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On the optimality of a class of designs with three concurences

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Abstract

In the present paper we consider a class of unequally replicated designs having concurrence range 2 and spectrum of the form $\mu_1(\mu_2)^{v-3}\mu_3$. Now, Jacroux's (1985) proposition 2.4 says that a design with spectrum of the above form, if satisfies some further conditions, is type 1 optimal. Unfortunately, this proposition does not apply to our designs since they have a poor status regarding E-optimality. Yet we are able to prove the A-optimality (in the general class) of these designs using majorisation technique. A method of construction of an infinite series of our A-optimal designs has also been given.

The first and only known infinite series of examples of designs satisfying Jacroux's conditions appears to be the first one in section 4.1 of Morgan and Srivastava (2000) -hitherto referred to as [MS]. In this paper we use majorisation technique to prove stronger optimality properties of the above mentioned designs of [MS] as well as to present simpler proof of another optimality result in [MS].

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1 Introduction

In the present paper we continue the search for optimal block designs. It is well-known that the "best" design (BIBD) must be binary and must have replication numbers as well as the concurrences all equal. It is also well-known that these equalities require certain divisibility conditions which are often not met. So, the follwing questions arise in one's mind. (a) "what is the best design in a set up where the divisibility conditions are not satisfied ?" (b) "if the divisibility conditions are satisfied, but a BIBD does not exist, then what is the best design ?" We take a glimpse at the status of our knowledge regarding question (a). For (b) we refer to Morgan and Srivastava (2000) and Morgan and Reck (2004).

It is reasonable to believe that in the situations when equal replication is possible but all the concurrences cannot be equal, a binary equireplicate design with concurrences differing by at most one would be optimal. This was conjectured by John and Mitchell (1977), who coined the name "regular Graph designs" (RGD) for such candidates. While this conjecture has been disproved regarding E-criterion, [see Bagchi (1988), Bagchi (1994), for instance] it is widely believed to be true for A- and D-criteria. In fact many RGDs have been proved to satisfy general optimality. [See Cheng (1978), Cheng and Bailey (1991), Bagchi and Bagchi (2001)].

Now suppose equal replication is not possible. Then a likely candidate for optimality is a binary design with replication numbers as well as concurrences differing by at most one. These were termed as semi-RGDs in Jacroux (1985), where many sufficient conditions for the optimality of RGDs and semi-RGDs were provided.

Next, let us consider the situations when neither RGD nor semi-RGD can exist. Morgan and Srivastava (2000) considered these. They defined nearly balanced incomplete block designs NBBD(m), which are binary designs with replication numbers differing by at most once and concurrences differing by at most m. They provided sufficient conditions for the optimality of NBBD(2)'s, using which they proved optimality of certain classes of NBBD(2)'s.

The two classes of NBBD(2)'s (say $\bar{d}_1, \bar{d}_2$) considered in section 4.1 of Morgan and Srivastava (2000) caught the attention of the present author for many reasons. Both are unequally replicated, but the spectra of their C-matrices are "very good". Of these $\bar{d}_1$ has spectrum $(\mu_1)^{v-2}\mu_2$ and it turned out to be, not surprisingly, generalised optimal of type 1 like the MBGDDs of type 1, [see Cheng (1978)]. On the other hand, $\bar{d}_2$ has spectra $\mu_1(\mu_2)^{v-3}\mu_3$. Now, in view of Proposition 2.4 of Jacroux (1985) many researchers in this area, including the present author, believe that a design with spectrum like this is must be optimal but no example was known. $\bar{d}_2$ seems to be the first example satisfying the hypothesis of above proposition and indeed it is optimal! This observation was so exciting that finding another example like this and verifying its optimality seemed to be very urgent. That led to the birth of the present paper. The example [see after (4.20)] may be thought of "opposite" of $\bar{d}_2$.

In section 3, we handle existing optimality results : extend one and provide simpler proof of another, both using using majorisation technique. In section 4 we prove A-optimality of d^* in the general class and present a method of construction of it in section 5.

2 preliminaries

NOTATION 2.1 *Consider a vector* $x = (x_1, x_2, \cdots x_n) \in I\!R^n$.

(a) $x \uparrow$ *and* $x \downarrow$ *will denote the vectors obained by rearranging the co-ordinates of* x *in the increasing and decreasing order respectively.*

(b) Suppose x *has* m *distinct entries* $(m < n)$. *Then* x *will be denoted by* $\prod_{i=1}^{m} x_i^{n_i}$, *if* x_i *has multiplicity* $n_i, i = 1, \cdots m, \sum_{i=1}^{m} n_i = n$.

DEFINITION 2.1 *(Marshall and Olkin (1979))*

For $x, y \in I\!R^n$, x *is said to be weakly majorized from above by* y *(in symbols,* $x \prec^w y$*) if*

$$\sum_{i=1}^{k} x^{\uparrow}{}_i \geq \sum_{i=1}^{k} y^{\uparrow}{}_i, \quad k = 1, 2, \cdots, n. \tag{2.1}$$

It is clear that $\prec^w$ is reflexive and transitive.

We begin with a trivial but useful result.

THEOREM 2.1 *Consider an* $n \times 1$ *vector* x. *Let* $\bar{x} = (\sum_{i=1}^{n} x_i)/n$. *Then*

$$\bar{x}^n \prec^w \Pi_{i=1}^n x_i.$$

We shall now state Tomic's theorem and derive a few results from it. For the proof of this theorem and other results on weak majorisation see Marshall and Olkin (1979).

THEOREM 2.2 *(Tomic(1949)* $x \prec^w y$ *if and only if*

$$\sum_{i=1}^{n} g(x_i) \leq \sum_{i=1}^{n} g(y_i)$$

for every convex decreasing function $g : I\!R \to I\!R$.

Applying Tomic's theorem, one can easily prove the following useful result.

THEOREM 2.3 *Suppose $x^{(1)}, y^{(1)}$ are $m \times 1$ and $x^{(2)}, y^{(2)}$ are $n \times 1$ vectors such that*

$$x^{(i)} \prec^w y^{(i)}, i = 1, 2.$$

Then,

$$x = (x^{(1)}|x^{(2)}) \prec^w y = (y^{(1)}|y^{(2)}).$$

Here $(p|q)$ is the juxtaposition of the vectors p and q.

Using the theorem above we derive a few results to be used in the later chapters of this paper.

THEOREM 2.4 *For an $n\times 1$ vector x, let $\tilde{x}(t)$ denote the $t\times 1$ vector $(x_1, x_2, \cdots x_t),\ t \leq n$. Consider two $n\times 1$ vectors x and y with entries arranged in ascending order and satisfying the following conditions.*

$$(i)\ \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} y_i$$

and

$$(ii)\ \tilde{x}(t) \prec^w \tilde{y}(t),$$

for some $t < n$.

Then, each of the following is a sufficient condition for $x \prec^w y$.

(a) $x_{t+1} = x_{t+2} = \cdots = x_n$.

(b) $x_{t+1} = x_{t+2} = \cdots = x_{n-1};\ y_n \geq x_n - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$.

Proof : (a) Take $q^{(1)} = \tilde{q}(t), q^{(2)} = (q_{t+1}, \cdots q_n), q = x$ or y. By assumption,

$$\sum_{i=1}^{t} x_i \geq \sum_{i=1}^{t} y_i.$$

If equality holds in the above relation, then we are done by Theorems 2.1 and 2.3. So, we assume strict inequality. Let $\delta = \sum_{i=1}^{t} x_i - \sum_{i=1}^{t} y_i$. Clearly $x^{(2)}$ is not majorised by $y^{(2)}$. We define a vector y^* as follows. $y_t^* = y_t + \delta,\ y_{t+1}^* = y_{t+1} - \delta,\ y_i^* = y_i, i \neq t, t+1$. Then, clearly,

$$\sum_{i=1}^{u} y_i^* = \sum_{i-1}^{u} y_i,\ u = t, n.$$

Thus, by Theorem 2.3

$$x \prec^w y^*.$$

But it is clear from the definition of y^* that $y^* \prec^w y$. Hence, the result follows from the transitivity of $\prec^w$.

(b) is proved by applying (a) on $\tilde{x}(n-1)$ and $\tilde{y}(n-1)$. $\square$.

The matrices considered here are all real symmetric matrices.

Notation 2.2 *Consider an $n \times n$ matrix A.*

(a) The principal submatrix bordered by the set of rows $i, j \cdots l$ of A will be denoted by $A(i, j \cdots l)$.

(b) $\mu(A)$ will denote the vector of eigenvalues of A . If A is nonnegative definite, then $\mu(A)$ will denote the vector of positive eigenvalues of A

We now present a few inequalities on the eigenvalues of symmetric matrices. The first one is an well-known result called Ky Fan's maximum principle [see problem I.6.15 of Bhatia (1997), for instance].

Theorem 2.5 *Consider a symmetric matrix A of order n. Suppose $x_1, x_2, \cdots x_k$ $(k < n)$ are orthonormal vectors $\in R^n$. Then,*

$$\sum_{j=1}^{k} \mu^{\uparrow}{}_j(A) \leq \sum_{j=1}^{k} x_j^T A x_j \leq \sum_{j=1}^{k} \mu^{\downarrow}{}_j. \qquad (2.2)$$

Theorem 2.6 *Consider a symmetric matrix A of order n and constant row sum s. If the average row sum of a principal submatrix B of order t is p then*

$$\mu^{\uparrow}{}_1(A) \leq (np - ts)/(n-t) \leq \mu^{\downarrow}{}_1(A).$$

Proof : W.l.g., let $B = A(1, 2, \cdots t)$. Let x denote the normalised version of the vector $(n-t)^t . (-t)^{n-t}$. Now apply Theorem 2.5 with $k = 1$. □

Putting $s = 0$ and $t = 1$ in the theorem above we get the following well-known [see Constantine (1981), for instance] and very useful result.

Corollary 2.1 *For a symmetric matrix A of order n with row sum zero the following equation holds for every $i, 1 \leq i \leq n$.*

$$\mu^{\uparrow}{}_1(A) \leq (n/(n-1))a_{ii} \leq \mu^{\downarrow}{}_1(A).$$

Theorem 2.7 *Consider a nonnegative definite matrix A with row sum zero.*

(a) Suppose A has mutually disjoint principal submatrices $B_1, B_2 \cdots B_m$, B_i of order t_i, $1 \leq i \leq m$, satisfying the following property. For every i there is a number u_i, $1 \leq u_i \leq t_i$ and a vector $z^i \in R^{+u_i}$. Also there is a set $y_{i,1}, y_{i,2}, \cdots y_{i,u_i}$, of orthonormal vectors in R^{t_i}, each of which is orthogonal to the all-one vector of appropriate order such that the following inequalities are satisfied.

$$\sum_{j=1}^{l} y_{i,j}{}^T B_i y_{i,j} \leq \sum_{j=1}^{l} z^{i\uparrow}{}_j, \; 1 \leq l \leq u_i. \qquad (2.3)$$

Suppose further A has another submatrix B_0 which is disjoint from each $B_i, 1 \leq i \leq m$ and has average row sum p . Let $u_0 = 1$ and $z^0 = np/(n - t_0)$. Let $h = \sum_{i=0}^{m} u_i$. Let z be an $h \times 1$ vector obtained by juxtaposing z^i's. Then the following hold.

$$\sum_{j=1}^{l} \mu(A)^{\uparrow}{}_j \leq \sum_{j=1}^{l} z^{\uparrow}{}_j, \ 1 \leq l \leq h. \tag{2.4}$$

(b) If $\geq$ holds in (2.3), then the following is true.

$$\sum_{j=1}^{l} \mu(A)^{\downarrow}{}_j \geq \sum_{j=1}^{l} z^{\downarrow}{}_j, \ 1 \leq l \leq h. \tag{2.5}$$

Using the above and Theorem 2.4, we obtain the following result.

THEOREM 2.8 *Consider a $v \times v$ matrix A with trace s. Suppose for some $m, n, \ m+n < v$, A, there are real numbers $z_i, 1 \leq i \leq m$ and $w_i, 1 \leq i \leq n$, such that*

(a) (2.4) holds with $h = m$,

(b) (2.5) holds with n for h and w_j for z_j, $j = 1, 2, \cdots n$ and

(c)$s_1 = \sum_{j=1}^{m} z_j + \sum_{j=1}^{n} w_j < s$.

Then

$$\prod_{i=1}^{m} z_i. \ (\bar{z})^u. \ \prod_{j=1}^{n} w_j \ \prec^w \mu(d).$$

Here $\bar{z} = s_1/(v - m - n)$.

Let us now consider a block design set up. All designs in this paper are connected block designs with constant block size. We present a set of notations which are commonly used.

NOTATION 2.3 *(i) $\mathcal{D} = \mathcal{D}_{b,k,v}$ denotes the class of all connected block designs with v treatments and b blocks of size k each.*

(ii) $\mathcal{D}^B_{b,k,v}$ denotes the class of binary designs in $\mathcal{D}_{b,k,v}$.

(iii) $r := [bk/v]$. $\lambda = [r(k-1)/(v-1)]$. Here $[x]$ is the smallest integer $\geq x$.

(iv) The replication number of the ith treatment in a design $d \in \mathcal{D}$ will be denoted by r_{di} $(1 \leq i \leq v)$. $R(d)$ will denote the diagonal matrix $diag(r_{d1}, \cdots r_{dv})$.

(vi) For a design $d \in \mathcal{D}, N(d)$ is the usual $(v \times b)$ treatment-block incidence matrix of d. $C(d)$ will denote the information matrix of d : $C(d) = R(d) - k^{-1}N(d)N(d)^T$. $\mu(d)$ will denote the vector of positive eigenvalues of $kC(d)$. $\lambda_{d(i,j)}$ will denote the (i, j)th entry of $N(d)N(d)^T$.

We shall drop d from the notations in (v) and (vi) when there is no scope of confusion as to which design is meant.

Next we present a definition introduced in Bagchi and Bagchi (2001).

DEFINITION 2.2 *A design $d_1 \in \mathcal{D}$ is said to be better than another design $d_2 \in \mathcal{D}$ in the sense of majorization (in short M-better) if (in terms of Notation 2.2)*

$$\mu(d_1) \prec^w \mu(d_2).$$

$d^* \in \mathcal{D}_{b,k,v}$ is said to be optimal in the sense of majorization in a subclass of $\mathcal{D}_{b,k,v}$ (or, in short, d^* is M-optimal in this subclass) if it is M-better than every member of this subclass.

Extending the well-known definition of A-optimality to vectors, we define the following.

DEFINITION 2.3 *An $n \times 1$ vector x is said to be A-better than another $n \times 1$ vector y if*

$$\sum_{i=1}^{n} x_i^{-1} < \sum_{i=1}^{n} y_i^{-1} \qquad (2.6)$$

Remark 2. As noted in Remark 3.1 of Bagchi and Bagchi (2001), if d_1 is M-better than d_2, then d_1 is A-better than d_2, apart from being better with regard to many other (convex) optimality criteria. In view of this, we have the following.

COROLLARY 2.2 *Suppose the C-matrix of a design d satisfies Theorem 2.7 for certain z's and w's. If further $\prod_{i=1}^{m} z_i \, (s/h)^h \, \prod_{j=1}^{n} w_j$ is A-worse than $\mu(d^*)$, then d is A-worse than d^*.*

3 A review of known results.

We shall refer to the paper Morgan and Srivastava (2000) as MS throughout this paper.

We first present a result which is a direct consequence of Proposition 2.4 of Jacroux (1985).

THEOREM 3.1 *Suppose d^* is a design in $\mathcal{D}^B_{b,k,v}$ satisfying the following properties.*
(i) C_{d^*} has spectrum of the form $\mu_1(\mu_2)^{v-3}\mu_3$.
(ii) d^* is E-optimal in $\mathcal{D}^B_{b,k,v}$,
(iii) d^* minimises $tr[(C_d)^2]$ over $d \in \mathcal{D}^B_{b,k,v}$,
Then d^* is Type 1 optimal in $\mathcal{D}^B_{b,k,v}$.

Next, we state an well-known result of Cheng (1987).

Theorem 3.2 *Suppose d^* is a design in $\mathcal{D}^B_{b,k,v}$ satisfying the following properties.*

(i) C_{d^*} has spectrum of the form $\mu_1(\mu_2)^{v-2}$.

(iii) d^* minimises $tr[(C_d)^2]$ over $d \in \mathcal{D}^B_{b,k,v}$,

Then d^* is Type 1 optimal in $\mathcal{D}^B_{b,k,v}$.

We now consider two series of optimal NBBD(2)'s of MS. The use of majorisation techniques yields stronger optimality result for one series and a simpler proof of the known result for the other which are presented below. Both of them satisfy

$$k = 3 \; v \equiv (2 \; mod \; 3) \; r(k-1) = (v-1)\lambda. \tag{3.7}$$

Here r, λ are as defined in Notation 2.3 (iii).

At first we consider the set up satisfying $bk = vr + 1$. More precisely, the parameters are

$$b = 3t^2 + 3t + 1, v = 3t + 2, r = v - 1, \lambda = 2. \tag{3.8}$$

An NBBD(2) $\bar{d}_1$ and a nonbinary design $\tilde{d}_1$ with completely symmetric C-matrix co-exist in this set up. Both of them are optimal with regard to some optimal criteria or other. [For the description, construction and other details see MS. and Morgan and Uddin (1995)]

Let us define

$$a = r(k-1) + \lambda = vr(k-1)/(v-1) \tag{3.9}$$

Then, the spectrums of $kC_{\bar{d}_1}$ and $kC_{\tilde{d}_1}$ are given by

$$spectrum[kC_{\bar{d}_1}] \;=\; (a-1)a^{v-3}(a+3) \tag{3.10}$$

$$spectrum[kC_{\tilde{d}_1}] \;=\; a^{v-1}. \tag{3.11}$$

Morgan and Uddin (1995) proved that $\tilde{d}_1$ is E-optimal in $\mathcal{D}_{b,k,v}$. Here we show that

Theorem 3.3 *$\bar{d}_1$ is E-optimal in $\mathcal{D}_{b,k,v} \setminus \{\tilde{d}_1\}$.*

Before going to the proof, we present a notation.

Notation 3.1 *(i) d will denote a competing design.*

(ii) $A = kC_d$, $\mu = \mu(d)$.

(iii) While using Theorem 2.7, the vectors $x_{i,j}$ are meant to be the normalised versions of those presented. For instance, $x_{1,1}$ of Lemma 4.1 is actually $(1/\sqrt{(6)})(2,-1,-1)^T$.

Proof of Theorem 3.3: Consider a design d in $\mathcal{D}_{b,k,v}$ $\{\tilde{d}\}$. Put $A = kC_d$ and $\mu = \mu(d)$.

Case 1 $a_{ii} < r(k-1)$ for some i, say $i = 1$.

In this case, $a_{11} \leq r(k-1) - 2$ and so applying (a) of Theorem 2.7 with $B = A(1)$, we get $\mu_1^{\uparrow} < a - 1$.

Case 2 $a_{ii} = r(k-1)$ for every i.

Since $d \neq \tilde{d}_1, \exists (i,j)$, such that $\lambda_{i,j} \leq \lambda - 1$. So, taking $B = A(i,j)$ and applying (b) of Theorem 2.7 we get $\mu_1^{\uparrow} \leq a - 1$ and the proof is complete. $\square$

In fact $\bar{d}_1$ satisfies stronger optimality as shown below.

THEOREM 3.4 *$\bar{d}_1$ is*

(a) Type 1 optimal in $\mathcal{D}_{b,k,v}^B$ and

(b) M-better than every non-binary design in $\mathcal{D}_{b,k,v}$ other than $\tilde{d}_1$.

Thus, we have

COROLLARY 3.1 *$\bar{d}_1$ is Type 1 optimal in $\mathcal{D}_{b,k,v} \setminus \{\tilde{d}_1\}$.*

Proof of Theorem 3.4 : (a) follows from the fact that $\bar{d}_1$ satisfies all the conditions of Theorem 3.1. So, we prove (b). Let d be an arbitrily fixed non-binary design other than $\tilde{d}_1$. By (b) of Theorem 2.4 and Theorem 3.3 it is enough to show that

$$\mu_1^{\downarrow} \geq a. \tag{3.12}$$

Let $u = max\{a_{ii}, 1 \leq i \leq v\}$.

Case 1 : $u > r(k-1)$.

In this case, $\exists i$ such that $a_{ii} \geq r(k-1) + 2$, so that $\mu_1^{\downarrow} \geq a + 2$. Hence we are done.

Case 2 : $u \leq r(k-1)$.

Case 2.1 : $a_{ii} = r(k-1)$ for all i.

Since $d \neq \tilde{d}, \exists (i,j)$, such that $\lambda_{i,j} \geq \lambda + 1$. So, taking $B = A(i,j)$ and applying (b) of Theorem 2.7 we get $\mu_1^{\downarrow} \geq a + 1$.

Case 2.2 : $a_{ii} < r(k-1)$ for at least one i.

Let m be the number of i's such that $kC_{dii} < r(k-1)$, i.e, $kC_{dii} \leq r(k-1) - 2$. Then,

$$\sum_{i=1}^{v-1} \mu_i \leq vr(k-1) - 2m = (v-1)a - 2m. \tag{3.13}$$

Case 2.2.a : $m = v$.

In this case $\mu_i \leq a - 2 \forall i$ and we are done.

Case 2.2.b : $m < v$.

In this case, $\exists i$, such that $a_{ii} = r(k-1)$. Thus, $\mu_1^{\downarrow} \geq a$. $\square$.

We now consider another set up satisfying (3.7) and having $bk = vr + 2$. This was first considered in Roy and Shah (1984) who provided the first example of a Type 1 optimal unequally replicated design, referred to as $\bar{d}_2$ here. $\bar{d}_2$ is an NBBD(2), according to the definition of MS. The set up of Roy and Shah (1984) is of the following nature.

$$v \equiv 5\ (mod\ 6), r = (v-1)/2, \lambda = 1. \tag{3.14}$$

In MS, a very similar set up is considered. This has

$$v \equiv 2\ (mod\ 3),\ r = 2(v-1),\ \lambda = 2. \tag{3.15}$$

MS constructed an NBBD(2) having the form of C-matrix as well as its spectrum similar to that of $\bar{d}_2$ and proved the same optimality property. They also found a non-binary design, termed $\tilde{d}_2$ here, which do not seem to satisfy any optimality property like $\tilde{d}_1$. It is not known whether a design corresponding to $\tilde{d}_2$ exists in the set up (3.14). The spectrums of these are as follows.

$$spectrum[\bar{d}_2] = a^{v-2}(a+4) \tag{3.16}$$
$$spectrum[\tilde{d}_2] = a^{v-1}. \tag{3.17}$$

Here a is as in (3.9). Looking at the spectrums, the following result is clear.

THEOREM 3.5 *$\bar{d}_2$ is M-better than $\tilde{d}_2$.*

In MS, the optimality property $\bar{d}_2$ has been derived from general lemmas. However, if we restrict to this particular set up and also use majorisation techniques, then the proofs becomes considerably simpler and transparent. This is what is done below. Henceforth, $\bar{d}_2$ would refer to both the designs of MS and Roy and Shah (1984)). We shall also refer to $\tilde{d}_2$, which may be a hypothetical design in the set up (3.14).

Let us first state an well-known result.

LEMMA 3.1 *Suppose $x_i, 1 \leq i \leq n$ are integers satisfying $\sum_{i=1}^n x_i = a$. Let u be the greatest integer $\leq a/n$ and $g = a - nu$. Then $\sum_{i=1}^n (x_i)^2$ is minimum if x_i's are "as nearly equal as possible". More precisely,*

(a)

$$\sum_{i=1}^n (x_i)^2 \geq (n-g)a^2 + g(a+1)^2 = m(x) say.$$

(b)Further, in case $\sum_{i=1}^n (x_i)^2 > m(x)$, then it is $\geq m(x) + 2$.

Now we present a proof of the crucial property of $\bar{d}_2$.

LEMMA 3.2 *$\bar{d}_2$ minimises $tr[(C_d)^2]$ over $d \in \mathcal{D}^B_{b,k,v}$*

Proof : Fix an arbitrary design $d \in \mathcal{D}^B_{b,k,v}$. Now, $[tr(C_d)^2] = \Sigma(C_{dij})^2 = T_1(d) + 2T_2(d)$, where

$$T_1(d) = (k-1)^2[\sum_{i=1}^{v}(r_i)^2] \text{ and } T_2(d) = \sum_{i<j}(\lambda_{ij})^2, \qquad (3.18)$$

as $\lambda_{ij} = \lambda_{ji}$.

Since $\sum\limits_{i<j} \lambda_{ij} = (1/2)v(v-1)\lambda + 2$, applying Lemma 3.1 on λ_{ij}'s, we find that $m(\lambda) = (1/2)v(v-1)(\lambda)^2 + 4\lambda + 2$ and $T_2(\bar{d}_2) = m(\lambda) + 2$.

From this and the expression for $T_1(d)$, it is clear that if the replication vector of d is different from that of $\bar{d}_2$, then $tr[(C_d)^2] > tr[(C_{\bar{d}_2})^2]$. Hence we assume

$$r_i = r, 1 \leq i \leq v-2,\ r_{v-1} = r_v = r+1.$$

In view of (b) of Lemma 3.1, all we have to show is $T_2(d) > m(\lambda)$. But to show that it is enough to show the following.

Claim : The expression for $T_2(d)$ always contain at least a term $(\lambda - 1)^2$.

Proof of the claim : Recall that

$$\sum_{j \neq i} \lambda_{ij} = r_i(k-1) \qquad (3.19)$$

So, $\sum\limits_{j \neq v} \lambda_{ij} = (v-1)\lambda + 2, i = v$ and$v-1$. So, $\sum\limits_{j \neq v} (\lambda_{vj})^2$ is minimum if $\lambda_{vm} = \lambda_{vl} = \lambda + 1$ for some $m, l < v$ and for all other j's, $\lambda_{vj} = \lambda$. Clearly, one of m, l has to be $\leq v-2$. W.l.g., let $l = 1$. Then, $\lambda_{1v} = \lambda + 1$, so that $\exists j$ such that $\lambda_{1j} = \lambda - 1$. This completes the proof of the claim and hence the proof of the lemma. □

A direct consequence of the preceeding lemma, in view of Theorem 3.2, is the following.

Corollary 3.2 *$\bar{d}_2$ is type 1 optimal in $d \in \mathcal{D}^B_{b,k,v}$*

We shal now consider the general class and prove :

Theorem 3.6 *$\bar{d}_2$ is type 1 optimal in $d \in \mathcal{D}_{b,k,v}$*

Proof : Fix an arbitrary non-binary design $d \in \mathcal{D}_{b,k,v}$. As usual, we put $A = kC_d$ and $\mu = \mu(d)$. In view of Corollary 3.2, is enough to show that $\bar{d}_2$ is M-better than d. As before, Let $A = kC_d$ and $\mu = \mu(d)$.

Case 1 : d has at least two non-binary blocks.

In this case, $tr[C_d] \leq tr[C_{\tilde{d}_2}]$ [for the description of $\tilde{d}_2$ see section 4 of MS]. Since $C_{\tilde{d}_2}$ is completely symmetric, d is M-worse than $\tilde{d}_2$ and hence M-worse than $\bar{d}_2$ by Theorem 3.5.

Case 2 : d has exactly one non-binary block.

Since $k = 3$, there is at most one non-binary treatment in the non-binary block. Thus, not both of the treatments v and $v-1$ can be non-binary. W.l.g., let us suppose that v is binary. Applying (a) of Theorem 2.7 with $B = kC_d(v)$, we get $\mu_1^{\downarrow} \geq v(r+1)(k-1)/(v-1) \geq a+2$. Therefore, $\sum_{i=1}^{v-2} \mu_i^{\uparrow} \leq (v-2)a$. Hence the result follows from (b) of Theorem 2.4. □.

4 A new Optimality Result.

We consider a set up where $k = 3$ and $bk - 1$ is divisible by v. Thus, $bk = vr - 1$. [See notation 2.3]. We further assume $r = (v-1)/2$, so that $\lambda = 1$. Thus, the parameters are of the folllowing form. For an integer $s \geq 1$,

$$b = 6s^2 + 9s + 3, v = 6s + 5, r = 3s + 2, k = 3. \qquad (4.20)$$

Let d^* denote the design with the following parameters. $r_1 = r - 1,\ r_i = r,\ 2 \leq i \leq v;\lambda_{1,2} = \lambda - 1 = \lambda_{1,3},\ \lambda_{2,3} = \lambda + 1$.

Let a be as in (3.9).

Lemma 4.1 *The spectrum of kC_{d^*} is as follows.*

$$\mu(d^*) = (a-3)^1 a^{v-3} (a+1)^1 \qquad (4.21)$$

Proof : By straightforward verification.

Remark : It is easy to verify that d^* satisfies conditions (i) and (iii) of Theorem 3.1. But it appears that it does not satisfy condition (ii), although we have not yet found a design E-better than d^*. Because of this, Theorem 3.1 could not be applied and general optimality of d^* could not be proved. We believe, d^* is also D-optimal, but the proof would be more involved.

We now present our main result.

Theorem 4.1 *d^* is A-optimal in $\mathcal{D}_{b,k,v}$ with b, k, v as in (4.20) provided* $a \geq 16$.

We prove this in two steps. First we show that

Theorem 4.2 *d^* is A-better than any non-binary design in $\mathcal{D}_{b,k,v}$, whenever* $a \geq 16$.

Next we prove

Theorem 4.3 *d^* is A-optimal in $\mathcal{D}^B_{b,k,v}$ if the parameters satisfy* $a \geq 16$.

Proof of Theorem 4.2 Consider a non-binary design $d \in \mathcal{D}_{b,k,v}$. Let

$$\delta = \max_{1 \le i \le v} a_{ii}.$$

Claim : If $\delta < r(k-1)$, then d is M-worse than d^*.

Proof of the claim : Suppose the hypothesis is true. Then, $a_{ii} < r(k-1) - 2$, for each i. This implies the following statements.

(a) $tr(A) < (v-1)(a-2)$ and

(b) $\exists(i,j)$ such that $\lambda_{ij} \le \lambda - 1$.

Now, (a) implies

$$\sum_{i=1}^{v-2} \mu_i^{\uparrow} \le (v-2)(a-2). \qquad (4.22)$$

Further, in view of (b), applying (a) of Theorem 2.7 with $B = A(i,j)$, we get $\mu_1^{\uparrow} \le a - 3$. This, together with (4.22) and (c) of Theorem 2.4 proves the claim.

So, we assume $\delta \ge r(k-1)$. This means $a_{ii} \ge r(k-1) + 2$, for some i. Thus, by Corollary 2.1, $\mu_1^{\downarrow} \ge a$. Again, as $r_1 \le r-1, \mu_1^{\uparrow} < a-2$, by the same corollary. These, in view of Theorem 2.8 implies that μ is M-worse than $v10$ which is A-worse than $\mu(d^*)$ by Lemma A.10. Hence, the proof is complete. □

Proof of Theorem 4.3

We first state a trivial but very useful result without proof.

Lemma 4.2 *Consider a design d. Fix a treatment i.*

(a) If $r_i < r$, then either $\lambda_{ij} \le \lambda - 2$ for some $j \ne i$, or there exist j_1, j_2, such that $\lambda_{i,j_u} \le \lambda - 1, u = 1, 2$.

(b) If $r_i = r$, and $\lambda_{i,j} >$ (resply $<$) λ for some j, then there exists l such that $\lambda_{i,l} <$ (resply) $> \lambda$.

We fix an arbitrary design $d \in \mathcal{D}^B_{b,k,v}$ with b, k, v as in (4.20) and show that it is A-worse than d^* under the condition of our Theorem.

LEMMA 4.3 *For any d, $\mu_1^{\downarrow} \geq a + 1$.*

Proof: W.l.g., let us assume that the replication numbers of d are in the increasing order.

Case 1 : The replication vector of d is different from that of d^*.

Then, $r_v > r$, that is $r_v \geq r + 1$. Now applying Corollary 2.1 with $i = v$, we get $\mu_1^{\downarrow} \geq a + 2$.

Case 2. : (Remaining case). The replication vector of d is same as that of d^*.

By (a) of Lemma 4.2, $\lambda_{1,j} < \lambda$ for some j. W.l.g., let $j = 2$. Then, by (b) of Lemma 4.2, there exist l, such that $\lambda_{2,l} \geq \lambda + 1$. Now we apply (a) of Theorem 2.7 with B_1 as the $a(2, l)$ and get the required result.

COROLLARY 4.1 *If $\mu_1^{\uparrow} \leq a - 3$, then d is M-worse than d^*.*

Proof : By (c) of Theorem 2.4.

THEOREM 4.4 *If the replication vector of d is not same as that of d^* then d is A-worse than d^*.*

Proof : It is enough to show that if $r_1 + r_2 \leq 2r - 2$, then d is A-worse than d^*.

Case 1. $r_1 \leq r - 2$.

In this case, applying Corollary 2.1 with $i = 1$, we get $\mu_1^{\uparrow} \leq a - 4$ and so the result follows from Corollary 4.1.

Case 2. $r_1 = r_2 = r - 1$. Clearly, $r_v \geq r + 1$.

First we take $B_1 = A(1, 2)$. Applying (a) of Theorem 2.7 if $\lambda_{1,2} \leq \lambda - 1$ and (a) of the same theorem if $\lambda_{1,2} \geq \lambda + 1$, we get $\mu_1 \leq a - 3$ and we are done by Corollary 4.1. Hence, we assume $\lambda_{1,2} = \lambda$.

Now, we take $i = v$ and apply Corollary 2.1. We get

$$\mu_1^{\downarrow} \geq a + 2. \qquad (4.23)$$

Again, by Lemma 4.2, there exist $j \neq 1, 2$, $l \neq 1, 2$ such that $\lambda_{1,j} \leq \lambda - 1$ and $\lambda_{2,l} \leq \lambda - 1$. We choose $B_1 = A(1, j)$, $B_2 = A(2, l)$. Now applying Theorem 2.7 (a) and using (4.23) we find that

$$(a-2)^2a^{v-4}(a+2) \prec^w \mu(d).$$

Now the result follows from Lemma **??** of Appendix. □

In view of the preceeding theorem, henceforth we assume that d has the same replication vector as d^*.

Next we obtain bounds on $\lambda_{i,j}$'s.

LEMMA 4.4 *If one of the following conditions hold then d is M-worse than d^*.*

(a) $\mid \lambda_{i,j} - \lambda \mid \geq 3$, for some (i,j), $i,j > 1$.

(b) $\mid \lambda_{1,j} - \lambda \mid \geq 2$, for some $j > 1$.

(c) $\lambda_{1,j} = \lambda_{1,l} = \lambda - 1$, $\lambda_{l,j} > \lambda$, for some $l,j > 1$.

Proof : In view of Corollary 4.1, it is enough to show that $\mu_1^{\uparrow} \leq a-3$.

Suppose condition (a) holds. Taking B_1 (respectively B_0) $= A(i,j)$ if $\lambda_{i,j} <$ (respectively $>$)λ and applying (a) (respectively (b)) of Theorem 2.7, we get the required condition.

Now suppose condition (b) holds. Recall that $r_1 = r-1$, so that $a_{1,1} = r(k-1)-2$. Proceeding on the line as above with $A(1,j)$ instead of $A(i,j)$, we get the result.

Finally, assume condition (c). We take $B_1 = A(1,j,l), x_{1,1} = (2,-1,-1)^T$. Now, applying (a) of Theorem 2.7 we get the required result. □

In view of our findings above and Lemma 4.2, we can assume w.l.g., that

$$\lambda_{1,2} = \lambda_{1,3} = \lambda - 1,\ \lambda_{2,3} \leq \lambda. \tag{4.24}$$

W.l.g. let $j=2,\ l=3$. Then,

LEMMA 4.5 *If $\mid \lambda_{i,j} - \lambda \mid \geq 2$, for some (i,j), $i,j > 1$, then d is A-worse than d^*.*

Proof : We shall show that if the condition holds then, $\mu(d)$ is M-worse than $v1$. Then the result would follow from Lemma A.1. Now, suppose the condition holds. Then, by Lemma 4.4, $\lambda_{(i,j)} = \lambda - 2$ or $\lambda + 2$.

Case 1 : $\lambda_{i,j} = \lambda - 2$.

Applying (b) of Theorem 2.7 on $C = A(i,j)$ we get

$$\mu_1^{\downarrow} \geq a+2. \tag{4.25}$$

Clearly, it is enough to show

$$\sum_{j=1}^{l} \mu_j \leq l(a-2), l = 1,2. \tag{4.26}$$

Case 1.1 : $\{i,j\} = \{2,3\}$.

Then we choose $B_1 = A(1,2,3)$, $x_{1,1} = (2,-1,-1)^T$ and $x_{1,2} = (0,1,-1)^T$. Applying (a) of Theorem 2.7 we get (4.26)

Case 1.2 : $\{i,j\}$ and $\{2,3\}$ are disjoint. Then take l to be anyone in $\{2,3\}$.

Case 1.3 : $\{i,j\}$ and $\{2,3\}$ has one element in common. Take l to be the element of $\{2,3\}$ which is not in $\{i,j\}$.

In both Case 2 and Case 3 we take $B_1 = A(i,j)$ and $B_2 = A(1.l)$. Clearly, B_1 and B_2 are disjoint. Now taking $x_{1,1} = x_{2,1} = (1,-1)^T$ and applying (a) of Theorem 2.7 on B_1, B_2, we find that (4.26) holds in these cases also. So, the proof for Case 1 is complete.

Case 2 : $\lambda_{i,j} = \lambda + 2$. If $\{i,j\} = \{2,3\}$, then we are done by Lemma 4.4. So, assume $\{i,j\} \neq \{2,3\}$.

We take cases 2.2 and 2.3 exactly like 1.2 and 1.3 respectively and chose l as there. Let m be the other element of $\{2,3\}$. Taking $B_0 = A(i,j), B_1 = A(1,l), x_{1,1} = (1,-1)^T$ and applying (b) of Theorem 2.7 on B_0, B_1, we find that (4.26) holds. Again, (a) of Theorem 2.7 on $B_2 = A(i,j)$ yields (4.25). Hence we are done in this case also. □

In view of the above, we assume the following.

$$\lambda_{i,j} \in \{\lambda, \lambda-1, \lambda+1\}. \qquad (4.27)$$

Using this, we are able to extend Lemma 4.2 as follows.

LEMMA 4.6 *Fix $i \geq 2$. Let $S = \{j \neq i : \lambda_{i,j} = \lambda - 1\}$ and $T = \{j \neq i : \lambda_{i,j} = \lambda + 1\}$. Then the sizes of S and T are equal.*

Proof is trivial.

For $\lambda_{1,i}$'s, we can say more as shown below.

LEMMA 4.7 *Let $S = \{i : \lambda_{1,i} = \lambda - 1\}$. If the size of S is ≥ 3 then d is A-worse than d^*.*

Proof : Suppose $\mid S \mid \geq 3$. W.l.g., let $S = \{2,3,4\cdots\}$. If $\lambda_{i,j} = \lambda + 1$, for some $i,j \in S$ then we are done by Lemma 4.4. So, assume $\lambda_{i,j} \leq \lambda$. Further, replace S by its subset $= \{2,3,4\}$. Let s be the number of pair (i,j) such that $\lambda_{i,j} = \lambda - 1$. Then, $s = 0,1,2,$ or 3. We take $B_1 = A(1,2,3,4)$ and apply Theorem 2.6. We get

$$\mu_1^{\downarrow} > a + 1 + s/2. \qquad (4.28)$$

Now, we consider the different values of s. In each case we apply (a) of Theorem 2.7 with B_1 and $u_1 = 1$ or 2 mutually orthogonal vectors among which $x_1 = (3,-1,-1,-1)^T$ is one. Thus, for every s,

$$\mu_1^{\uparrow} \leq a - 3 + s/6.$$

So, in view of Corollary 4.1, if $s = 0$ then we are done. If $s = 3$, then the inequality above together with (4.28) yields that $\mu(d)$ is M-worse than $v3$.

Now we consider $s = 1$ and 2. We assume, w.l.g., that $\lambda_{3,4} = \lambda - 1$ when $s = 1$ and $\lambda_{2,3} = \lambda_{2,4} = \lambda - 1$ when $s = 2$. Let $x_2 = (0, 2, -1, -1)^T$, $x_3 = (0, 0, 1, -1)^T$. Now we apply Theorem 2.7, with the vectors x_1, x_3 if $s = 1$ and x_1, x_2 if $s = 2$. We find that $\mu(d)$ is M-worse than $v4$ if $s = 1$ and $v1$ if $s = 2$. So, by lemmas A1, A3 and A4 our proof is complete. $\square$

In view of the lemma above, we assume the following.

$$\lambda_{1,2} = \lambda_{1,3} = \lambda - 1;\ \lambda_{1,j} = \lambda, j \geq 4;\ \lambda_{2,3} = \lambda \text{or } \lambda - 1 \qquad (4.29)$$

THEOREM 4.5 *If $\lambda_{2,3} = \lambda$ then d is A-worse than d^*.*

Proof : By Lemma 4.2, $\exists j_1, j_2$ such that $\lambda_{2,j_1} = \lambda_{3,j_2} = \lambda + 1$.

Case 1: $j_1 = j_2$. W.l.g., let $j_1 = 4$.

By Lemma 4.2, $\exists j_3, j_4$ such that $\lambda_{4,j_3} = \lambda_{4,j_4} = \lambda - 1$. W.l.g., let $j_3 = 5,\ j_4 = 6$. But then $\exists j$ such that $\lambda_{5,j} = \lambda + 1$.

Case 1a: $j = 2$. Take $B_1 = A(1, 2, 3), B_2 = A(4, 5)$ and $B_3 = A(2, 4, 5)$. Further, we take $x_{1,1} = (2, -1, -1)^T$, $x_{2,1} = (1, -1)^T$ and $x_{3,1} = (2, -1, -1)^T$. Now, applying (a) of Theorem 2.7 with B_1, B_2 and (b) of the same theorem with B_3, we get the following equations.

$$\sum_{i=1}^{l} \mu_i^{\uparrow} \leq \sum_{i=1}^{l} z_l,\ l = 1, 2 \qquad (4.30)$$

$$\mu_1^{\downarrow} \geq w. \qquad (4.31)$$

Here $z_1 = a - 8/3, z_2 = a - 1, w = a + 5/3$. It follows that $v2 \prec^w \mu(d)$. Now the result follows from Lemma A2. The case $j = 3$ can be handled in the same way.

Case 1b: $j > 5$. Take $B_0 = A(5, j), B_1 = A(1, 2, 3, 4)$; $x_{1,1} = (5, -2, -2, -1)^T$ $B_2 = A(2, 3, 4), x_{2,1} = (1, 1, -2)^T$. Applying (a) of Theorem 2.7 on B_0, B_1 and (b) of the same theorem on B_2, we get similar equations like (4.30) and (4.31) with $z_1 = a - 23/8, z_2 = a - 1, w = a + 4/3$.

It follows that $v8 \prec^w \mu(d)$ and the result follows from Lemma A8. Thus, Case 1 is settled.

Case 2: $j_1 \neq j_2$. W.l.g., we assume $j_1 = 4,\ j_2 = 5$.

If $\lambda_{3,4} = \lambda + 1$, then we are reduced to Case 1. So, $\lambda_{3,4} = \lambda$ or $\lambda - 1$. Similarly $\lambda_{2,5} = \lambda$ or $\lambda - 1$.

Case 2a: At least one of $\lambda_{2,5}$ and $\lambda_{3,4}$ is $\lambda - 1$. W.l.g., Let $\lambda_{3,4} = \lambda - 1$.

Case 2a.1: $\lambda_{4,5} = \lambda + 1$.

We take $B_0 = A(4,5)$, B_1 and $x_{1,1}$ as in case 1(a), $B_2 = A(5,3,4), x_{2,1} = x_{3,1}$ of case 1(a). Now applying (a) of Theorem 2.7 on B_0, B_1 and (b) of the same theorem on B_2, we get the same equations as in Case 1a. Thus this case is settled.

Case 2a.2: $\lambda_{4,5} = \lambda$ or $\lambda - 1$.

Taking $B_1 = A(1,2,3,4,5), x_{1,1} = (2,-1,-1,0,0)^T, x_{1,2} = (0,1,-1,1,-1)^T$; $B_3 = A(2,4), B_4 = A(3,5)$ and applying (a) of Theorem 2.7 on B_1, B_2 we get (4.30) with $z_1 = a - 8/3, z_2 \leq a - 3/2$. Again applying (b) of the same theorem on B_3, B_4 we see that

$$\sum_{i=1}^{m} \mu_i^{\downarrow} \geq \sum_{i=1}^{m} w_i, \ m = 1,2, \tag{4.32}$$

where $w_1 = w_2 = a + 1$. Thus, $v5 \prec^w \mu(d)$. Now Lemma A5 settles this case.

Case 2b:$\lambda_{2,5} = \lambda_{3,4} = \lambda$

Case 2b.1: $\lambda_{4,5} = \lambda$. We take B_3, B_4 as in case 2a.2. so that (4.32) holds. We also take $B_1, x_{1,1}$ as in case 1a.

Now, by Lemma 4.6, $\exists j, l$ such that $\lambda_{4,j} = \lambda - 1$ and $\lambda_{5,l} = \lambda - 1$.

If $j = l$, we take $B_2 = A(4,5,j), x_{2,1} = (1,1,-2)^T$. If $j \neq l$, we take $B_2 = A(4,j), B_5 = A(5,l)$. We apply (a) of Theorem 2.7 on B_1, B_2 in the former case and B_1, B_2, B_5 in the later case. We obtain (4.30) with the same z_1. In the former case we have the same range for l and $z_2 = a - 4/3$. In the later, $l = 1,2,3$; $z_2 = z_3 = a - 1$. These, together with (4.32) says (in view of Theorem 2.8) that $v5$ (resply) $v6 \prec^w \mu(d)$ in the former (resply later) case and we are done by Lemmas A5 and A6 respectively.

Case 2b.2: $\lambda_{4,5} = \lambda + 1$.

By Lemma 4.2, $\exists j > 5$ such that $\lambda_{4,j} = \lambda - 1$. We take $B_1 = A(1,2,3,4,5), u_1 = 1, x_{1,1} = (8,-3,-3,-1,-1)^T$; $B_2 = A(4,j), u_2 = 1, x_{2,1} = (1,-1)^T$; $B_3 = A(2,3,4,5), x_{3,1} = (1,-1,-1,1)^T$. Now applying (a) of Theorem 2.7 on B_1, B_2 and (b) of the same theorem on B_3 we get equations similar to (4.30) and (4.31) with $z_1 = a - 17/6, z_2 = a - 1, w = a + 3/2$.

This implies $v4 \prec^w \mu(d)$ which completes the proof of this case by Lemma A4.

Case 2b.3: $\lambda_{4,5} = \lambda + 1$.

Take $B_1, x_{1,1}$ as before, $B_2 = A(4,5)$ and $B_3 = A(1,2,3,4,5), x_{3,1} = (2,2,2,-3,-3)^T$. Now applying (a) of Theorem 2.7 on B_1, B_2 and (b) of the same theorem on B_3 we get the same inequalities on μ's as in Case 1a and this completes the proof of this theorem. □

Now we consider the remaining possibility in the next and last theorem.

THEOREM 4.6 *If $\lambda_{2,3} = \lambda - 1$ then d is A-worse than d^*.*

Proof : By Lemma 4.6, $\exists j_1, j_2, j_3, j_4$ such that $\lambda_{2,j_1} = \lambda_{2,j_2} = \lambda_{3,j_3} = \lambda_{3,j_4}\lambda + 1$.

Case 1: One element is common between $\{j_1, j_2\}$ and $\{j_3, j_4\}$. W.l.g., let $j_1 = j_3 = 4$.

Take $B_1 = A(1,2,3,4), x_{1,1} = (3,-1,-1,-1)^T, x_{1,2} = (0,1,-1,0)^T$ and $B_2 = A(2,3,4), x_{2,1} = (1,1,-2)^T$. Applying (a) of Theorem 2.7 on B_1 and (b) of the same theorem on B_2 we get the same inequalities on μ as in Case 1a of Theorem 4.5 and hence we are done.

Case 2: No element is common between $\{j_1, j_2\}$ and $\{j_3, j_4\}$. W.l.g., we assume $j_1 = 4, j_2 = 5, j_3 = 6, j = 4 = 7$.

For $i \in \{4,5\}, j \in \{6,7\}$, $\lambda_{i,j}$ can not be $\lambda + 1$, as that would be case 1. So $\lambda_{i,j} = \lambda$ or $\lambda - 1$.

Case 2.1: $\lambda_{3,4} = \lambda - 1$.

By Lemma 4.6, $\exists l$ such that $\lambda_{6,l} = \lambda - 1$.

Case $l = 2$ (or 3).

Take $B_1 = A(1,2,4,6), x_{1,1} = (2,-1,-1,0,0)^T, x_{1,2} = (0,1,-1,1,-1)^T; B_2 = A(2,3,4,6), x_{2,1} = (1,-1,-1,1)^T$. Now, applying (a) of Theorem 2.7 on B_1 and (b) of the same theorem on B_2 we find that $v1 \prec^w \mu(d)$ and thus we are done by Lemma A1.

Case $l > 6$ Taking $B_1 = A(1,2,3,4), x_{1,1} = (3,-1,-1,-1)^T, x_{1,2} = (0,1,-2,1)^T;\ B_2 = A(6,l), x_{2,1} = (1,-1)^T$. Now, applying (a) of Theorem 2.7 on B_1, B_2 we get equations similar to (4.30) with $l \le 3, z_1 = a - 7/3, z_2 = a - 5/3, z_3 = a - 1$.

Next we take $B_3 = A(2,4,5)$ and $B_4 = (3,6,7)$. In view of Corollary A11 we can assume $\lambda_{4,5} = \lambda_{6,7} = \lambda$. Then, applying Theorem 2.7 on $B_3, x_{3,1} = (2,-1,-1)^T; B_4, x_{4,1} = x_{3,1}$ we get an equation same as (4.32) with $w_1 = w_2 = a + 4/3$. Thus, $v7 \prec^w \mu(d)$. Hence we are done by Lemma A7.

Case 2.2: $\lambda_{2,i} = \lambda_{3,j} = \lambda; i = 6,7,\ j = 4,5$.

Take $B_1 = A(1,2,3,4,6), x_{1,1} = (4,-1,-1,-1,-1)^T$.

Case 2.2a: $\lambda_{4,6} = \lambda + 1$.

By Lemma 4.6 there is a j, such that $\lambda_{4,j} = \lambda - 1$. Take $x_{1,2} = (0,2,-2,1,-1)^T;\ B_2 = A(4,j), x_{2,1} = (1,-1)^T$. Now applying (a) Theorem 2.7 on B_1, B_2, we get equations similar to (4.30) with $l \le 3, z_1 = a - 13/5, z_2 = a - 7/5, z_3 = a - 1$. We keep the same B_3, B_4 and hence obtain the same equation as in Case 2.1 Now, using the fact that $(a - 13/5, a - 7/5) \prec^w (a - 7/3, a - 5/3)$ we note that $v7 \prec^w \mu(d)$ and Lemma A7 settles this case.

Case 2.2b: $\lambda_{4,6} = \lambda$.

We proceed exacly like case 2.1a, except that we take $x_{1,2} = (0,1,-1,1,-1)^T$. We get an equation similar to (4.30) with $l \le 3, z_1 = a - 5/2, z_2 = a - 3/2, z_3 = a - 1$ and an equation similar to (4.32) with w_1, w_2 as in Case 2.1. Since in this case also $(z_1, z_2) \prec^w (a - 7/3, a - 5/3)$, the same argument goes through.

Case 2.2c: $\lambda_{4,6} = \lambda - 1$.

We take $B_1, x_{1,1}, x_{1,2}$ as in Case 2.1b. But we take $B_2 = A(2,3,4,6)$. Now applying

(a) of Theorem 2.7 on B_1 and (b) of the same theorem on B_2, we find that $v1 \prec^w \mu(d)$. Hence this case is also settled. $\Box$.

5 Construction

In this section we present an infinite series of designs d^*, [see the statement following (4.20)] with $\lambda = 1$.

NOTATION 5.1 *(a) p is an odd integer, say $p = 2s + 1$. P is the set of integers modulo p. $P^* = P\ 0$.*

(b) $I = \{0, 1, 2\}$.

(c) $V_0 = I \times P$.

(d) $V = V_0 \cup \{\alpha, \beta\}$ is the set of treatments.

We first construct a Steiner triple system $(\tilde{d})$ on the treatment set V_0 as follows.

Let $B_t = \{(0, t), (0, -t), (1, 0)\}, t \in P^*$ and $C = \{(0, 0), (1, 0), (2, 0)\}$. $\tilde{d}$ will consist of blocks of two types. The blocks of type 1 are generated by adding (i, j) to each member of each of B_t and those of type two by adding $(0, j)$ to C. Here the addition is $mod3$ for the first and $modp$ to the second co-ordinate. Thus, altogether, there are $\tilde{b} = 3sp + p = 6s^2 + 5s + 1$ blocks.

THEOREM 5.1 *d^* with $\lambda = 1$ exists for all s, whenever P^* defined above can be partitioned into two sets Q, R such that $\mid Q \mid = \mid R \mid = s$ and for every $u \in Q$, $2u \in R$.*

Proof : Let us assume the condition on P. We construct d^* from $\tilde{d}$ as follows. We replace (1) all the p blocks of type 2 by a set S_1 of $2p$ blocks and (2) each one of a certain subset of the type 1 blocks by two blocks. All these new blocks contain either α or β. Finally, we add the block

$$\{(1, 0), (2, 0), \alpha\}.$$

.

We now describe the new blocks. For each $j \in P$, there are two blocks in S_1, namely

$$\{(0, j), (1, j), \alpha\}, \text{ and}\{(0, j), (2, j), \beta\}, \text{ if } j \in Q \cup \{0\},$$

$$\{(0, j), (2, j), \alpha\}, \text{ and } \{(0, j), (1, j), \beta\},\ j \in R\}.$$

These gives $2p = 4s + 2$ blocks.

Now, let us consider the following set of $p - 1 (= 2s)$ blocks of type 1 of $\tilde{d}$.

$$L_t = \{(1, 0), (1, 2t), (2, t)\}, t \in P^*.$$

We replace L_t by two blocks, say D_t and E_t, which are as follows.

$$\begin{aligned} D_t &= \{(1,0),(1,t),(2,t)\}, t \in P^*. \\ E_t &= \{(1,2t),(2,t),\alpha\}, \text{ if } t \in Q, \\ E_t &= \{(1,2t),(2,t),\beta\}, \text{ if } t \in R. \end{aligned}$$

This gives $2(p-1) = 4s$ blocks. Clearly, the total number of blocks is $\tilde{b} - p + p - 1 + 2p + 1 = 6s^2 + 9s + 3$.

It is easy to verify that

(i) every pair of treatments of V_0 appear together exactly once,

(ii) α occurs twice with $(1,0)$ once with all other treatments of V_0,

(iii) β does not occur with $(1,0)$ and if occurs with all other treatments of V_0 exactly once,

(iv) α, β do not occur together.

These completes the proof of our theorem. $\square$

Concluding Remark : So, far we have seen two series of designs with spectrum and $Tr(C_d)^2$ satisfying Jacroux's (1985) condition [See Theorem 3.1]. These are $\bar{d_1}$ of MS and d^* of this paper. It is interesting that the spectrum of one is "opposite" of the other.

$$spectrum[kC_{\bar{d_1}}] = (a-1)a^{v-3}(a+3) \quad (5.33)$$

$$spectrum[kC_{d^*} = (a-3)^1 a^{v-3}(a+1)^1 \quad (5.34)$$

$\bar{d_1}$ has a comparetively bigger value of its minimum eigenvalue and consequently satisfies general optimality of (within the binary class) while d^* does not seem to be likely to satisfy general optimality.

Now, does there exist a design with spectrum

$$(a-2)^1 a^(v-3)(a+2)?$$

If it does, then it is very likely to be type 1 optimal as the proofs of this paper indicate.

6 Appendix

We list a few vectors in R^{v-1}, and show that each of them is A-worse than $\mu(d^*)$ for sufficiently large a.

NOTATION 6.1 *(0)* $v0 = \mu(d^*) = (a-3)^1 a^{n-2}(a+1)$

(1) $v1 = (a-2)^2 a^{n-3}(a+2)$

(2) $v2 = (a-8/3)(a-1)a^{n-3}(a+5/3)$

(3) $v3 = (a-5/2)(a-1)^2 a^{n-4}(a+5/2)$

(4) $v4 = (a-17/6)(a-1)(a+t)^{n-3}(a+3/2)$

(5) $v5 = (a-8/3)(a-4/3)a^{n-4}(a+1)^2$

(6) $v6 = (a-8/3)(a-1)^2(a+t)^{n-5}(a+1)^2$, *where* $(n-5)t = 2/3$.

(7) $v7 = (a-7/3)(a-5/3)(a-1)(a+t)^{n-5}(a+4/3)^2$, *where* $(n-5)t = 1/3$. *(8)* $v8 = (a-23/8)(a-1)(a+t)^{n-3}(a+4/3)$, *where* $(n-3)t = 13/24$

(9) $v9 = (a-2)(a-h)^{n-2}a$, *where* $(n-2)h = g \geq 2$

NOTATION 6.2 *For two vectors* $x, y \in R^n$, *we define*

$$\psi = \sum_{i=1}^{n} x_i^{-1} - \sum_{i=1}^{n} y_i^{-1}.$$

Lemma A1 $v1$ is A-worse than $v0$ if $a \geq 12$.

Proof : Let $x = v2$, $y = v0$. So, we are to show that $psi > 0$ if $a \geq 12$. Put $D_1 = (a-2)(a-3)$, $D_2 = a(a-2)$, $D_3 = (a+1)(a+2)$. It follows that $\psi = D_2^{-1} - D_3^{-1} - (D_1^{-1} - D_2^{-1})$, which implies $\psi.D_1.D_2.D_3/(a-2) = 2a^2 - 22a - 8$ which is ≥ 0, if $a \geq 12$. □

Lemma A2 The vector $v2$ is A-worse than $v0$, whenever $a \geq 16$.

Proof: Let $x = v1$, $y = v0$.

Then, $\psi = 2(D_2^{-1} - D_3^{-1}) - (D_1^{-1} - D_2^{-1})$, where $D_1 = (a-8/3)(a-3)$, $D_2 = a(a-1)$ and $D_3 = (a+1)(a+5/3)$. Thus, $3\psi D_1.D_2.D_3 > (8/3)a^2(a-16)$, on simplification. Hence the result.□

Lemma A3 $v3$ is A-worse than $v0$ if $a \geq 12$.

Proof : We shall show that $v3$ is A-worse than $v1$ if $a \geq 1$, from which the result will follow in view of Lemmma A1.

Take $x = v5$ and $y = v1$. So, $\psi = T_1 - T_2 + T_3$, where $T_1 = D_1^{-1} - D_2^{-1}$, $T_2 = D_2^{-1} - D_3^{-1}$ and $T_3 = D_3^{-1} - D_4^{-1}$, with $D_1 = (a-2)(a-5/2)$, $D_2 = (a-1)(a-2)$, $D_3 = a(a-1)$, $D_4 = (a+2)(a+5/2)$. Calculation shows $T_1 - T_2 = (5-a/2)/(D_1.D_3)$, so that $2\psi.D_1.D_3 D_4 = (5-a/2)D_4 + ((11/2)a+5)D_1$, which is ≥ 0 if $a \geq 4$. □

Lemma A4 $v4$ is A-worse than $v0$ if $a \geq 16$.

Proof : By Lemma **??** it is enough to show that $v4$ is A-worse than $v1$ if $a \geq 5$. Take $x = v4$ and $y = v1$. Clearly, $6\psi = D_1^{-1} - D_2^{-1} - (D_2^{-1} - D_3^{-1})$, where $D_1 = (a-17/6)(a-8/3)$, $D_2 = a(a+t)$, $D_3 = (a+3/2)(a+5/3)$. Since D_i's are in increasing order, $\psi \geq 0$, whenever $D_2 - D_1 \geq D_3 - D_2$, which holds whenever $a \geq 5$. □

Lemma A5 $v5$ is A-worse than $v0$ if $a \geq 16$.

Proof : It is enough to show that $v5$ is A-worse than $v2$ if $a \geq 1$.

Take $x = v5$ and $y = v1$. Clearly, $\psi = D_1^{-1} - D_2^{-1} - 2(D_2^{-1} - D_3^{-1})$, where $D_1 = (a-1)(a-4/3)$, $D_2 = a(a+1)$, $D_3 = (a+1)(a+5/3)$. It is easy to check that $2\psi D_1.D_2.D_3 > 0$ for $a \geq 1$. □

Lemma A6 $v6$ is A-worse than $v0$ if $a \geq 16$.

Proof : We shall show that $v4$ is A-worse than $v2$ if $a \geq 5$. We take $x = v6$, $y = v2$. We find that $\psi = D_1^{-1} - D_3^{-1} - (2/3)(D_2^{-1} - D_4^{-1})$. Here $D_1 = a(a-1)$, $D_2 = a(a+t)$, $D_3 = a(a+1)$, $D_4 = (a+1)(a+5/3)$. Here t is as in (6) of Notation 6.1. So, it is enough to show that $D_3 - D_1 > (2/3)(D_4 - D_2)$ which holds whenever $a \geq 5$.□

Lemma A7 $v7$ is A-worse than $v0$ if $a \geq 12$.

Proof : We prove that $v7$ is A-worse than $v1$ if $a \geq 4$. We take $x = v7, y = v1$.

We get $\psi = T_1 + 3T_2 - T_3 - T_4$, where $T_1 = D_1^{-1} - D_2^{-1}$, $T_2 = D_3^{-1} - D_5^{-1}$, $T_3 = D_4^{-1} - D_6^{-1}$ and $T_4 = D_5^{-1} - D_6^{-1}$. Here $D_1 = (a-7/3)(a-2), /; D_2 = (a-5/3)(a-2), D_3 = a(a-1)$, $D_4 = a(a+t)$, $D_5 = a(a+4/3)$, $D_6 = (a+4/3)(a+2)$. Here t is as in (7) of Notation 6.1.

Clearly, it suffices to show that $3T_2 - T_3 - T_4 \geq 0$, which holds if $a \geq 4$.□

Lemma A8 $v8$ is A-worse than $v0$ if $a \geq 16$.

Proof : We prove that $v8$ is A-worse than $v2$ if $a \geq 2$. We take $x = v8, y = v2$.

We find $\psi = (5/24)D_1^{-1} - (13/24)D_2^{-1} + (1/3)D_3^{-1}$. Here $D_1 = (a-23/8)(a-8/3)$, $D_2 = a(a+t)$, $D_3 = (a+4/3)(a+5/3)$. It is enough to show that $5(D_2 - D_1) - 8(D_3 - D_2) \geq 0$ for $a \geq 2$, which can be verified by simple calculation.□

Lemma A9$v9$ is A-worse than $v0$ if $a \geq 16$.

Proof : We take $x = v9, y = v0$. Then, $\psi = -(D_1)^{-1} + g.(D_2)^{-1} + (D_3)^{-1}$. So, $\psi.D_1.D_2 > g.D_1 - D_2 > a[(g-1)a - (5g+1)] > 0$ as $a \leq 16$. □

Lemma A10 Consider the following vectors x, y of the same dimension and the same sum of entries.

$$\begin{aligned} y &= \prod_{i=1}^{m} u_i \prod_{j=1}^{n} w_j \ (a+4/3)(a+t_1)^{l+1} \\ x &= \prod_{i=1}^{m} u_i \prod_{j=1}^{n} w_j \ (a+1)^2(a+t_2),. \end{aligned}$$

Here $a > 0$ and

$$(l+1)t_1 + 4/3 = lt_2 + 2. \tag{6.35}$$

If $0 < t_1 < 1/6$, then x is A-worse than y.

Proof : Let $\psi = \sum_{i=1}^{k} x_i^{-1} - \sum_{i=1}^{k} y_i^{-1}$, where $k = m+n+l+2$. It is enough to show that $\psi \geq 0$. Now,

$$\psi = l(t_1 - t_2)/(D_1) - (1-t_1)/D_2 + (1/3)/D_3,$$

where $D_1 = (a+t_1)(a+t_2),\ D_2 = (a+1)(a+t_1),\ D_3 = (a+1)(a+4/3)$. Now note that (6.35) says that $t_2 < t_1$. So,

$$\psi.D_1.D_3 > (a+4/3)(2/3-t_1)(1-t_1) - (1/3)(a+t_1)(4/3-t_1).$$

Now simple calculation shows that under the given condition, the coefficient of a as well as the term independent of a in the above expresssion is > 0. Hence the result follows as the D_i's are > 0. □

Corollary A11 Consider a matrix A with constant diagonal element a. Suppose it has a principal submatrix B of order 3 as follows. $b_{1,2} = b_{1,3} = -1, b_{2,3} = t, t = -1, 0, \text{or} 1$. Then $\mu(A)$ is A-best if $t = 0$.

Prof : Let $x_1 = (2, -1, -1)^T$ and $x_2 = (0, 1, -1)^T$.

Now we apply Theorem 2.7 with B as one of the submatrices, the corresponding vector(s) being x_1 if $t = 0$ or 1 and x_1, x_2 if $t = -1$. It follows that $x \prec^w \mu(A)$ if $t = 0, 1$ and $y \prec^w \mu(A)$ if $t = -1$. Here x, y are the vectors of Lemma A10. Now the result follows from Lemma A10. □

7 References

1. Bagchi, B. and Bagchi, S. (2001). Optimality of partial geometric designs, Ann. Stat. vol. 29, pp : 577-594.

2. Bagchi, S. (1987). A class of non-binary unequally replicated E-optimal designs, Metrika 35, 1-12.

3. Bagchi, S. (1994). Optimality and construction of some rectangular designs, Metrika 41, 29-41.

4. Bhatia, R. (1997). Matrix Analysis, Springer.

5. Cheng, C.S. (1978). Optimality of certain asymmetrical experimental designs, Ann. Statist. 6, 1239-1261.

6. Cheng, C.S. (1987). An optimisation problem with application to optimal design theory, Ann. Statist. 15, 712-723.

7. Cheng, C.S. & Bailey, R.A. (1991), Optimality of some Two-Associate-Class Partially Balanced Incomplete Block Designs, Annals of Statistics, Vol. 19, No 3, 1667-1671.

8. Dey, A. (1986), Theory of Block designs, Wiley Eastern series.

9. Jacroux, M. (1980), On the E-optimality of regular graph designs, Jour. Roy. Stat. Soc. B, 42, 205-209.

10. Jacroux, M. (1985), Some sufficient conditions for the type I optimality of block designs, Jour. Stat. Plan. Inf., 11, 385-396.

11. Jacroux, M. (1989), Some sufficient conditions for the type I optimality with applications to regular graph designs, Jour. Stat. Plan. Inf., 23, 195-215.

12. John, J.A. & Mitchell, T. (1977), Optimal incomplete block designs, Jour. Roy. Stat. Soc. B, 39, 39-43

13. Kiefer (1975), Construction and optimality of generalized Youden designs, In A Survey of Statistical Designs and Linear Models, Ed, J.N.Srivastava, 333-353.

14. Marshall, A.W. & Olkin, I. (1979) , Inequalities : Theory of Majorisation and its Applications, Academic Press.

15. Morgan, J.P. & Srivastava, S.K. (2001). On the Type-1 optimality of nearly balanced incomplete block designs with small concurrence range, Stat. Sin., 10, 1091-1116.

16. Morgan, J.P. & Uddin (1995). Optimal, nonbinary, variance-balanced designs, Stat. Sin., 5, 535-546.

17. Morgan, J.P. & Reck, B. (2004) Optimal designs in irregular settings. Preprint.