\rangle_{\ell^2_m}}$ which is an anti-holomorphic function on Ω. Thus $k^i_{\boldsymbol{z}}$ is a weakly anti-holomorphic function and therefore $\boldsymbol{z} \longrightarrow k^i_{\boldsymbol{z}}$ is strongly anti-holomorphic. Finally, since the functions $\{k^i_{\boldsymbol{z}}\}$ span $\mathcal{R}^\perp_{\boldsymbol{z}}$ for $\boldsymbol{z}$ in Ω, we see that $\bigcup_{\boldsymbol{z}\in\Omega} \mathcal{R}^\perp_{\boldsymbol{z}}$ is an anti-holomorphic Hermitian rank m vector bundle over Ω.

We record this result as

Lemma 4. *For $\mathcal{R}$ a finite rank m quasi-free Hilbert module, $\bigcup_{\boldsymbol{z}\in\Omega} \mathcal{R}^\perp_{\boldsymbol{z}}$ is a Hermitian rank m anti-holomorphic vector bundle over Ω.*

With the additional assumption of a "closedness of range" condition, this result is established in [7]. Also, the above proof can be rephrased in terms of the ordinary notion of kernel function and rests on the holomorphicity of the functions in $\mathcal{R}$. Note that we have assumed the local uniformed boundedness of evaluation to reach the conclusion of Lemma 4. On the other hand, as mentioned earlier, if the space is known to consist of holomorphic functions, then this property follows from the uniform boundedness principle. It would be of interest to understand better the relation of this notion to that of the closedness of range condition. In particular, one knows that the latter property does not always hold.

There is one final question concerning the relationship of these concepts. Does there exist a finite matrix-valued kernel function defining a Hilbert space satisfying 2) and 3) of the definition of quasi-free Hilbert module but which is not holomorphic in the first variable and anti-holomorphic in the second? Evaluation could not be locally uniformly bounded for such an example, which would probably be only a curiosity for the theory developed in this paper.

2. Representations of Module Maps

Next we state a result familiar in settings such as the one provided by that of quasi-free Hilbert modules, which we essentially used in the preceding section to define δ^*.

Lemma 5. *If $\mathcal{R}$ and $\mathcal{R}'$ are finite rank quasi-free Hilbert modules over $A(\Omega)$ relative to the generating sets $\{f_i\}_{i=1}^m$ and $\{g_i\}_{i=1}^m$, $1 \le m < \infty$, and X is a module map from $\mathcal{R}$ to $\mathcal{R}'$, then there exists $\Psi = \{\psi_{ij}\}$ in $Hol_m(\Omega)$ such that*

$$Xf_i = \sum_{j=1}^m \psi_{ij} g_j, \quad \textit{for} \quad 1 \le i \le m.$$

Proof. For $\boldsymbol{z}$ in Ω, both $\{f_i(\boldsymbol{z})\}_{i=1}^m$ and $\{g_i(\boldsymbol{z})\}_{i=1}^m$ are bases for ℓ_m^2 and hence there exists a unique matrix $\{\psi_{ij}(\boldsymbol{z})\}_{i,j=1}^m$ such that

$$(Xf_i)(\boldsymbol{z}) = \sum_{j=1}^m \psi_{ij}(\boldsymbol{z}) g_j(\boldsymbol{z}) \quad \text{for} \quad i = 1, 2, \ldots, m.$$

Since the functions $\{(Xf_i)(\boldsymbol{z})\}_{i=1}^m$ and $\{g_i(\boldsymbol{z})\}_{i=1}^m$ are all holomorphic, it follows from Cramer's rule that $\Psi = \{\psi_{ij}\}_{i,j=1}^m$ is in $\mathrm{Hol}_m(\Omega)$ which completes the proof. □

Although we obtain a holomorphic matrix function defining a module map between distinct quasi-free Hilbert modules, this function is not very useful unless the modules and the generating sets are the same. That is because the matrix representing a linear transformation relative to different bases captures little information about the norm of it or the eigenvalues of its absolute value.

Before continuing, we want to show that the multiplier representation for a module map also extends to its localization.

Lemma 6. *If $\mathcal{R}$ and $\mathcal{R}'$ are rank m quasi-free Hilbert modules with generating sets $\{f_i\}$ and $\{g_i\}$, respectively, and $X\colon \mathcal{R} \to \mathcal{R}'$ is the module map from $\mathcal{R}$ to $\mathcal{R}'$ represented by $\Psi = \{\psi_{ij}\}$ in $Hol_m(\Omega)$, then*

$$(X \otimes_A 1_{\mathbb{C}_{\boldsymbol{z}}})(f_i \otimes_A 1_{\boldsymbol{z}}) = \sum_{j=1}^m \psi_{ij}(\boldsymbol{z})(g_j \otimes_A 1_{\boldsymbol{z}}) \textit{ for } \boldsymbol{z} \textit{ in } \Omega.$$

Proof. Let $\{k'^i_{\boldsymbol{z}}\}$ be the set of kernel functions dual to the generating set $\{g_i\}$. Then for a fixed $\boldsymbol{z}$ the span of the set $\{k'^i_{\boldsymbol{z}}\}_{i=1}^m$ is the orthogonal complement of $[A_{\boldsymbol{z}}\mathcal{R}']$ and we can identify $\mathcal{R}' \otimes_A \mathbb{C}_{\boldsymbol{z}}$ with the quotient module $\mathcal{R}'/[A_{\boldsymbol{z}}\mathcal{R}']$. Calculating we see that the vector $Xf_i - \sum\limits_{i=1}^m \psi_{ji}(\boldsymbol{z}) g_j$ is orthogonal to each $k'^i_{\boldsymbol{z}}$, $1 \le \ell \le m$, and hence is in $[A_{\boldsymbol{z}}\mathcal{R}']$. Therefore, we have that

$$(X \otimes_A 1_{\mathbb{C}_{\boldsymbol{z}}})(f_i \otimes_A 1_{\boldsymbol{z}}) = (Xf_i) \otimes_A 1_{\boldsymbol{z}} = \sum_{j=1}^m \psi_{ij}(z)(g_j \otimes_A 1_{\boldsymbol{z}}) \quad \text{for} \quad 1 \le i \le m,$$

which completes the proof. □

Note that this result also holds for the localization of δ. Also, if the ranks of $\mathcal{R}$ and $\mathcal{R}'$ are finite integers m and m' but not equal, then we obtain the same result for a holomorphic $m' \times m$ matrix-valued function.

Although, as we mentioned above, this representation has limited value, it does enable us to investigate the nature of the sets of constancy for the local rank of a module map X between two quasi-free Hilbert modules $\mathcal{R}$ and $\mathcal{R}'$. The previous lemma shows that, this local behavior is the same as that of a holomorphic matrix-valued function. In particular, each singular set Σ_k of $X \otimes_A 1_{\boldsymbol{z}}$, that is, the subset of Ω on which the rank of $X \otimes_A 1_{\boldsymbol{z}}$ is k, is an analytic subvariety of Ω. Thus we have established

Theorem 1. *If $\mathcal{R}$ and $\mathcal{R}'$ are finite rank quasi-free Hilbert modules and X is a module map $X\colon\ \mathcal{R} \to \mathcal{R}'$, then the singular sets Σ_k of $X \otimes_A 1_{\boldsymbol{z}}$ are analytic subvarieties of Ω.*

We intend to use this fact to relate our work to that of Harvey–Lawson [15] in the future. In particular, we expect their formulas for singular connections to be useful in obtaining invariants from resolutions such as those exhibited in [11].

3. Holomorphic Structure

Recall that the spectral sheaf of a Hilbert module $\mathcal{M}$ over $A(\Omega)$ is defined to be $Sp(\mathcal{M}) = \bigcup\limits_{\boldsymbol{z}\in\Omega} \mathcal{M} \otimes_A \mathbb{C}_{\boldsymbol{z}}$ with the collection of sections $\{f \otimes_A 1_{\boldsymbol{z}} \mid f \in \mathcal{M}\}$. A priori the fibers of $Sp(\mathcal{M})$ are isomorphic to the Hilbert modules $\mathbb{C}_{\boldsymbol{z}} \otimes \ell^2_{m_{\boldsymbol{z}}}$, where the dimension $m_{\boldsymbol{z}}$ can vary from point to point and $0 \le m_{\boldsymbol{z}} \le \infty$. If $\mathcal{R}$ is a quasi-free rank m Hilbert module, then $m_{\boldsymbol{z}} = m$ for all $\boldsymbol{z}$, but we would like more. Namely, we would like to define a canonical structure on $Sp(\mathcal{R})$ making it into a holomorphic vector bundle relative to which the sections are holomorphic. We would also like to understand better the relation between the spectral sheaf $Sp(\mathcal{R})$ and the anti-holomorphic vector bundle $\bigcup\limits_{\boldsymbol{z}\in\Omega} \mathcal{R}_{\boldsymbol{z}}^{\perp}$.

Although it might seem straightforward that the spectral sheaf $Sp(\mathcal{R}) = \bigcup\limits_{\boldsymbol{z}\in\Omega} \mathcal{R} \otimes_A \mathbb{C}_{\boldsymbol{z}}$, for a finite rank quasi-free Hilbert module $\mathcal{R}$, is a Hermitian holomorphic vector bundle, it is worth considering how one exhibits such structure and shows that it is well-defined.

Let $\{f_i\}_{i=1}^n$ be a subset of $\mathcal{R}$ relative to which $\mathcal{R}$ is quasi-free and define the map $F(\boldsymbol{z})$ from $\mathcal{R} \otimes_A \mathbb{C}_{\boldsymbol{z}}$ to ℓ^2_m such that $F(\boldsymbol{z})\left(\sum\limits_{i=1}^n \lambda_i (f_i \otimes_A 1_{\boldsymbol{z}})\right) = \sum\limits_{i=1}^n \lambda_i f_i(\boldsymbol{z})$. By the quasi-freeness of $\mathcal{R}$ relative to the generating set $\{f_i\}_{i=1}^m$, it follows that this map is well-defined, one-to-one and onto. Its inverse F^{-1} defines a map from the trivial vector bundle $\Omega \times \ell^2_m$ to the spectral sheaf $Sp(\mathcal{R})$ of $\mathcal{R}$ which can be used to make $Sp(\mathcal{R})$ into a holomorphic vector bundle. It is clear that the sections $f_i \otimes_{A(\Omega)} 1_{\boldsymbol{z}}$ are holomorphic relative to this structure. We see later that the same is true for all k in $\mathcal{R}$. The only issue now is whether the intrinsic norm on the fibers of $Sp(\mathcal{R})$ yields a real-analytic metric on this bundle, which is necessary for $Sp(\mathcal{R})$ to be a Hermitian holomorphic vector bundle.

To show that, consider $F(z)^{-1}\colon\ \ell^2_m \to \mathcal{R} \otimes_A \mathbb{C}_{\boldsymbol{z}}$. We need to know that the function $\boldsymbol{z} \to \langle F(z)^{-1}x, F(z)^{-1}y\rangle_{\mathcal{R}\otimes_A\mathbb{C}_{\boldsymbol{z}}}$ is real-analytic for vectors x and y in ℓ^2_m. Since the functions $\{f_i(\boldsymbol{z})\}$ are holomorphic, the map from a fixed basis $\{e_i\}$ in ℓ^2_m to ℓ^2_m defined by $e_i \to f_i(\boldsymbol{z})$ is holomorphic. Hence, the question rests on the behavior of the Grammian $\{\langle f_i \otimes_A 1_{\boldsymbol{z}}, f_j \otimes_A 1_{\boldsymbol{z}}\rangle_{\mathcal{R}\otimes_A \mathbf{C}_{\boldsymbol{z}}}\}$. Using the dual set of kernel functions $\{k_{\boldsymbol{z}}^{\ell}\}_{\ell=1}^m$ for the generating set $\{f_i\}$, we see that $f_i \otimes_A 1_{\boldsymbol{z}}$, viewed as a vector in $\mathcal{R}$, is the projection of f_i onto $\mathcal{R}_{\boldsymbol{z}}^{\perp}$, the span of the $\{k_{\boldsymbol{z}}^{\ell}\}_{\ell=1}^m$. Now consider the identity involving the inner products $\langle f_i, k_{\boldsymbol{z}}^{\ell}\rangle_{\mathcal{R}} = \langle f_i(\boldsymbol{z}), f_\ell(\boldsymbol{z})\rangle_{\ell^2_m}$ obtained using the defining property of the dual set $\{k_{\boldsymbol{z}}^{\ell}\}$. We see that $\boldsymbol{z} \to \langle f_i, k_{\boldsymbol{z}}^{\ell}\rangle_{\mathcal{R}}$ is real-analytic.

Therefore, inner products of the projections of f_i and f_j onto the span of the $\{k^\ell_{\boldsymbol{z}}\}_{i=1}^m$ are also real-analytic which completes the proof. (Because of linear independence, the expressions can't vanish.)

Now we must consider what happens if we use a different generating set $\{g_i\}_{i=1}^n$ relative to which $\mathcal{R}$ is quasi-free. Using Lemma 5, we see that the map which sends f_i to g_i, $i = 1, 2, \ldots, m$, is defined by a holomorphic $m \times m$ matrix-valued function $\Psi(\boldsymbol{z})$ in $\mathrm{Hol}_m(\Omega)$. That is, we have $g_i(\boldsymbol{z}) = \sum_{j=1}^m \psi_{ij}(\boldsymbol{z}) f_j(\boldsymbol{z})$ for $\boldsymbol{z}$ in Ω and hence $\Psi(\boldsymbol{z})$ defines a holomorphic bundle map which intertwines the holomorphic structures defined by the generating sets $\{f_i\}_{i=1}^n$ and $\{g_i\}_{i=1}^n$. Thus, we have proved:

Theorem 2. *For $\mathcal{R}$ a finite rank quasi-free Hilbert module over $A(\Omega)$, there is a unique, well-defined holomorphic structure on $Sp(\mathcal{R})$ relative to which the functions $\boldsymbol{z} \to k \otimes_A 1_{\boldsymbol{z}}$ are holomorphic sections for each k in $\mathcal{R}$.*

Proof. The only part requiring proof is the last statement. Clearly, this is true for any f_i in a generating set $\{f_i\}_{i=1}^m$ for $\mathcal{R}$. Similarly, it follows for any linear combination $\sum_{i=1}^m \varphi_i f_i$ for $\{\varphi_i\} \subset A(\Omega)$, that we obtain a holomorphic section. Finally, the $\mathcal{R}$-norm limit of such a sequence will converge uniformly locally and hence to a holomorphic section of $Sp(\mathcal{R})$ which completes the proof. $\square$

There is another approach to the holomorphic structure on $Sp(\mathcal{R})$ which was essentially used in [6], [7]. If the space $A_{\boldsymbol{z}}\mathcal{R}$ is closed and the rank of $\mathcal{R}$ is finite, then the projection onto $[A_{\boldsymbol{z}}\mathcal{R}]^\perp$ can be shown to define an anti-holomorphic map and hence the quotient $\mathcal{R}/[A_{\boldsymbol{z}}\mathcal{R}]$ is holomorphic. Since $\mathcal{R}/[A_{\boldsymbol{z}}\mathcal{R}] \cong \mathcal{R} \otimes_A \mathbb{C}_{\boldsymbol{z}}$, this is another way of establishing a holomorphic structure on $Sp(\mathcal{R})$. The smoothness of sections is straightforward in this case. However, the proof of Theorem 2 is valid without the assumption of "closed range" but does require the local uniform boundedness of evaluation or equivalently, that the module consists of holomorphic functions.

This identification of a holomorphic structure on the spectral sheaf of a finite rank quasi-free Hilbert module raises a series of questions regarding the situation for the spectral sheaf of a general Hilbert module. In particular, although we have called $Sp(\mathcal{M}) = \bigcup_{\boldsymbol{z}\in\Omega} \mathcal{M} \otimes_A \mathbb{C}_{\boldsymbol{z}}$ a sheaf, is it?

Although we can adopt the preceding approach to attempt to identify $\bigcup_{\boldsymbol{z}\in\Gamma} \mathcal{M} \otimes_A \mathbb{C}_{\boldsymbol{z}}$ with the trivial bundle $\Gamma \times \mathbb{C}^m$ on an open subset Γ of Ω on which the fiber dimension is constant, the utility of this identification depends on being able to show that the transition functions on an overlap $\Gamma_1 \cap \Gamma_2$ are holomorphic. This would show that $Sp(\mathcal{M})$ is a holomorphic bundle for the "easy case," that is, a Hilbert module $\mathcal{M}$ for which the fiber dimension of $\mathcal{M} \otimes_A \mathbb{C}_{\boldsymbol{z}}$ is constant and finite on all of Ω. Until that case is decided, it is pointless to speculate about the general case of an $\mathcal{M}$ with finite but different dimensional fibers.

There is additional information about the behavior of the Grammian for the $\{f_i \otimes_A 1_{\boldsymbol{z}}\}$ that we can obtain from a modification of the preceding arguments. Let $\{f_i\}$ be a generating set for the finite rank quasi-free Hilbert module $\mathcal{R}$. We introduce a related notion of dual generating set which we will denote by $\{g_{\boldsymbol{z}}^i\}$ so that $\langle h, g_{\boldsymbol{z}}^i\rangle_{\mathcal{R}} = \langle h \otimes_A 1_{\boldsymbol{z}}, f_i \otimes_A 1_{\boldsymbol{z}}\rangle_{\mathcal{R}\otimes_A \mathbb{C}_{\boldsymbol{z}}}$ for all i and $\boldsymbol{z}$ in Ω and h in $\mathcal{R}$. If $P_{\boldsymbol{z}}$ denotes the orthogonal projection of $\mathcal{R}$ onto $\mathcal{R}_{\boldsymbol{z}}^{\perp}$, then one sees that $g_{\boldsymbol{z}}^i = P_{\boldsymbol{z}} f_i$ for all i and $\boldsymbol{z}$ in Ω since we can identify $f_i \otimes_A 1_{\boldsymbol{z}}$ with $P_{\boldsymbol{z}} f_i$. Since $\bigcup\limits_{\boldsymbol{z}\in\Omega} \mathcal{R}_{\boldsymbol{z}}^{\perp}$ is an anti-holomorphic Hermitian rank m vector bundle, we see that the $\{g_{\boldsymbol{z}}^i\}$ form an anti-holomorphic frame for it. Moreover, we have

$$\langle f_i \otimes_A 1_{\boldsymbol{z}}, f_j \otimes_A 1_{\boldsymbol{z}}\rangle_{\mathcal{R}\otimes_A \mathbb{C}_{\boldsymbol{z}}} = \langle P_{\boldsymbol{z}} f_i, P_{\boldsymbol{z}} f_j\rangle_{\mathcal{R}} = \langle g_{\boldsymbol{z}}^i, g_{\boldsymbol{z}}^j\rangle_{\mathcal{R}}$$

or that the Grammian for the localization at $\boldsymbol{z}$ in Ω of the generating set $\{f_i\}$ agrees with that of the anti-holomorphic frame $\{g_{\boldsymbol{z}}^i\}$ for the anti-holomorphic Hermitian rank m vector bundle $\bigcup\limits_{\boldsymbol{z}\in\Omega} \mathcal{R}_{\boldsymbol{z}}^{\perp}$. This allows us to obtain the following result which will be used in the next section.

Theorem 3. *If $\mathcal{R}$ and $\mathcal{R}'$ are finite rank quasi-free Hilbert modules for the generating sets $\{f_i\}$ and $\{f_i'\}$ so that the Grammians $\{\langle f_i \otimes_A 1_{\boldsymbol{z}}, f_j \otimes_A 1_{\boldsymbol{z}}\rangle_{\mathcal{R}\otimes_A \mathbb{C}_{\boldsymbol{z}}}\}$ and $\{\langle f_i' \otimes_A 1_{\boldsymbol{z}}, f_j' \otimes_A 1_{\boldsymbol{z}}\rangle_{\mathcal{R}'\otimes_A \mathbb{C}_{\boldsymbol{z}}}\}$ are equal, then $\delta(\mathcal{R}, \mathcal{R}')$ is an isometric module map and $\mathcal{R}$ and $\mathcal{R}'$ are unitary equivalent.*

Proof. Proceeding as above we obtain anti-holomorphic frames $\{g_{\boldsymbol{z}}^i\}$ and $\{{g'}_{\boldsymbol{z}}^i\}$ for $\bigcup\limits_{\boldsymbol{z}\in\Omega} \mathcal{R}_{\boldsymbol{z}}^{\perp}$ and $\bigcup\limits_{\boldsymbol{z}\in\Omega} \mathcal{R}_{\boldsymbol{z}}'^{\perp}$, respectively. The mapping taking one anti-holomorphic frame to the other defines an anti-holomorphic unitary bundle map, call it Ψ, and hence the bundles are equivalent. Appealing to the Rigidity Theorem in [6], we obtain a unitary operator $U\colon\ \mathcal{R} \to \mathcal{R}'$ which agrees with the bundle map, that is, $\Psi(\boldsymbol{z}) = P_{\boldsymbol{z}}' U|_{\mathcal{R}_{\boldsymbol{z}}^{\perp}}$ for $\boldsymbol{z}$ in Ω. Moreover, since the action of M_φ^* on $\mathcal{R}_{\boldsymbol{z}}^{\perp}$ and ${\mathcal{R}'}_{\boldsymbol{z}}^{\perp}$ is multiplication by $\overline{\varphi(\boldsymbol{z})}$, where M_φ denotes the module actions of φ on $\mathcal{R}$ and $\mathcal{R}'$, respectively, we see that U^* is a module map from $\mathcal{R}'$ to $\mathcal{R}$ and hence $U = (U^*)^{-1}$ is a module map, which concludes the proof. □

4. Equivalence of Quasi-Free Hilbert Modules

We now state our first result about equivalence and the modulus.

Theorem 4. *If the finite rank quasi-free Hilbert modules $\mathcal{R}$ and $\mathcal{R}'$ over $A(\Omega)$ are unitarily equivalent, then the modulus $\mu(\mathcal{R}, \mathcal{R}')$ is the absolute value of a function Ψ in $Hol_m(\Omega)$.*

Proof. Let $V\colon\ \mathcal{R}' \to \mathcal{R}$ be a unitary module map. We consider localization of the triangle

$$\begin{tikzcd}
\mathcal{R} \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{z}} \arrow[rr, "(V\delta)\otimes_{A(\Omega)} 1_{\boldsymbol{z}}"] \arrow[dr, "\delta \otimes_{A(\Omega)} 1_{\boldsymbol{z}}"'] & & \mathcal{R} \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{z}} \\
& \mathcal{R}' \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{z}} \arrow[ur, "V \otimes_{A(\Omega)} 1_{\boldsymbol{z}}"'] &
\end{tikzcd}$$

which yields $(V\delta) \otimes_{A(\Omega)} 1_{\boldsymbol{z}} = (V \otimes_{A(\Omega)} 1_{\boldsymbol{z}})(\delta \otimes_{A(\Omega)} 1_{\boldsymbol{z}})$. Since $(V\delta) \otimes_{A(\Omega)} 1_{\boldsymbol{z}}$ is in $Hol_m(\Omega)$ by Lemmas 5 and 6, it is sufficient to show that $V \otimes_{A(\Omega)} 1_{\boldsymbol{z}}$ is unitary.

Again, by considering the factorization $I_{\mathcal{R}} \otimes_{A(\Omega)} 1_{\boldsymbol{z}} = (V^{-1} \otimes_{A(\Omega)} 1_{\boldsymbol{z}})(V \otimes_{A(\Omega)} 1_{\boldsymbol{z}})$ and in view of the fact that both $\|V^{-1} \otimes_{A(\Omega)} 1_{\boldsymbol{z}}\| \le \|V^{-1}\| = 1$ and $\|V \otimes_{A(\Omega)} 1_{\boldsymbol{z}}\| \le \|V\| = 1$, we see that $V \otimes_{A(\Omega)} 1_{\boldsymbol{z}}$ is unitary and the result is proved since $\mu(\mathcal{R}, \mathcal{R}')$ is the absolute value of $\delta(\mathcal{R}, \mathcal{R}')$. □

Note that if we use V^{-1} from $\mathcal{R}$ to $\mathcal{R}'$ we see that the other square root, $\mu(\mathcal{R}',\mathcal{R})$ is also the modulus of a holomorphic function in $Hol_m(\Omega)$.

The argument in this theorem raises a question about a bounded module map V between finite rank quasi-free Hilbert module $\mathcal{R}'$ and $\mathcal{R}$ such that the localization $V\otimes_{\mathrm{A}(\Omega)} 1_{\boldsymbol{z}}$ is unitary for $\boldsymbol{z}$ in Ω. We see by Theorem 3 that such a map must be unitary if it has dense range by choosing a generating set $\{f_i\}$ for $\mathcal{R}'$ and the generating set $\{Vf_i\}$ for $\mathcal{R}$. If θ is a singular inner function, then the module map from the Hardy module $H^2(\mathbb{D})$ to itself defined by multiplication by θ is locally one to one but does not have dense range. However, it is not locally a unitary map. It would seem likely that maps that are locally unitary must have dense range but we have been unable to prove this. Some of these issues would also seem to be related to the proof of Theorem 2.4 in [3]. This is the reference we made earlier to the use in this work of localization at $\boldsymbol{z}$ in addition to the origin.

What about the converse to the theorem? Suppose there exists a function Ψ in $Hol_m(\Omega)$ such that $\Psi(\boldsymbol{z})^*\Psi(\boldsymbol{z}) = \mu(\boldsymbol{z})^2 = (\delta\otimes_{A(\Omega)} 1_{\boldsymbol{z}})^*(\delta\otimes_{A(\Omega)} 1_{\boldsymbol{z}})$. Since $\mu(\boldsymbol{z})$ is invertible, we see that $\Psi(\boldsymbol{z})^{-1}$ exists. Multiplying on the left by $(\Psi(\boldsymbol{z})^{-1})^*$ and on the right by $\Psi(\boldsymbol{z})^{-1}$, we obtain

$$I = [(\delta\otimes_{A(\Omega)} 1_{\boldsymbol{z}})\Psi(\boldsymbol{z})^{-1}]^* = [(\delta\otimes_{A(\Omega)} 1_{\boldsymbol{z}})\Psi(\boldsymbol{z})^{-1}].$$

Thus the function $(\delta\otimes_{A(\Omega)} 1_{\boldsymbol{z}})\Psi(\boldsymbol{z})^{-1} = U(\boldsymbol{z})$ is unitary-valued. We would like to show under these circumstances that $\mathcal{R}$ and $\mathcal{R}'$ are unitarily equivalent. The obvious approach is to consider the operator on $\mathcal{R}$ defined to be multiplication by Ψ^{-1} followed by δ. Unfortunately, we know little about the growth of Ψ^{-1} as a function of $\boldsymbol{z}$ and hence we don't know if the operator defined by multiplication by Ψ is densely defined.

Suppose we assume that Ω is starlike relative to the point $\boldsymbol{\omega}_0$ in Ω, that is, the line segment $\{t\boldsymbol{\omega}_0 + (1-t)\boldsymbol{\omega} \mid 0\le t\le 1\}$ is contained in Ω for each $\boldsymbol{\omega}$ in Ω. Without loss of generality, we can assume that $\boldsymbol{\omega}_0 = \mathbf{0}$. Then we can define the function $\Psi_t^{-1}\colon\ \Omega\to\mathfrak{L}(\ell^2_m)$ for $0<t\le 1$ by $\Psi_t^{-1}(\boldsymbol{z}) = \Psi^{-1}(t\boldsymbol{z})$ for $\boldsymbol{z}$ in Ω. Now the family $\{\Psi_t^{-1}\}$ converge uniformly to Ψ^{-1} on compact subsets of Ω. (Actually, not only do the functions, which comprise the matrix entries, converge but so do all of their partial derivatives converge on compact subsets of Ω.) Moreover, the matrix entries for $\{\Psi_t^{-1}\}$ for $0<t<1$ are in $A(\Omega)$ and thus we can define multiplication by Ψ_t^{-1} on $\mathcal{R}$ and also $\delta\Psi_t^{-1}$. Moreover, $\delta\Psi_t^{-1}$ is a closed module transformation which has the same domain and range as δ.

Theorem 5. *If Ω is starlike and the modulus $\mu(\mathcal{R},\mathcal{R}')$ for two finite rank quasi-free Hilbert modules over $A(\Omega)$ is the absolute value of a function in $Hol_m(\Omega)$, then $\mathcal{R}$ and $\mathcal{R}'$ are unitarily equivalent.*

Proof. By Lemma 2 the localizations of both δ and $\delta\Psi_t^{-1}$ are well-defined and can be evaluated using the identifications of $\mathcal{R}\otimes_{A(\Omega)}\mathbb{C}_{\boldsymbol{z}}$ and $\mathcal{R}'\otimes_{A(\Omega)}\mathbb{C}_{\boldsymbol{z}}$ with $\mathcal{R}/\mathcal{R}_{\boldsymbol{z}}$ and $\mathcal{R}'/\mathcal{R}'_{\boldsymbol{z}}$, respectively. For Φ a function in $Hol_m(\Omega)$ with entries from $A(\Omega)$, the operator M_Φ in $\mathfrak{L}(\mathcal{R})$ defined to be multiplication by Φ, using generating sets for $\mathcal{R}$ and $\mathcal{R}'$, is well-defined and $M_\Phi\otimes_{A(\Omega)} 1_{\boldsymbol{z}} = \Phi(\boldsymbol{z})$

for $\boldsymbol{z}$ in Ω. Next we consider the localization of the factorization of $\delta\Psi_t^{-1}$ to obtain

$$\begin{aligned}(\delta\Psi_t^{-1})\otimes_{A(\Omega)} 1_{\boldsymbol{z}} &= (\delta\otimes_{A(\Omega)} 1_{\boldsymbol{z}})(\Psi_t^{-1}\otimes_{A(\Omega)} 1_{\boldsymbol{z}})\\ &= (\delta\otimes_{A(\Omega)} 1_{\boldsymbol{z}})\Psi_t^{-1}(\boldsymbol{z})\\ &= U(\boldsymbol{z})[\Psi(\boldsymbol{z})\Psi_t^{-1}(\boldsymbol{z})].\end{aligned}$$

Since $U(\boldsymbol{z}) = (\delta\otimes_{A(\Omega)} 1_{\boldsymbol{z}})\Psi^{-1}(\boldsymbol{z})$ is unitary, we see that the map $(\delta\Psi_t^{-1})\otimes_{A(\Omega)} 1_{\boldsymbol{z}}$, which acts between the local modules $\mathcal{R}\otimes_{A(\Omega)}\mathbb{C}_{\boldsymbol{z}}$ and $\mathcal{R}'\otimes_{A(\Omega)}\mathbb{C}_{\boldsymbol{z}}$, is almost a unitary module map. Since $\lim_{t\to 1}[\Psi(\boldsymbol{z})\Psi_t^{-1}(\boldsymbol{z})] = I_{\ell^2_m}$, we see that the two local modules are unitarily equivalent. But for $m > 1$ this is not enough.

For $\mathcal{M}$ a Hilbert module and n a positive integer, let $\mathcal{M}^n_{\boldsymbol{z}}$ denote the closure of $(A(\Omega)^n_{\boldsymbol{z}})\mathcal{M}$, where $A(\Omega)^n_{\boldsymbol{z}}$ is the ideal of $A(\Omega)$ generated by the products of n functions in $A(\Omega)_{\boldsymbol{z}}$. (The quotient $\mathcal{M}/\mathcal{M}^n_{\boldsymbol{z}}$ can also be identified as the module tensor product of $\mathcal{M}$ with some finite dimensional module with support at $\boldsymbol{z}$. It is not straightforward, however, to identify the correct norm on the local module.) In Theorem 3.12 [4], X. Chen and the first author established that a class of Hilbert modules, which includes the finite rank quasi-free Hilbert modules, are determined up to unitary equivalence by the collection of local modules $\mathcal{M}/\mathcal{M}^n_{\boldsymbol{z}}$ for $\boldsymbol{z}$ in Ω, where n depends on the rank of $\mathcal{R}$. To apply this result to $\mathcal{R}$ and $\mathcal{R}'$ we require the unitary equivalence of the higher order local modules $\mathcal{R}/\mathcal{R}^n_{\boldsymbol{z}}$ and $\mathcal{R}'/\mathcal{R}'^n_{\boldsymbol{z}}$. This is accomplished by noting that the localization of $[\Psi(\boldsymbol{z})\Psi_t^{-1}(\boldsymbol{z})]$ to $\mathcal{R}'/\mathcal{R}'^n_{\boldsymbol{z}}$ depends on the values of the partial derivatives of the entries of this matrix function up to some fixed order depending on n. Since the latter functions all converge to the appropriate entries for the identity matrix on $\mathcal{R}'/\mathcal{R}'^n_{\boldsymbol{z}}$, we conclude that $\mathcal{R}/\mathcal{R}^n_{\boldsymbol{z}}$ and $\mathcal{R}'/\mathcal{R}'^n_{\boldsymbol{z}}$ are unitarily equivalent as $A(\Omega)$-modules. Thus, we conclude that $\mathcal{R}$ and $\mathcal{R}'$ are unitarily equivalent as $A(\Omega)$-modules. □

Arguments such as the preceding one are familiar in several complex variables. An early instance of it using starlike domains occurs in Douady's thesis [9]. Actually Ω being starlike is not necessary. What is required for the preceding argument to work is that one can approximate the function Ψ by matrix functions with entries from $A(\Omega)$ in a very strong sense. That is, one must be able to control not only the convergence of the function entries but also the convergence of their partial derivatives and their inverses. By Montel's Theorem uniform convergence on compact subsets of Ω is sufficient. One can show using various techniques (cf. [17] and Thm. 3.5.1 in [16]) that such approximation is possible for Ω a bounded strongly pseudo-convex domain which allows us to state:

Corollary 6. *If Ω is a bounded strongly pseudo-convex domain in $\mathbb{C}^m$ and the modulus $\mu(\mathcal{R},\mathcal{R}')$ for two finite rank quasi-free Hilbert modules over $A(\Omega)$ is the absolute value of a function in $Hol_m(\Omega)$, then $\mathcal{R}$ and $\mathcal{R}'$ are unitarily equivalent.*

If we actually know that the mapping $\delta\Psi^{-1}$ is densely defined, we can use Theorem 3 which means appealing to the Rigidity Theorem of [6] rather than involving curvature and its partial derivatives.

Now one knows that a non-negative real-valued function $h(\boldsymbol{z})$ on as simply connected domain Ω is the absolute value of a function holomorphic on Ω if and only if the two-form-valued

Laplacian of the logarithm of it vanishes identically on Ω. Hence, we could restate Theorems 4 and 5 for the rank one case using this fact. However, we can go even further.

Recall we saw in Theorem 2 that a rank m quasi-free Hilbert module $\mathcal{R}$ determines a Hermitian holomorphic rank m vector bundle $E_{\mathcal{R}} = \bigcup_{\boldsymbol{z}\in\Omega} \mathcal{R} \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{z}}$ over Ω. Moreover, on such a bundle there is a canonical connection and hence a curvature which is a two-form valued matrix function on Ω (cf. [6]). In the rank one case, we obtain a line bundle and if $\gamma(\boldsymbol{z})$ is the holomorphic section $f \otimes_{A(\Omega)} 1_{\boldsymbol{z}}$ of it, then the curvature $K_{\mathcal{R}}$ can be calculated so that

$$K_{\mathcal{R}}(\boldsymbol{z}) = -\frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \|\gamma(\boldsymbol{z})\| dz_i \wedge d\bar{z}_j.$$

Now let us return to the case of two rank one quasi-free Hilbert modules over Ω. If $\gamma'(\boldsymbol{z})$ is the holomorphic section $g \otimes_{A(\Omega)} 1_{\boldsymbol{z}}$ for $E_{\mathcal{R}'}$, then $(\delta\gamma)(\boldsymbol{z})$ is the holomorphic section $\gamma'(\boldsymbol{z})$ for $\mathcal{R}' \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{z}}$. Moreover, a calculation shows that

$$\|\gamma'(\boldsymbol{z})\| = \|(\delta\gamma)(\boldsymbol{z})\| = |(\delta \otimes_{A(\Omega)} 1_{\boldsymbol{z}})| \|\gamma(\boldsymbol{z})\|.$$

Theorem 7. *If $\mathcal{R}$ and $\mathcal{R}'$ are rank one quasi-free Hilbert modules and μ is the modulus, $\mu(\mathcal{R}, \mathcal{R}')$, then*

$$-\frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \mu(\boldsymbol{z}) dz_i \wedge d\bar{z}_j = K_{\mathcal{R}} - K_{\mathcal{R}'}.$$

Proof. If $\gamma(\boldsymbol{z})$ and $\gamma'(\boldsymbol{z})$ are the holomorphic sections of $E_{\mathcal{R}}$ and $E_{\mathcal{R}'}$ given above, then we have

$$K_{\mathcal{R}} = -\frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \|\gamma(\boldsymbol{z})\| dz_i \wedge d\bar{z}_j \text{ and } K_{\mathcal{R}'} = -\frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log |\delta \otimes_{A(\Omega)} 1_{\boldsymbol{z}}| \|\gamma(\boldsymbol{z})\| dz_i \wedge d\bar{z}_j.$$

The proof is completed by using Lemma 5 to conclude that $\mu(\boldsymbol{z}) = |(\delta \otimes_{A(\Omega)} 1_{\boldsymbol{z}})|$ for $\boldsymbol{z}$ in Ω. □

Formulas such as this one appeared first for specific examples in [13] and for general quotient modules in [10] where they are used to obtain invariants for the quotient module. Here, of course, there is no quotient module involved.

Finally, one can rephrase this result to state that for rank one quasi-free Hilbert modules the modulus is the square of the absolute value of a holomorphic function if and only if their respective curvatures coincide.

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