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Contractive and Completely Contractive Maps Over Planar Algebras

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CONTRACTIVE AND COMPLETELY CONTRACTIVE HOMOMORPHISMS OF PLANAR ALGEBRAS

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ABSTRACT. Given a planar domain Ω, we first consider the algebra $\mathrm{Rat}(\Omega)$ of rational functions with poles off Ω and equipped with the norm $\|r\| = \sup\{|r(z)| : z \in \Omega\}$ for $r \in \mathrm{Rat}(\Omega)$ and then its closure which we denote by $\mathcal{A}$. We investigate which contractive homomorphisms of the algebra $\mathcal{A}$ are necessarily completely contractive. We start with homomorphisms $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ for which $\dim(\mathcal{A}/\ker\rho) = 2$ and show that such a homomorphism is the direct integral of homomorphisms ρ_T induced by operators on a two dimensional Hilbert space via a suitable functional calculus $\rho_T : f \mapsto f(T),\ f \in \mathcal{A}$. It is well-known that contractive homomorphisms ρ_T, where T is from $\mathbb{C}^2$ to $\mathbb{C}^2$ are necessarily completely contractive. Consequently, they possess a dilation. We construct this dilation explicitly. In view of recent examples discovered by Dritschel and McCullough, we know that not all contractive homomorphisms are completely contractive even when $\mathcal{H}$ is finite-dimensional. We explore the possibility, in a certain special case, of constructing a dilation for contractive homomorphisms ρ_T where T is a finite dimensional operator. We construct an operator space which is naturally associated with the problem.

1. INTRODUCTION

All our Hilbert spaces are over complex numbers and are separable. Let $T \in \mathcal{B}(\mathcal{H})$, the algebra of bounded operators on $\mathcal{H}$. Given a rational function $r = p/q$ with no poles in the spectrum $\sigma(T)$, there is the natural functional calculus $r(T) = p(T)q(T)^{-1}$. Thus T induces a unital homomorphism $\rho_T = r(T)$ on the algebra of rational functions $\mathrm{Rat}(\sigma(T))$ with poles off $\sigma(T)$. By von Neumann's inequality [13] ρ_T is contractive, i.e., $\|\rho_T\| \le 1$ if and only if the operator T is a contraction. Thus in this case, contractivity of the homomorphism ρ_T is equivalent to the operator T being a contraction.

As is well known, Sz.-Nagy [18] showed that a contraction T on a Hilbert space $\mathcal{H}$ dilates to a unitary operator U on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, i.e., $PU^n h = T^n h$ for all $h \in \mathcal{H}$ and $n \in \mathbb{N}$, where $P : \mathcal{K} \to \mathcal{H}$ is the projection. Of course the unitary operator U has a continuous functional calculus and hence induces a $*$ - homomorphism $\varphi_U : C(\sigma(U)) \to \mathcal{L}(\mathcal{K})$. It is easy to check that $P[(\varphi_U)_{|\mathcal{A}(\mathbb{D})}(f)]_{|\mathcal{H}} = \rho_T(f)$, for f in $\mathcal{A}(\mathbb{D})$.

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Let Ω be a domain in $\mathbb{C}$ and let $\mathcal{A}(\Omega)$ be the closure of $\mathrm{Rat}(\Omega)$ with respect to the supremum norm on $\bar{\Omega}$. A bounded linear operator T on $\mathcal{H}$ with spectrum $\sigma(T) \subseteq \bar{\Omega}$ induces the homomorphism $\rho_T : \mathcal{A}(\Omega) \to \mathcal{L}(\mathcal{H})$ as above. The homomorphism ρ_T is said to be *dilatable* if there exists a normal operator N on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ with $\sigma(N) \subseteq \partial\bar{\Omega}$ such that the induced homomorphism $\varphi_N : C(\sigma(N)) \to \mathcal{L}(\mathcal{K})$, via the functional calculus for the normal operator N, satisfies the relation

$$P(\varphi_N)_{|\mathcal{A}(\Omega)}(f)h = \rho_T(f)h, \tag{1.1}$$

for h in $\mathcal{H}$ and f in $\mathcal{A}(\Omega)$.

The observations about the disk prompt two basic questions:

(i) When is ρ_T contractive;
(ii) do contractive homomorphisms ρ_T necessarily dilate?

For the disc algebra, the answer to the first question is given by von Neumann's inequality while the answer to the second question is affirmative – Sz.-Nagy's dilation theorem. Clearly if the domain Ω is simply connected these questions can be reduced to that of the disc (cf. [17]).

If the domain Ω is the annulus, while no satisfactory answer to the first question is known, the answer to the second question was shown to be affirmative by Agler (cf. [4]).

If $\rho_T : \mathcal{A}(\Omega) \to \mathcal{M}_2$ is a homomorphism induced by a two dimensional operator then it is possible to obtain a characterization of contractivity and then use it to show that the second question has an affirmative answer. We do this in Section 3. In Section 2, we show that a larger class of contractive homomorphisms, we call them contractive homomorphisms of *rank 2*, dilate. This is done by proving that the rank 2 homomorphisms are direct integrals of homomorphisms induced by two dimensional operators.

However, the existence of a dilation involves a theorem due to Arveson ([5] and [6]) which reduces it to complete contractivity of the homomorphism. We will recall some of these notions in greater detail in section 4. We will then show how one may proceed to possibly construct an example of a contractive homomorphism of the algebra $\mathcal{A}(\Omega)$ which does not dilate.

In the final section of the paper, we obtain a general criterion for contractivity. This involves a factorization of a certain positive definite kernel. This criterion is somewhat intractable to be of any practical value. More importantly, we outline a scheme for constructing dilations of homomorphism $\rho_T : \mathcal{A}(\Omega) \to \mathcal{M}_n$ induced by an operator T with distinct eigenvalues. This scheme is a generalization of the construction of the dilation in section 3.

2. Homomorphisms of Rank Two

A homomorphism $\rho : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{H})$ is said to be of *rank n* if it has the property $\dim\big(\mathcal{A}(\Omega)/\ker\rho\big) = n$. In this section, we shall begin construction of dilation for homomorphisms of rank 2. Nakazi and Takahashi showed that contractive homomorphisms $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of rank 2 are completely contractive for any uniform sub-algebra of the algebra of continuous functions $C(\bar{\Omega})$ (see [12]). In what follows, we construct explicit dilations for homomorphisms from $\mathcal{A}(\Omega)$ to $\mathcal{B}(\mathcal{H})$ of rank two.

We first show that any homomorphism ρ of rank 2 is the direct integral of homomorphisms of the form ρ_T as defined in the introduction, where $T \in \mathcal{M}_2$. The existence of dilation of a contractive homomorphism ρ_T induced by a two dimensional operator T is established in [11] by showing that the homomorphism ρ_T must be completely contractive. It then follows that every contractive homomorphism ρ of rank 2 must be completely contractive. This implies by Arveson's theorem that they possess a dilation. However, it is not always easy to construct the dilation whose existence is guaranteed by the theorem of Arveson. In this case, we shall explicitly construct the dilation of a homomorphism of rank 2. This is achieved by constructing the dilation of a contractive homomorphism of the form ρ_T for a two dimensional operator T.

LEMMA 1. *If $\rho_T : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{L})$ is a homomorphism of rank two, then up to unitary equivalence, the Hilbert space $\mathcal{L}$ is a direct integral*

$$\mathcal{L} = \int_\Lambda^\oplus \mathcal{L}_\lambda d\lambda$$

where each $\mathcal{L}_\lambda$ is two-dimensional. In this decomposition, the operator T is of the form

$$T = \int_\Lambda^\oplus \begin{pmatrix} z_1 I_{\mathcal{H}} & \lambda \\ 0 & z_2 I_{\mathcal{K}} \end{pmatrix} d\lambda.$$

Proof. To begin with, it is easy to see (see Lemma 1 of [12]) that $\mathcal{L}$ is a direct sum of two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and the operator $T : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K}$ is of the form:

$$\begin{pmatrix} z_1 I_{\mathcal{H}} & C \\ 0 & z_2 I_{\mathcal{K}} \end{pmatrix}, \text{ with } z_1,\ z_2 \in \Omega \text{ or } \begin{pmatrix} z I_{\mathcal{H}} & C \\ 0 & z I_{\mathcal{K}} \end{pmatrix}, \text{ with } z \in \Omega,$$

where C is a bounded operator from $\mathcal{K}$ to $\mathcal{H}$. Now if we put $\mathcal{K}_0 = (\ker C)^\perp$, $\mathcal{K}_1 = \ker C$, $\mathcal{H}_0 = \overline{\text{Ran}}\ C$ and $\mathcal{H}_1 = (\text{Ran}\ C)^\perp$, then with respect to the decomposition $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$ and $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, we have

$$C = \begin{pmatrix} \tilde{C} & 0 \\ 0 & 0 \end{pmatrix},$$

where the operator $\tilde{C}$ is from $\mathcal{K}_0$ to $\mathcal{H}_0$. The polar decomposition of $\tilde{C}$ then yields $\tilde{C} = VP$, where the operator V is unitary and P is positive. We apply the spectral theorem to the positive operator P and conclude that there exists a unitary operator $\Gamma : \int_\Lambda^\oplus \mathcal{H}_\lambda d\lambda \to \mathcal{K}_0$ which intertwines the multiplication operator M on the Hilbert space $\int_\Lambda^\oplus \mathcal{H}_\lambda d\lambda$ and P.

Now notice that the operator $T : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K}$ can be rewritten as

$$\begin{pmatrix} z_1 I_{\mathcal{H}_1} & 0 & \tilde{C}_{\mathcal{K}_0 \to \mathcal{H}_0} & 0 \\ 0 & z_1 I_{\mathcal{H}_0} & 0 & 0 \\ 0 & 0 & z_2 I_{\mathcal{K}_0} & 0 \\ 0 & 0 & 0 & z_2 I_{\mathcal{K}_1} \end{pmatrix}.$$

Interchanging the third and the second column and then the second and third row, which can be effected by a unitary operator, we see that the operator T is unitarily equivalent to the direct sum of a diagonal operator D and an operator $\tilde{T}$ of the form $\begin{pmatrix} z_1 I_{\mathcal{H}_0} & \tilde{C}_{\mathcal{K}_0 \to \mathcal{H}_0} \\ 0 & z_2 I_{\mathcal{K}_0} \end{pmatrix}$, where $\tilde{C}$ has dense range. It is clear that if we conjugate the operator $\tilde{T}$ by the operator $I_{\mathcal{H}_0} \oplus U_{\mathcal{H}_0 \to \mathcal{K}_0}$, where U is any unitary operator identifying $\mathcal{H}_0$ and $\mathcal{K}_0$ then we obtain a unitarily equivalent copy of $\tilde{T}$ (again, denoted by $\tilde{T}$) which is of the form $\begin{pmatrix} z_1 I_{\mathcal{H}_0} & \tilde{C}_{\mathcal{K}_0 \to \mathcal{H}_0} U_{\mathcal{H}_0 \to \mathcal{K}_0} \\ 0 & z_2 I_{\mathcal{K}} \end{pmatrix}$. Now, if we apply the polar decomposition to $\tilde{C}$ then we see that the off diagonal entry is a positive operator on $\mathcal{H}_0$. One then sees that $\tilde{T}$ is unitarily equivalent to $\begin{pmatrix} z_1 I_{\int_\Lambda^\oplus \mathcal{H}_\lambda d\lambda} & M \\ 0 & z_2 I_{\int_\Lambda^\oplus \mathcal{H}_\lambda d\lambda} \end{pmatrix}$ via conjugation using the operator $\Gamma \oplus \Gamma$. We need to conjugate this operator one more time using the unitary W that identifies $\int_\Lambda^\oplus \mathcal{H}_\lambda d\lambda \oplus \int_\Lambda^\oplus \mathcal{H}_\lambda d\lambda$ and $\int_\Lambda^\oplus \mathcal{H}_\lambda \oplus \mathcal{H}_\lambda d\lambda$, where $W(s_1 \oplus s_2)(\lambda) = s_1(\lambda) \oplus s_2(\lambda)$ for $s_1 \oplus_2 \in \int_\Lambda^\oplus \mathcal{H}_\lambda d\lambda$. It is easy to calculate $W\tilde{T}W^*$ and verify the claim. ∎

In view of the Lemma above, it is now enough to consider dilations of homomorphisms ρ_T where T is a linear transformation on $\mathbb{C}^2$. Let Ω be a bounded domain in $\mathbb{C}$ containing the eigenvalues of T. Consider any reproducing kernel Hilbert space $\mathcal{H}_K$ of holomorphic functions on Ω with $K : \Omega \times \Omega \to \mathbb{C}$ as the kernel. Assume that the multiplication operator M_z by the independent variable z is bounded. Then

$$M_z^*(K(\cdot, z)) = \bar{z}K(\cdot, z)$$

and it is clear by differentiation that

$$M_z^* \bar{\partial}_z K(\cdot, z) = K(\cdot, z) + \bar{z}\bar{\partial}_z K(\cdot, z).$$

The matrix representation of the operator M_z^* restricted to the subspace $\mathcal{M}$ spanned by the two vectors $K(\cdot, z_1)$ and $K(\cdot, z_2)$ has two distinct eigenvalues $\bar{z}_1$ and $\bar{z}_2$. Similarly, the operator M_z^* restricted to the subspace $\mathcal{N}$ spanned by the two vectors $K(\cdot, z)$ and $\bar{\partial}_z K(\cdot, z)$ has only one eigenvalue $\bar{z}$ of multiplicity 2. In the lemma below, we identify ceratin 2 dimensional subspaces of $\mathcal{H}_K \oplus \mathcal{H}_K$ which are invariant under the multiplication operator M_z^* and then find out the form of the matrix. The reproducing kernel K satisfies:

$$K(z_1, z_2) = \langle K(\cdot, z_2), K(\cdot, z_1)\rangle, \; z_1, z_2 \in \Omega, \tag{2.1}$$

$$(\partial_z K)(z, u) = \langle K(\cdot, u), \bar{\partial}_z K(\cdot, z)\rangle, \; u, \, z \in \Omega. \tag{2.2}$$

Using (2.1) and applying the Gram-Schmidt orthogonolization process to the set $\{K(\cdot, z_1), K(\cdot, z_2)\}$, we get the orthonormal pair of vectors

$$e(z_1) = \frac{K(\cdot, z_1)}{K(z_1, z_1)^{1/2}} \text{ and } f(z_1, z_2) = \frac{K(z_1, z_1)K_\alpha(\cdot, z_2) - K(z_2, z_2)K(\cdot, z_1)}{K(z_1, z_1)\|^{1/2}(K(z_1, z_1)K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2}}.$$

Now for any $\mu \in \bar{\mathbb{D}}$, the pair of vectors

$$h_1(z_1, z_2) = \begin{pmatrix} 0 \\ e(z_1) \end{pmatrix} \text{ and } h_2(z_1, z_2) = \begin{pmatrix} (1 - |\mu|^2)^{1/2} e(z_2) \\ \mu f(z_1, z_2) \end{pmatrix}$$

are orthonormal in $\mathcal{H}_K \oplus \mathcal{H}_K$. Similarly, using (2.2), orthonormalization of the pair of vectors $\{K(\cdot, z), \bar{\partial}_z K(\cdot, z)\}$ produces the orthonormal set $\{e(z), f(z)\}$, where

$$\begin{aligned} e(z) &= \frac{K(\cdot, z)}{K(z, z)^{1/2}} \text{ and} \\ f(z) &= \frac{K(z, z)\bar{\partial}_z K(\cdot, z) - \langle \bar{\partial}_z K(\cdot, z), K(\cdot, z)\rangle K(\cdot, z)}{K(z, z)^{1/2}\big(K(z, z)\, \|\bar{\partial}_z K(\cdot, z)\|^2 - |\langle \bar{\partial}_z K(\cdot, z), K(\cdot, z)\rangle|^2\big)^{1/2}}. \end{aligned}$$

and then for any $\lambda \in \bar{\mathbb{D}}$,

$$k_1(z) = \begin{pmatrix} 0 \\ e(z) \end{pmatrix} \text{ and } k_2(z) = \begin{pmatrix} (1-|\lambda|^2)^{1/2} e(z) \\ \lambda f(z) \end{pmatrix}$$

form a set of two orthonormal vectors in $\mathcal{H}_K \oplus \mathcal{H}_K$.

Note that from the definition of M_z^* it follows that $M_z^* e(z_1) = \bar{z}_1 e(z_1)$ for all $z_1 \in \Omega$. Therefore we have $(M_z^* \oplus M_z^*) h_1(z_1, z_2) = \bar{z}_1 h_1(z_1, z_2)$. Now,

$$\begin{aligned} M_z^* f(z_1, z_2) &= \frac{K(z_1,z_1)\bar{z}_2 K_\alpha(\cdot, z_2) - K(z_2,z_2)\bar{z}_1 K(\cdot, z_1)}{K(z_1,z_1)^{1/2}(K(z_1,z_1)K(z_2,z_2) - |K(z_1,z_2)|^2)^{1/2}} \\ &= \bar{z}_2 f(z_1,z_2) + \frac{(\bar{z}_2 - \bar{z}_1)K(z_1,z_2)}{(K(z_1,z_1)K(z_2,z_2) - |K(z_1,z_2)|^2)^{1/2}} e(z_1). \end{aligned}$$

It follows that $\mathcal{M}$ is invariant under $M_z^* \oplus M_z^*$. In particular, we have

$$\begin{aligned} (M_z^* \oplus M_z^*) h_2(z_1,z_2) &= \begin{pmatrix} (1-|\mu|^2)^{1/2} M_z^* e(z_2) \\ \mu M_z^* f(z_1,z_2) \end{pmatrix} \\ &= \begin{pmatrix} (1-|\mu|^2)^{1/2} \bar{z}_2 e(z_2) \\ \mu \bar{z}_2 f(z_1,z_2) + \mu \frac{(\bar{z}_2 - \bar{z}_1)K(z_1,z_2)}{(K(z_1,z_1)K(z_2,z_2) - |K(z_1,z_2)|^2)^{1/2}} e(z_1) \end{pmatrix} \\ &= \bar{z}_2 f(z_1,z_2) + \mu \frac{(\bar{z}_2 - \bar{z}_1)K(z_1,z_2)}{(K(z_1,z_1)K(z_2,z_2) - |K(z_1,z_2)|^2)^{1/2}} e(z_1). \end{aligned}$$

Thus the matrix representation of $M_z^* \oplus M_z^*$ on the invariant two-dimensional subspace $\mathcal{M}$ is

$$\begin{pmatrix} \bar{z}_1 & \frac{\mu(\bar{z}_2 - \bar{z}_1)K(z_1,z_2)}{(K(z_1,z_1)K(z_2,z_2) - |K(z_1,z_2)|^2)^{1/2}} \\ 0 & \bar{z}_2 \end{pmatrix}.$$

Now recall that $(M_z^* - \bar{z})K(\cdot, z) = 0$. Differentiating with respect to $\bar{z}$, we obtain, $M_z^* \bar{\partial}_z K(\cdot, z) = K(\cdot, z) + \bar{z}\bar{\partial}_z K(\cdot, z)$. Thus the subspace $\mathcal{N}$ spanned by the vectors $k_1(z), k_2(z)$ is invariant under M_z^*. A little more computation, similar to the computation in the first part of the proof, shows that the restriction of the operator M_z^* to the subspace $\mathcal{N}$ has the matrix form

$$\begin{pmatrix} \bar{z} & \frac{\lambda K(z,z)}{(K(z,z)\|\bar{\partial}_z K(\cdot,z)\|^2 - |\langle \bar{\partial}_z K(\cdot,z), K(\cdot,z)\rangle|^2)^{1/2}} \\ 0 & \bar{z} \end{pmatrix}.$$

So we have proved the following lemma.

Lemma 2. *The two-dimensional space $\mathcal{M}$ spanned by the two vectors $h_1(z_1,z_2), h_2(z_1,z_2)$ is an invariant subspace for the operator $M_z^* \oplus M_z^*$ on $\mathcal{H}_K \oplus \mathcal{H}_K$ and the restriction of this operator to the subspace $\mathcal{M}$ has the matrix representation*

$$\begin{pmatrix} \bar{z}_1 & \frac{\mu(\bar{z}_2 - \bar{z}_1)K(z_1,z_2)}{(K(z_1,z_1)K(z_2,z_2) - |K(z_1,z_2)|^2)^{1/2}} \\ 0 & \bar{z}_2 \end{pmatrix}.$$

Similarly, the two-dimensional space $\mathcal{N}$ spanned by the two vectors $k_1(z), k_2(z)$ is an invariant subspace for the operator $M_z^* \oplus M_z^*$ on $\mathcal{H} \oplus \mathcal{H}$ and the restriction of this operator to the *subspace $\mathcal{N}$ has the matrix representation*

$$\begin{pmatrix} \bar{z} & \frac{\lambda K(z,z)}{(K(z,z)\|\bar{\partial}_z K(\cdot,z)\|^2 - |\langle \bar{\partial}_z K(\cdot,z), K(\cdot,z)\rangle|^2)^{1/2}} \\ 0 & \bar{z} \end{pmatrix}.$$

We carry out the dilation in the next section because we need a result of Abrahamse extending the Nevanlinna-Pick interpolation from the disk to a multi-connected domain.

3. Dilations and Abrahamse-Nevanlinna-Pick Interpolation

Associated with a multi-connected domain Ω in $\mathbb{C}$ is a natural family of Hilbert spaces $H^2_{\boldsymbol{\alpha}}(\Omega)$ consisting of modulus automorphic holomorphic functions on Ω. This family is indexed by $\boldsymbol{\alpha} \in \mathbb{T}^m$, where m is the number of bounded connected components in $\mathbb{C} \setminus \Omega$. Each $H^2_{\boldsymbol{\alpha}}(\Omega)$ has a reproducing kernel which we denote by $K_{\boldsymbol{\alpha}}(z,w)$ and a deep result due to Widom (cf. [10, page 140]) shows that the map $\boldsymbol{\alpha} \mapsto K_{\boldsymbol{\alpha}}(z,w)$ is continuous for any fixed pair (z,w) in $\Omega \times \Omega$. The generalization of Nevanlinna-Pick theorem due to Abrahamse states that given n points $w_1, w_2, \dots, w_n$ in the open unit disk, there is a holomorphic function $f : \Omega \to \mathbb{C}$ with $f(z_i) = w_i$ for $i = 1, 2, \dots, n$ if and only if the matrix

$$M(\underline{w}, \boldsymbol{\alpha}) \overset{\text{def}}{=} \big(\!\big((1 - w_i \bar{w}_j) K_\alpha(z_i, z_j)\big)\!\big) \tag{3.1}$$

is positive semidefinite. To get hold of an $\boldsymbol{\alpha}$ for constructing the dilation space, we need to consider the following sets which will also be of much use later when we construct an operator space structure.

Definition 3. Given $z_1, z_2, \dots, z_n$ in Ω, let $I^k_{\underline{z}}$ be the subset of $\mathbb{C}^n \otimes \mathcal{M}_k$ defined by

$$\begin{aligned} I^k_{\underline{z}} &= \{(W_1, W_2, \dots, W_n) : W_i = R(z_i) \text{ for } i = 1, 2, \dots, n \\ &\qquad \text{where } R \text{ is a rational function from } \Omega \text{ to } \mathcal{M}_k \text{ with } \|R(z)\| \le 1 \text{ and } z \in \Omega\}. \end{aligned}$$

In case $k = 1$, we write $I_{\underline{z}}$ instead of $I^1_{\underline{z}}$.

The sets of the form $I^k_{\underline{z}}$ were first studied, in the case $k = 1$, by Cole and Wermer [8]. They called such sets hyperconvex. In a more recent work [16], Paulsen points out that the sequence of sets $I^k_{\underline{z}} \subseteq \mathbb{C}^n \otimes \mathcal{M}_k$ determines an operator space structure on $\mathbb{C}^m$ and calls this sequence matricially hyperconvex. In particular, for each $k > 0$, the set $I^k_{\underline{z}}$ determines a norm $\|\cdot\|_{\underline{z},k}$ in $C^n \otimes \mathcal{M}_k$ such that $I^k_{\underline{z}}$ is the closed unit ball in this norm.

Lemma 4. *Let Ω be an open connected subset of the complex plane bounded by $n+1$ disjoint analytic simple closed curves. Let $z_1, z_2, \dots, z_n$ be n points from Ω. The set $I_{\underline{z}}$ defined in Definition 3 is a compact set.*

Proof. Clearly, $I_{\underline{z}}$ is a subset of $\bar{D}^n$. Since $\bar{\mathbb{D}}^n$ is a compact set, it is enough to show that $I_{\underline{z}}$ is a closed subset. To that end, recall that the smallest eigenvalue $\lambda_{\min}(A)$ of a hermitian matrix A is a continuous function of the hermitian matrix A (see for example, [7, Corollary III.2.6]). For any $\boldsymbol{\alpha} \in \mathbb{T}^m$, recall that $M(\underline{w}, \boldsymbol{\alpha})$ denotes the $n \times n$ matrix whose (i,j)th.

entry is $(1 - w_i\bar{w}_j)K_{\boldsymbol{\alpha}}(z_i, z_j)$. It is a hermitian matrix. Now by Abrahamse-Nevanlinna-Pick interpolation theorem (see [3]),

$$\begin{aligned} I_{\underline{z}} &= \{(w_1, w_2, \ldots, w_n) \in \overline{\mathbb{D}}^n : \text{ the matrix } M(\underline{w}, \boldsymbol{\alpha}) \text{ is positive semidefinite for all } \boldsymbol{\alpha} \in \mathbb{T}^m\} \\ &= \{(w_1, w_2, \ldots, w_n) \in \overline{\mathbb{D}}^n : \lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha})) \geq 0 \text{ for all } \boldsymbol{\alpha} \in \mathbb{T}^n\} \\ &= \cap_{\boldsymbol{\alpha} \in \mathbb{T}^m}\{(w_1, w_2, \ldots, w_n) \in \bar{\mathbb{D}}^n : \lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha})) \geq 0\} \\ &= \cap_{\boldsymbol{\alpha} \in \mathbb{T}^m} (\lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha})))^{-1}([0, \infty)) \end{aligned}$$

For any $\boldsymbol{\alpha} \in \mathbb{T}^m$, the set $(\lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha})))^{-1}([0, \infty))$ is closed because $\lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha}))$ is a continuous function of the entries of $M(\underline{w}, \boldsymbol{\alpha})$, and hence of $(w_1, w_2, \ldots, w_n)$. Since arbitrary intersection of closed sets is closed, the proof is complete. ■

THEOREM 5. *Given $\{z_1, z_2, \ldots, z_n\} \subseteq \Omega$, there is a single $\boldsymbol{\alpha} \in \mathbb{T}^m$ depending only on $\underline{z}$ such that $\underline{w} = (w_1, w_2, \ldots, w_n) \in I_{\underline{z}}$ if and only if the matrix $M(\underline{w}, \boldsymbol{\alpha})$ is positive semidefinite.*

Proof. The hermitian matrix $M(\underline{w}, \boldsymbol{\alpha})$ is positive semidefinite if and only if $\lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha}))$ is non-negative.

For any (i, j), the function $\boldsymbol{\alpha} \to K_{\boldsymbol{\alpha}}(z_i, z_j)$ is a continuous function of $\boldsymbol{\alpha}$ by [10, Page 140]. The (i, j)th. entry of the matrix $M(\underline{w}, \boldsymbol{\alpha})$ is a product of this continuous function with a polynomial function in $\underline{w}$. Thus the function $(\underline{w}, \boldsymbol{\alpha}) \to M(\underline{w}, \boldsymbol{\alpha})$ is a continuous function. It follows that the smallest eigenvalue $\lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha}))$ of the hermitian matrix $M(\underline{w}, \boldsymbol{\alpha})$ is a continuous function of $(\underline{w}, \boldsymbol{\alpha})$ on the compact set $I_{\underline{z}} \times \mathbb{T}^n$. So there is a point $(\underline{w}_0, \boldsymbol{\alpha}_0) \in I_{\underline{z}} \times \mathbb{T}^n$ where its infimum is attained.

If for a given $\underline{w}$, the matrix $M(\underline{w}, \boldsymbol{\alpha}_0)$ is positive semidefinite, then for any $\boldsymbol{\alpha} \in \mathbb{T}^n$, we have $\lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha})) \geq \lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha}_0)) \geq 0$. Thus $M(\underline{w}, \boldsymbol{\alpha})$ is positive semidefinite for every $\boldsymbol{\alpha} \in \mathbb{T}^m$ and by Abrahamse-Nevanlinna-Pick condition, $\underline{w} \in I_z$. ■

We now have enough material to construct the dilation for a homomorphism $\rho_T : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is two dimensional. In this case, T is a 2×2 matrix with spectrum in Ω. Since we can apply a unitary conjugation to make T upper-triangular, it is enough to exhibit the dilation for $T = \begin{pmatrix} z_1 & 0 \\ t\mu(z_1 - z_2) & z_2 \end{pmatrix}$, where $\mu \in \mathbb{C}$ and

$$t = \frac{1}{\sup\{|r(z_1)|^2 : r \in \text{ Rat}(\Omega) \text{ and } r(z_2) = 0\}} - 1. \tag{3.2}$$

By Theorem 5, choose an $\boldsymbol{\alpha}$ depending on z_1 and z_2 such that $(w_1, w_2) \in I_{\underline{z}}$ if and only if the matrix $M(\underline{w}, \boldsymbol{\alpha})$ is positive semidefinite. Let the subspace $\mathcal{M}$ of $H^2_{\boldsymbol{\alpha}} \oplus H^2_{\boldsymbol{\alpha}}$ be as in the first part of Lemma 2. For brevity, let

$$m^2 = 1 - \frac{|K_{\boldsymbol{\alpha}}(z_1, z_2)|^2}{K_{\boldsymbol{\alpha}}(z_1, z_1)K_{\boldsymbol{\alpha}}(z_2, z_2)}.$$

which is a non-negative number in $\mathbb{D}$. Then of course

$$M(\underline{w}, \boldsymbol{\alpha}) = \begin{pmatrix} K_{\boldsymbol{\alpha}}(z_1, z_1) & K_{\boldsymbol{\alpha}}(z_1, z_2) \\ K_{\boldsymbol{\alpha}}(z_2, z_1) & (1 - m^2)K_{\boldsymbol{\alpha}}(z_2, z_2) \end{pmatrix}$$

is positive semidefinite by definition of m. So by Theorem 5 there is a holomorphic function $f : \Omega \to \mathbb{D}$ such that $f(z_1) = 0$ and $f(z_2) = m$. Moreover, if g any holomorphic function from

Ω to $\mathbb{D}$ such that $g(z_1) = 0$, then the matrix $M((0, g(z_2)), \boldsymbol{\alpha})$ is positive semidefinite, which implies that

$$|g(z_2)|^2 \leq 1 - \frac{|K_{\boldsymbol{\alpha}}(z_1, z_2)|^2}{K_{\boldsymbol{\alpha}}(z_1, z_1)K_{\boldsymbol{\alpha}}(z_2, z_2)}.$$

Thus $|f(z_2)| = \sup\{|g(z_2)| : g$ is a holomorphic function from Ω to $\mathbb{D}$ and $g(z_1) = 0\}$.

Hence

$$t = \frac{1}{m^2} - 1 = \frac{|K_{\boldsymbol{\alpha}}^{21}|^2}{K_{\boldsymbol{\alpha}}^{11}K_{\boldsymbol{\alpha}}^{22} - |K_{\boldsymbol{\alpha}}^{21}|^2}.$$

So by the first part of Lemma 2, we have that the matrix of restriction of the operator $M_z^* \oplus M_z^*$ to the subspace $\mathcal{M}$ in the orthonormal basis $\{h_1(z_1, z_2), h_2(z_1, z_2)\}$ has the matrix representation T^*.

Having constructed the dilation, it is natural to find out what the characteristic function is when $\Omega = \mathbb{D}$. In this case, the general form of the matrix T discussed above is

$$T = \begin{pmatrix} z_1 & 0 \\ \lambda(1 - |z_1|^2)^{1/2}(1 - |z_2|^2)^{1/2} & z_2 \end{pmatrix}. \tag{3.3}$$

where z_1 and z_2 are two points in the open unit disk $\mathbb{D}$ and $\lambda \in \mathbb{C}$. We are using the explicit value of t for the unit disc.

Lemma 6. *For $i = 1, 2$, let $\varphi_i(z) = (z - z_i)/(1 - \bar{z}_i z)$. The characteristic function of T is*

$$\theta_T(z) = \begin{pmatrix} (1 - |\lambda|^2)^{1/2}\varphi_2 & -\lambda \\ \bar{\lambda}\varphi_1\varphi_2 & (1 - |\lambda|^2)^{1/2}\varphi_1 \end{pmatrix}$$

Proof: Recall that $\mathcal{M}$ is the subspace spanned by the orthonormal vectors $h_1(z_1, z_2)$ and $h_2(z_1, z_2)$. Since the compression of $M_z \oplus M_z$ to the co-invariant subspace $\mathcal{M}$ is T, by Beurling-Lax-Halmos theorem, we need to only find up to unitary coincidence (see [19], page 192 for definition) the inner function whose range is $\mathcal{M}^\perp$. So let $\begin{pmatrix} f \\ g \end{pmatrix}$ be a vector in the orthogonal complement of $\mathcal{M}$. The condition of orthogonality to h_1 implies that $g(z_1) = 0$ which is equivalent to $g = \varphi_1 \xi$ for arbitrary $\xi \in H^2(\mathbb{D})$. Now the orthogonality condition to h_2 implies that $(1 - |\lambda|^2)^{1/2} f(z_2) + \lambda \xi(z_2) = 0$, which is the same as

$$(1 - |\lambda|^2)^{1/2}\varphi_1(z_2)f(z_2) + \lambda g(z_2) = 0. \tag{3.4}$$

This implies that there is an $\eta_1 \in H^2(\mathbb{D})$ such that

$$(1 - |\lambda|^2)^{1/2} f + \lambda g' = \varphi_2 \eta_1.$$

It is obvious that conversely if $\begin{pmatrix} f \\ g \end{pmatrix}$ is a function from $H^2(\mathbb{D}) \oplus H^2(\mathbb{D})$ such that g is in range of φ and satisfies (3.4), then it is in the orthogonal complement of $\mathcal{M}$.

Now let $\eta_2 = (1 - |\lambda|^2)^{1/2}\xi - \bar{\lambda} f$. Then

$$\theta \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} (1 - |\lambda|^2)^{1/2}\varphi_2\eta_1 - \lambda\eta_2 \\ \bar{\lambda}\varphi_1\varphi_2\eta_1 + (1 - |\lambda|^2)^{1/2}\varphi_1\eta_2 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Thus if $\begin{pmatrix} f \\ g \end{pmatrix}$ satisfies (3.4), then it is in the range of θ. Conversely, it is easy to see that any element in the range of θ will satisfy (3.4). Thus the orthogonal complement of $\mathcal{M}$ in $\mathcal{H}$ is the range of θ. So θ is the characteristic function of the given matrix. ∎

We would like to remark here that for $z_1 = z_2$, the characteristic function obtained above can also be obtained by using the definition of characteristic function along with the formula for transformation of characteristic function under a biholomorphic automorphism of the unit disk.

Denoting the matrix defined in (3.3) by T_λ, note that if T_λ and T_μ are two such matrices with $|\lambda| = |\mu|$, then

$$\begin{aligned} \theta_{T_\lambda}(z) &= \begin{pmatrix} (1-|\lambda|^2)^{1/2}\varphi_2 & -\lambda \\ \bar{\lambda}\varphi_1\varphi_2 & (1-|\lambda|^2)^{1/2}\varphi_1 \end{pmatrix} = \\ &= \begin{pmatrix} (1-|\mu|^2)^{1/2}\varphi_2 & -e^{i\psi}\mu \\ e^{-i\psi}\bar{\mu}\varphi_1\varphi_2 & (1-|\mu|^2)^{1/2}\varphi_1 \end{pmatrix} \text{ for some } \psi \in [0, 2\pi] \\ &= \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix} \begin{pmatrix} (1-|\mu|^2)^{1/2}\varphi_2 & -\mu \\ \bar{\mu}\varphi_1\varphi_2 & (1-|\mu|^2)^{1/2}\varphi_1 \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix} \\ &= \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix} \theta_{S_\mu}(z) \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix}, \end{aligned}$$

and hence their characteristic functions coincide. So they are unitarily equivalent. Conversely, if T_λ and T_μ are unitarily equivalent, then their characteristic functions coincide and hence the singular values of the characteristic functions are same. Note that when $z_1 \neq z_2$, we have

$$\theta_{T_\lambda}(z_1)\theta_{T_\lambda}(z_1)^* = \begin{pmatrix} (1-|\lambda|^2)|\omega|^2 + |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix}$$

for some $\omega \in \mathbb{C}$ (independent of λ). When $z_1 = z_2$, then

$$\theta_{T_\lambda}(z_1)\theta_{T_\lambda}(z_1)^* = \begin{pmatrix} 0 & |\lambda|^2 \\ 0 & 0 \end{pmatrix}.$$

In either case, coincidence of θ_{T_λ} and θ_{T_μ} mean that $|\lambda| = |\mu|$. Thus using the explicit characteristic function we have proved the following.

Theorem 7. *Two matrices T_λ and T_μ as defined in (3.3) are unitarily equivalent if and only if $|\lambda| = |\mu|$.*

We now shift our attention to the construction of dilation when the homomorphism ρ_T is induced by a 2×2 matrix T with equal eigenvalues. So $\sigma(T) = \{z\}$. The domain Ω has its associated Szego kernel which is denoted by $\hat{K}_\Omega(z, w)$. If

$$a = (\sup\{|r'(z)| : r \in \text{ Rat}(\Omega) \text{ and } r(z) = 0\})^{-1},$$

then a generalization due to Ahlfors to multiply connected domain of Schwarz lemma says that $a = \hat{K}_\Omega(z, z)^{-1}$. Let $\partial\Omega$ be the topological boundary of Ω and let $|d\nu|$ be the arc-length measure on $\partial\Omega$. Consider the measure $dm = |\hat{K}_\Omega(\nu, z)|^2|d\nu|$, and let the associated Hardy space $H^2(\Omega, dm)$ be denoted by $\mathcal{H}$. The measure dm is mutually absolutely continuous with respect to the arc length measure. Thus the evaluation functionals on $\mathcal{H}$ are bounded and hence $\mathcal{H}$ possesses a reproducing kernel K. Then it is known that K satisfies the property:

$$\frac{K(z,z)}{(K(z,z)\|\partial_{\bar z}K(\cdot,z)\|^2 - |\langle \partial_{\bar z}K(\cdot,z), K(\cdot,z)\rangle|^2)^{1/2}} = \hat{K}_\Omega(z,z)^{-1},$$

see [11, Theorem 2.2]. Now the dilation for $A = \begin{pmatrix} z & 0 \\ a\lambda & z \end{pmatrix}$, where $\lambda > 0$, is the operator $M_z \oplus M_z$ on the Hilbert space $\mathcal{H} \oplus \mathcal{H}$. This is easily verified since the restriction of $M_z^* \oplus M_z^*$ to the subspace $\mathcal{N}$ which was described in the second part of Lemma 2 is A^*.

4. The Operator Space

In this section, we shall show that a natural operator space structure is associated with the problem that we are considering. We begin by recalling the relevant definitions.

Given a Banach space X, if $\|\cdot\|_k$ is a norm on the algebraic tensor product $X \otimes \mathcal{M}_k$ for each $k \in \mathbb{N}$ which satisfies

(1) $\|A((x_{ij}))B\|_k \leq \|A\|\|((x_{ij}))\|_k\|B\|$ for any pair of scalar matrices A, B in $\mathcal{M}_k$ and $((x_{ij}))$ in $X \otimes \mathcal{M}_k$, $k = 1, 2, \ldots$ and
(2) $\left\|\left(\begin{smallmatrix} R & 0 \\ 0 & S \end{smallmatrix}\right)\right\|_k = \max\{\|R\|_k, \|S\|_\ell\}$, for $R \in X \otimes \mathcal{M}_k$ and $S \in X \otimes \mathcal{M}_\ell$

then X is said to be an operator space. Now, if X is an operator space and $\rho : X \to \mathcal{H}$ is a linear map and for each $k \in \mathbb{N}$, the map $\rho \otimes I_k : (X, \|\cdot\|_k) \to \mathcal{L}(\mathcal{H} \otimes \mathbb{C}^k)$ is contractive then ρ is said to be *completely contractive.* If $\mathcal{C}$ is a C^* - algebra then $\mathcal{C} \otimes \mathcal{M}_k$ is again a C^* - algebra with respect to the unique C^* - norm $\|\cdot\|_k$ on $\mathcal{C}$. This provides a canonical operator space structure $(\mathcal{C}, \|\cdot\|_k)$. In particular, if $\mathcal{C}$ is a commutative C^* - algebra and $\mathcal{A}$ is a uniform sub-algebra of $\mathcal{C}$ then it inherits a natural operator space structure from that of $\mathcal{C}$. The uniform algebra $\mathcal{A}$ equipped with this operator space structure is called the MIN operator space. A celebrated theorem of Arveson says that a contractive homomorphism $\rho_T : \mathcal{A}(\Omega) \to \mathcal{L}(\mathcal{H})$ dilates if and only if it is completely contractive.

Given an operator T on the finite dimensional Hilbert space $\mathbb{C}^n$, we may assume, after a unitary conjugation, that T is upper triangular. We impose the additional condition that the eigenvalues $\underline{z} = \{z_1, z_2, \ldots, z_n\}$ of T are distinct.

Lemma 8. *There are $n \times n$ matrices $V_1, V_2, \ldots, V_n$ (depending on the eigenvalues $z_1, z_2, \ldots, z_n$) such that the map $\rho_T \otimes I : \mathcal{A}(\Omega) \otimes \mathcal{M}_k \to \mathcal{M}_n \otimes \mathcal{M}_k$ is of the form*

$$(\rho_T \otimes I)R = V_1 \otimes R(z_1) + V_2 \otimes R(z_2) + \cdots + V_n \otimes R(z_n)$$

for any $R \in \mathcal{A}(\Omega) \otimes \mathcal{M}_k$ and any $k = 1, 2, \ldots$.

Proof: First let us note that the homomorphism ρ_T does not distinguish between functions in $\mathcal{A}(\Omega) \otimes \mathcal{M}_k$ which agree on the subset $\{z_1, z_2, \ldots, z_n\}$. Let F and G be two such functions in $\mathcal{A}(\Omega)$. Then $H = F - G$ is in $\mathcal{A}(\Omega) \otimes \mathcal{M}_k$ and H vanishes at the points $z_1, z_2, \ldots z_n$. Thus on Ω, the function H can be written as the product $(z - z_1)(z - z_2) \ldots (z - z_n)R(z)$ where R again is in $\mathcal{A}(\Omega) \otimes \mathcal{M}_k$. By multiplicativity of the functional calculus, we see that $\rho_T(H) =$

$(T-z_1)(T-z_2)\dots(T-z_n)R(T)$. Note that $(z-z_1)(z-z_2)\dots(z-z_n)$ is the characteristic polynomial of T and by Cayley-Hamilton theorem, $(T-z_1)(T-z_2)\dots(T-z_n)=0$. Thus $\rho_T(H)=0$ and consequently, $\rho_T(F)=\rho_T(G)$.

Now define the following n elements of $\mathcal{A}(\Omega)$:

$$r_i(z)=\frac{(z-z_1)\dots(z-z_{i-1})(z-z_{i+1})\dots,(z-z_n)}{(z_i-z_1)\dots(z_i-z_{i-1})(z_i-z_{i+1})\dots,(z_i-z_n)}.\ i=1,2,\dots,n$$

and let

$$V_i=\rho_T(r_i)\text{ for } i=1,2,\dots,n.$$

Given $R\in\mathcal{A}(\Omega)\otimes\mathcal{M}_k$, consider the function

$$S(z)=R(z_1)r_1(z)+R(z_2)r_2(z)+\dots+R(z_n)r_n(z)\in\mathcal{A}(\Omega)\otimes\mathcal{M}_k.$$

Then R and S agree on the set $\{z_1,z_2,\dots.z_n\}$ and hence

$$\rho_T(R)=\rho_T(S)=R(z_1)\rho_T(r_1)+R(z_2)\rho_T(r_2)+\dots+R(z_n)\rho_T(r_n)=R(z_1)V_1+R(z_2)V_2+\dots+R(z_n)V_n.$$

■

The contractivity and the complete contractivity of the map ρ_T amounts to respectively

$$\sup\{\|w_1V_1+w_2V_2+\dots+w_nV_n\|_{\rm op}:w=(w_1,w_2,\dots,w_n)\in I_{\underline{z}}\}\leq 1 \tag{4.1}$$

and

$$\sup\{\|\sum_{i=1}^n W_i\otimes V_i\|_{\rm op}:W_i\in\mathcal{M}_k\text{ and }W=(W_1,W_2,\dots,W_n)\in I^k_{\underline{z}}\text{ for }k\geq 1\}\leq 1. \tag{4.2}$$

The norm $\|R\|_\infty=\sup_{z\in\Omega}\|R(z)\|$ coincides with the norm of the MIN operator space structure that $\mathcal{A}(\Omega)$ inherits as a uniform sub-algebra of the commutative C^*-algebra $C(\bar\Omega)$. We let $\mathrm{HC}_{\Omega,\underline{z}}(\mathbb{C}^n)$ denote the operator space structure determined by the sequence of norms $\|\cdot\|_{\underline{z},k}$. Again, if $k=1$, we will write $\|\cdot\|_{\underline{z}}$ rather than $\|\cdot\|_{\underline{z},1}$. The proof of the following Lemma is self evident.

Lemma 9. *The contractive homomorphism $\rho_T:\mathcal{A}(\Omega)\to\mathcal{M}_n$ is completely contractive if and only if the contractive linear map $L_T:(\mathbb{C}^n,\|\cdot\|_{\underline{z}})\to\mathcal{M}_n$ defined by $L_T(\underline{w})=w_1V_1+w_2V_2+\dots+w_nV_n$, $\underline{w}\in I_{\underline{z}}$ is completely contractive on the operator space $\mathrm{H}_{\Omega,\underline{z}}(\mathbb{C}^n)$.*

Paulsen [15] has shown that there is a contractive map $\mathrm{MIN}(\mathbb{C}^m)\to\mathcal{M}_n$ which is not completely contractive for $m\geq 5$. Pisier in his book [14, Exercise 3.7] points out the existence of such maps for $m\geq 3$. However, the answer to the following question is not obvious.

Question 10. Does the Hyperconvex operator space $\mathrm{HC}_{\Omega,\underline{z}}(\mathbb{C}^n)$ ever coincide with the $\mathrm{MIN}(\mathbb{C}^n)$?

Remark 11. *If the answer to this question was affirmative for some domain Ω in $\mathbb{C}$ and a subset $\{z_1,z_2,\dots,z_n\}$ of Ω then the existence of a contractive homomorphism $\rho_T:\mathcal{A}(\Omega)\to\mathcal{M}_n$ which is not completely contractive would follow. Agler [4] proves that all contractive homomorphisms of the algebra $\mathcal{A}(\mathbb{A})$, where $\mathbb{A}=\{z\in\mathbb{C}:r<|z|<1\}\subseteq\mathbb{C}$ is the annulus, dilates. Or, equivalently, all contractive homomorphisms of $\mathcal{A}(\mathbb{A})$ are completely contractive. This result has the unexpected implication that $\mathrm{HC}_{\Omega,\underline{z}}(\mathbb{C}^n)$ cannot be the MIN operator space $\mathrm{MIN}(\mathbb{C}^n)$ when Ω is the annulus $\mathbb{A}$, $n\in\mathbb{N}$. Also, since all contractive homomorphisms of $\mathcal{A}(\Omega)$ into the*

2×2 matrix algebra $\mathcal{M}_2$ are completely contractive, the possibility of the Hyperconvex operator space $\mathrm{HC}_{\Omega,\underline{z}}(\mathbb{C}^2)$ coinciding with the MIN operator space $\mathrm{MIN}(\mathbb{C}^2)$ is ruled out.

We had initially hoped that we could show that $\mathrm{HC}_{\Omega,\underline{z}}(\mathbb{C}^n)$ agrees with $\mathrm{MIN}(\mathbb{C}^n)$ for some plain domain $\Omega \subseteq \mathbb{C}$. If that were the case, the existence of a contractive homomorphism of the algebra $\mathcal{A}(\Omega)$ which is not completely contractive would follow from the Theorem of Paulsen [15] discussed in the previous paragraph. This does not appear to be a very tractable approach at the moment. However, Dritschel and McCullough have found such an example recently [9].

We now make a couple of remarks. The first one of them is about zeroes of the kernel $K_{\boldsymbol{\alpha}}$. Pick an $\boldsymbol{\alpha}$ as obtained from the theorem above, fix it and for brevity, suppress it. So we denote $K_{\boldsymbol{\alpha}}$ by K. For any $1 \leq i, j \leq n$, the value $K(z_i, z_j)$ can not be zero. Indeed, if say, $K(z_1, z_2) = 0$, then we shall get a function $f : \Omega \to \mathbb{D}$ such that $|f(z_1)| = 1$ which is impossible in view of Maximum Modulus Theorem.

The second remark concerns contractivity of the following operator. Fix an n-tuple $\underline{w} = (w_1, \ldots, w_n)$ with elements from $\mathbb{D}$ and define an operator T on the n-dimensional space spanned by $K(\cdot, z_1), \ldots, K(\cdot, z_n)$ by

$$TK(\cdot, z_i) = \bar{w}_i K(\cdot, z_i),\ 1 \leq i \leq n.$$

This operator is a contraction if and only if for an arbitrary set $c_1, c_2, \ldots, c_n$ of n complex numbers, we have

$$\|T(\sum c_j K(\cdot, z_j))\| \leq \|\sum c_j K(\cdot, z_j)\|.$$

This is equivalent to

$$\sum_{i,j} c_i \bar{c}_j \overline{w}_i w_j K_{\boldsymbol{\alpha}}(z_j, z_i) \leq \sum_{i,j} c_i \bar{c}_j K_{\boldsymbol{\alpha}}(z_j, z_i)$$

and finally

$$\sum_{i,j} c_i \bar{c}_j (1 - \bar{w}_i w_j) K_{\boldsymbol{\alpha}}(z_j, z_i) \geq 0.$$

This last condition is equivalent to the Abrahamse-Nevanlinna-Pick matrix being non-negative definite. Thus an n-tuple $\underline{w}$ from $\mathbb{D}^n$ is in $I_{\underline{z}}$ if and only if the operator T defined above is a contraction.

Lemma 12. *There is a $\underline{w}$ in $I_{\underline{z}}$ for which the function mapping $(z_1, z_2, \ldots, z_n)$ to $(w_1, w_2, \ldots, w_n)$ is unique.*

Proof: Consider the continuous function $\underline{w} \to \lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha}))$, where $\boldsymbol{\alpha} \in \mathbb{T}^m$ is a fixed element whose existence is guaranteed by Theorem 5 and $\underline{w} \in \mathbb{C}^n$. As $I_{\underline{z}}$ is compact, let U be a large enough open ball containing it. There are points $\underline{w}'$ and $\underline{w}''$ in U such that $\lambda_{\min}(M(\underline{w}', \boldsymbol{\alpha}))$ is positive and $\lambda_{\min}(M(\underline{w}'', \boldsymbol{\alpha}))$ is negative. Continuous image of a connected set is connected, so $(\lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha})))(U)$ is an interval E in the real line with positive and negative numbers in it. So $0 \in E$, that is, $\lambda_{\min}(M(\underline{w}, \boldsymbol{\alpha})) = 0$ for some $\underline{w}$. By definition of the set $I_{\underline{z}}$, this $\underline{w}$ is in $I_{\underline{z}}$. As the Abrahamse-Nevanlinna-Pick determinant vanishes for the chosen $\boldsymbol{\alpha}$, there is a unique function f satisfying $f(z_i) = w_i$ for $i = 1, 2, \ldots, n$. ∎

The argument above also yields the following result. It is easy to see that the set $I_{\underline{z}}$ is convex and balanced, so it is the closed unit ball of some norm on $\mathbb{C}^n$. The corresponding norm is given by the Minkowski functional

$$\|\underline{w}\| = \inf\{t > 0 : \frac{1}{t}\underline{w} \in I_{\underline{z}}\}.$$

The unit sphere in this norm consists of all $\underline{w}$ such that

$$\det\left(((1 - w_i\overline{w_j})K(z_i, z_j))\right) = 0.$$

This can be seen as follows. The unit sphere is by definition

$$\begin{aligned} I_{\underline{z}} \cap \overline{I_{\underline{z}}^c} &= (\lambda_{\min}(M(\underline{w})))^{-1}[0,\infty) \cap (\lambda_{\min}(M(\underline{w})))^{-1}[-\infty, 0] \\ &= (\lambda_{\min}(M(\underline{w})))^{-1}\{0\}. \end{aligned}$$

The minimum eigenvalue of a positive semidefinite matrix vanishes if and only if the matrix has determinant zero. Of course, this determinant is zero if and only if one of the principal minors is zero. Thus as an application of Theorem 5, we get

LEMMA 13. *An element $\underline{w}$ of $\mathbb{C}^n$ has norm 1 (the Minkowski norm) if and only if one of the principal minors of the matrix*

$$\left((1 - w_i\bar{w}_j)K_{\boldsymbol{\alpha}}(z_i, z_j)\right)$$

vanishes where $\boldsymbol{\alpha}$ is the element of $\mathbb{T}^m$ whose existence is guaranteed by Theorem 5.

Now we introduce an equivalence relation on $\mathcal{A}(\Omega)$ by setting f to be equivalent to g if their values at the points $z_1, z_2, \ldots z_n$ agree. The set of all functions in $\mathcal{A}(\Omega)$ which vanish at the points $z_1, z_2, \ldots z_n$ is a closed subspace. Thus the quotient space under the above equivalence relation is a Banach space under the natural quotient norm. We denote the quotient Banach space by $\mathcal{F}$. On the other hand, let V consist of all $\underline{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$ such that there is a holomorphic function $f : \Omega \to \mathbb{C}$ satisfying $f(z_i) = w_i$ for $i = 1, 2, \ldots, n$. Endow V with a norm by declaring $I_{\underline{z}}$ as its unit ball. We are sure that the following lemma must be known to experts. However, we have added it here for completeness.

LEMMA 14. *The Banach space V (with $I_{\underline{z}}$ as its unit ball) is an operator space. Indeed, V is completely isometrically isomorphic to the subspace $\mathcal{F}$ of the quotient C^*-algebra $C(\overline{\Omega})/X$.*

Proof. It is easy to see that there is a bijective linear transformation from the subspace V of $\mathbb{C}^m$ on to $\mathcal{F}$ which sends $\underline{w}$ to the equivalence class $[f]$ of a holomorphic function $f : \Omega \to \mathbb{C}$ mapping z_i to w_i for $i = 1, 2, , \ldots, n$. The norm of $[f]$ is

$$\begin{aligned} \|[f]\| &= \inf\{\|f + g\| : g(z_i) = 0 \text{ for all } i = 1, 2, \ldots, m\} \\ &= \inf\{\|h\| : h(z_i) = w_i \text{ for all } i = 1, 2, \ldots, m\} \end{aligned}$$

and the norm of $\underline{w}$ is

$$\|\underline{w}\| = \inf\{t > 0 : \frac{\underline{w}}{t} \in I_{\underline{z}}\}.$$

Now $\underline{w}/t \in I_{\underline{z}}$ if and only if there i a function g such that $tg(z_i) = w_i$ for all $i = 1, 2, \ldots, m$ and $\|g\| \leq 1$. Note that if $h(z_i) = w_i$ for all $i = 1, 2, \ldots, m$, then $\underline{w}/\|h\| \in I_{\underline{z}}$ because $\|h\| \frac{h}{\|h\|}(z_i) = w_i$ for all $i = 1, 2, \ldots, m$. So

$$\{\|h\| : h(z_i) = w_i \text{ for all } i = 1, 2, \ldots, m\} \subseteq \{t > 0 : \frac{w}{t} \in I_{\underline{z}}\}.$$

Thus $\|[f]\| \geq \|\underline{w}\|$. Suppose $\|[f]\| > \|\underline{w}\|$. Then there exists $c > 0$ such that

$$\inf\{\|h\| : h(z_i) = w_i \text{ for all } i = 1, \ldots, m\} > c > \inf\{t > 0 : \underline{w} = tg(\underline{z}) \text{for some} g \text{satisfying} \|g\| \leq 1\}.$$

So there is a $t < c$ such that $w_i = tg(z_i)$ for all $i = 1, 2, \ldots, m$, for some g with $\|g\| \leq 1$. Now let $h = tg$. Then $h(z_i) = tg(z_i)w_i$ for all $i = 1, \ldots, m$ and $\|h\| = t\|g\| < c$. We therefore arrive at a contradiction. So $\|[f]\| = \|\underline{w}\|$.

The same argument, when repeated with matrix valued functions rather than the scalar valued ones, gives the complete isometry. ■

LEMMA 15. *The homomorphism ρ induces a linear transformation $\tilde{\rho}$ from the operator space V into $\mathcal{L}(\mathbb{C}^m)$ such that ρ is contractive (respectively completely contractive) if and only if $\tilde{\rho}$ is contractive (respectively completely contractive).*

Proof. The homomorphism ρ induces a unique linear map $\tilde{\rho} : V \to \mathcal{L}(\mathbb{C}^m)$ since ρ is constant on the subspace X of $\mathcal{A}(\Omega)$ and since the quotient vector space $\mathcal{F}$ is isomorphic to the vector space V. Then the conclusion follows from completely isometric isomorphism of V and $\mathcal{F}$. ■

We end this section with a characterization of contractivity of the homomorphism ρ_T or which is the same as the contractivity of the linear map L_T.

PROPOSITION 16. *The linear map $L_T : (\mathbb{C}^n, \|\cdot\|_z) \to \mathcal{M}_n$ is contractive if and only if*

$$\sup\{\|w_1 V_1 + \cdots w_n V_n\|_z : M(\underline{w}, \alpha) \geq 0 \textit{ for all } \boldsymbol{\alpha}, \det M(\underline{w}, \boldsymbol{\alpha}) = 0 \textit{ for some } \boldsymbol{\alpha}\} \leq 1.$$

Proof. The operator L_T being finite dimensional, it has a maximizing vector $\underline{w}_0$ of unit length, that is, $\|L_T(\underline{w}_0)\| = \|L_T\|$. But we have observed that the length $\|\underline{w}\|_{\underline{z}} = 1$ if and only if $\det M(\underline{w}, \boldsymbol{\alpha}) = 0$ for some $\boldsymbol{\alpha}$. This completes the proof of the Lemma. ■

5. A FACTORIZATION CONDITION

Let T be a linear transformation on an n dimensional vector space V with distinct eigenvalues $z_1, z_2, \ldots, z_n$. Suppose T^* has n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$. If $\sigma = \{z_1, z_2, \ldots, z_n\}$, then define a positive definite function $K : \sigma \times \sigma \to \mathbb{C}$ by setting

$$K(z_j, z_i) = \langle v_i, v_j \rangle, \quad i = 1, 2, \ldots, n. \tag{5.1}$$

If Ω is a bounded domain containing $\sigma(T)$, then $\rho_T : \mathcal{A}(\Omega) \to \mathcal{L}(V)$ is the homomorphism induced by T. Suppose there exists a dilation of the homomorphism ρ_T. Then it follows from [1, Theorem 2] that there is a flat unitary vector bundle $\mathcal{E}$ of rank n (see [2] for definitions and complete results on model theory in multiply connected domains) such that

$$\tilde{\rho}_T : C(\partial\Omega) \to \mathcal{B}(H^2_{\mathcal{E}}(\Omega)) \tag{5.2}$$

is a dilation of ρ_T in the sense of (1.1). The Hilbert space $H^2_{\mathcal{E}}(\Omega)$ has the $\mathcal{B}(\mathcal{E})$ valued reproducing kernel $K_{\mathcal{E}}$. So the fact that ρ_T dilates implies in particular that the linear transformation T can be realized as the compression of the operator M_z on $H^2_{\mathcal{E}}(\Omega)$ to an n-dimensional co-invariant subspace, say M. Indeed, let us take M to be the span of the n eigenvectors $K_{\mathcal{E}}(\cdot, z_i)x_i$ for $i = 1, 2, \ldots, n$ of M_z^* on $H^2_{\mathcal{E}}(\Omega)$ where $x_1, x_2, \ldots x_n$ are from $\mathcal{E}$. Now the map which sends v_i to $K_{\mathcal{E}}(\cdot, z_i)x_i$ for every $i = 1, 2, \ldots, n$ is an isometry. Clearly, this map intertwines T^* and the restriction of M_z^* to M. Hence

$$\langle v_i, v_j\rangle = \langle K_{\mathcal{E}}(\cdot, z_i)x_i, K_{\mathcal{E}}(\cdot, z_j)x_j\rangle = \langle K_{\mathcal{E}}(z_j, z_i)x_i, x_j\rangle.$$

Conversely, if there is a flat unitary vector bundle $\mathcal{E}$ and n vectors $x_1, x_2, \ldots, x_n$ in $\mathcal{E}$ satisfying (5), then ρ_T obviously dilates. So we have proved that

THEOREM 17. *The homomorphism ρ_T is dilatable to a homomorphism $\tilde{\rho}$ if and only if the kernel k, as defined in (5.1), can be written as*

$$K(z_i, z_j) = \langle K_{\mathcal{E}}(z_j, z_i)x_i, x_j\rangle$$

where $K_{\mathcal{E}}$ is the reproducing kernel corresponding to the Hardy space $H^2_{\mathcal{E}}(\Omega)$ associated with a flat unitary vector bundle $\mathcal{E}$.

It is interesting to see how contractivity of ρ_T is related to the above theorem. Note that ρ_T is contractive if and only if $\|f(T)^*\| \le \|f\|_\infty$ by definition of ρ_T. Since $T^*v_i = \bar{z}_i v_i$ we note that $f(T)^*v_i = \overline{f(z_i)}v_i$, for $1 \le i \le n$ and $f \in \mathrm{Rat}(\Omega)$. It then follows that

$$\begin{aligned} \|\rho_T(f)^*\|^2 &= \sup\{\|f(T)^*\big(\sum_{i=1}^n \alpha_i v_i\big)\|^2 : \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}\} \\ &= \|\sum_{i=1}^n \alpha_i \overline{f(z_i)} v_i\|^2 = \sum_{i,j=1}^n \alpha_i\bar{\alpha}_j \overline{f(z_i)} f(z_j)\langle v_i, v_j\rangle. \end{aligned}$$

Therefore, $\|f(T)^*\| \le \|f\|_\infty$ if and only if

$$\sum_{i,j=1}^n \alpha_i\bar{\alpha}_j \overline{f(z_i)} f(z_j)\langle v_i, v_j\rangle \le \sum_{i,j=1}^n \alpha_i\bar{\alpha}_j\langle v_i, v_j\rangle.$$

Thus contractivity of ρ_T is equivalent to no-negative definiteness of the matrix

$$\Big(\!\!\Big((1 - \overline{f(z_i)}f(z_j))K(z_j, z_i)\Big)\!\!\Big)_{i,j=0}^n. \tag{5.3}$$

If ρ_T is dilatable then the theorem above tells us that

$$\Big(\!\!\Big((1 - \overline{f(z_i)}f(z_j))K(z_j, z_i)\Big)\!\!\Big)_{i,j=0}^n = \Big(\!\!\Big(\langle(1 - \overline{f(z_i)}f(z_j))K_{\mathcal{E}}(z_j, z_i)x_i, x_j\rangle\Big)\!\!\Big)_{i,j=0}^n. \tag{5.4}$$

The last matrix is non-negative definite because M_z on $H^2_{\mathcal{E}}(\Omega)$ induces a contractive homomorphism.

The interesting point to note here is that our construction of the dilation of ρ_T when T is a 2×2 matrix proves that the general dilation in that case is of the form $H^2_\alpha(\Omega) \otimes \mathbb{C}^2$.

In such a case, that is when ρ_T has a dilation of the form

$$\tilde{\rho}_T : C(\partial\Omega) \to H^2_\alpha(\Omega) \otimes \mathbb{C}^n \tag{5.5}$$

for some α, the multiplication operator $M_z \otimes I$ on $H^2_\alpha(\Omega) \otimes \mathbb{C}^n$ is a dilation of T. Since the eigenvectors $\{v_1, v_2, \ldots, v_n\}$ for T^* span V and the set of eigenvectors of $M_z^* \otimes I : H^2_\alpha(\Omega) \otimes \mathbb{C}^n \to H^2_\alpha(\Omega) \otimes \mathbb{C}^n$ at z_i is the set of vectors $\{K_\alpha(\cdot, z_i) \otimes a_j : a_j \in \mathbb{C}^n, 1 \leq j \leq n\}$ for $1 \leq i \leq n$, it follows that any map $\Gamma : V \to H^2_\alpha(\Omega)$ that intertwines T^* and M_z^* must be defined by $\Gamma(v_i) = K_\alpha(\cdot, z_i) \otimes a_i$ for some choice of a set of n vectors $a_1, a_2, \ldots, a_n$ in $\mathbb{C}^n$. Now Γ is isometric if and only if

$$(5.6) \qquad \big(\!\big(K(z_j, z_i)\big)\!\big) = \big(\!\big(\langle v_i, v_j\rangle\big)\!\big) = \big(\!\big(K_\alpha(z_j, z_i)\langle a_i, a_j\rangle\big)\!\big).$$

Clearly, this means that $\big(\!\big(K(z_j, z_i)\big)\!\big)$ admits $\big(\!\big(K_\alpha(z_j, z_i)\big)\!\big)$ as a factor in the sense that $\big(\!\big(K(z_j, z_i)\big)\!\big)$ is the Schur product of $\big(\!\big(K_\alpha(z_j, z_i)\big)\!\big)$ and a positive definite matrix, namely, the matrix $A = \big(\!\big(\langle a_i, a_j\rangle\big)\!\big)$.

Conversely, the contractivity assumption on ρ_T does not guarantee that K_α is a factor of K. Suppose we make this stronger assumption, that is, we assume there exists a positive definite matrix A such that $\big(\!\big(K(z_j, z_i)\big)\!\big) = \big(\!\big(K_\alpha(z_j, z_i) a_{ij}\big)\!\big)$, where $A = \big(\!\big(a_{ij}\big)\!\big)$. Since A is positive, it follows that $A\big(\!\big(\langle a_i, a_j\rangle\big)\!\big)$ for some set of n vectors $a_1, \ldots, a_n$ in $\mathbb{C}^n$. Therefore if we define the map $\Gamma : V \to H^2_\alpha(\Omega) \otimes \mathbb{C}^n$ to be $\Gamma(v_i) = K_\alpha(\cdot, z_i) \otimes a_i$ for $1 \leq i \leq n$ then Γ is clearly unitary and is an intertwiner between T and M_z^*. Thus the theorem above has the corollary:

COROLLARY 18. *The homomorphism ρ_T is dilatable to a homomorphism $\tilde{\rho}$ of the form (5.5) if the kernel K, as defined in (5.1), is Schur product of K_α for some α and a positive definite kernel.*

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