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Abstract

We construct a certain 'cobordism category' $\mathcal{D}$ whose morphisms are suitably decorated cobordism classes between similarly decorated closed oriented 1-manifolds, and show that there is essentially a bijection between (1+1-dimensional) *unitary* topological quantum field theories (TQFTs) defined on $\mathcal{D}$, on the one hand, and Jones' subfactor planar algebras, on the other.

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1 Introduction

It was shown a while ago, via a modification by Ocneanu of the Turaev-Viro method, that 'subfactors of finite depth' give rise to 2+1-dimensional TQFTs. On the other hand, it has also been known that 1+1-dimensional TQFTs (on the cobordism category denoted by **2Cob** in [Koc]) are in bijective correspondence with 'Frobenius algebras'.

The purpose of this paper is to elucidate a relationship between 'unitary 1+1-dimensional TQFTs defined on a suitably decorated version $\mathcal{D}$ of **2Cob**' and 'subfactor planar algebras'. These latter objects are a topological/diagrammatic reformulation - see [Jon] - of the so-called 'standard invariant' of an 'extremal finite-index subfactor'. These planar algebras may be described as *algebras over the coloured operad of planar tangles* which satisfy some 'positivity conditions'. (We shall use the terminology and notation of the expository paper [KS1].) For this paper, the starting point is adopting the point of view that planar tangles are the special building blocks of more complicated gadgets, which are best thought of as compact oriented 2-manifolds, possibly with boundary, which are suitably 'decorated', and obtained by patching together many planar tangles. These gadgets are the 'morphisms' in the category $\mathcal{D}$, which is the subject of §2. In order to keep proper track of various things, it becomes necessary to regard the morphisms of this category as equivalence classes of the more easily and geometrically described 'pre-morphisms'. A lot of the subsequent work lies in ensuring that various constructions on pre-morphisms 'descend' to the level of morphisms.

§3 is devoted to showing how a subfactor planar algebra gives rise to a TQFT defined on $\mathcal{D}$, which is *unitary* in a natural sense, while §4 establishes that every unitary TQFT defined on $\mathcal{D}$ arises from a subfactor planar algebra as in §3 provided only that it satisfies a couple of (necessary and sufficient) restrictions.

The final §5 is a 'topological appendix', which contains several topological facts needed in proofs of the results of §3.

2 The category $\mathcal{D}$

All our manifolds will be compact oriented smooth manifolds, possibly with boundary. We will be concerned here with only one and two-dimensional manifolds, although we will be interested in suitably 'decorated' versions thereof. We shall find it convenient to write $\mathcal{C}(X)$ to denote the set of components of a space X.

DEFINITION 2.1 *A* **decoration** *on a closed 1-manifold σ is a triple $\delta = (P, *, sh)$ - where*

(i) P is a finite subset of σ,

(ii) $* : \mathcal{C}(\sigma) \to P \cup \{B, W\}$, *and*

(iii) $sh : \mathcal{C}(\sigma \backslash P) \to \{B, W\}$ *-*

with these three ingredients being required to satisfy the following conditions:

(a) if $J \in \mathcal{C}(\sigma)$*, then* $|J \cap P|$ *is even, and*

$$*(J) \in \begin{cases} J \cap P & if \ J \cap P \neq \emptyset \\ \{B, W\} & if \ J \cap P = \emptyset \end{cases} \text{, } and$$

(b) if $p \in P$*, then*

$$\{sh(J) : J \in \mathcal{C}(\sigma \backslash P), p \in J^-\} = \{B, W\} \ ;$$

thus, 'sh' yields a 'checkerboard shading' of $\sigma \setminus P$*.*

If $\phi : \sigma \to \sigma'$ is a diffeomorphism of one-manifolds, and if $\delta = (P, *, sh)$ is a decoration of σ, define $\phi_*(\delta)$ to be the **transported decoration** $\delta' = (P', *', sh')$ of σ', where

$$\begin{aligned} P' &= \phi(P) \\ sh' &= sh \circ \phi^{-1} \\ \{*'(\phi(J))\} &= \begin{cases} \{*(J)\} & \text{if } \phi \text{ is orientation preserving} \\ & \text{or } J \cap P \neq \emptyset \\ \{B, W\} \setminus \{*(J)\} & \text{if } \phi \text{ is orientation reversing} \\ & \text{and } J \cap P = \emptyset \end{cases} \end{aligned}$$

Finally, if $\phi : \sigma \to \sigma'$ is an *orientation-preserving* diffeomorphism of one-manifolds, and if $\delta = (P, *, sh)$ is a decoration of σ, then we shall consider the two 'decorated 1-manifolds' (σ, δ) and $(\sigma', \phi_*(\delta))$ as being equivalent.

Define sets C and Col by

$$\begin{aligned} C &= \{0_+, 0_-, 1, 2, 3, ...\} \\ Col &= \{k : k \in C\} \coprod \{\bar{k} : k \in C\} \end{aligned}$$

(We shall refer to the elements of Col as 'colours'.)

Suppose now that (σ, δ) is a decorated one-manifold, with $\delta = (P, *, sh)$, and that σ is non-empty and connected. Define $k(\sigma, \delta) = \frac{1}{2}|P|$. We define an associated **col(**σ, δ**)** by considering two cases, according as whether $k(\sigma, \delta)$ is positive or not.

Case (1) $k = k(\sigma, \delta) > 0$:

In this case, we define

$$col(\sigma, \delta) = k \text{ (resp., } \overline{k})$$

if, as one proceeds along σ in the given orientation and crosses the point labelled $*(\sigma)$, one moves from a black region into a white region (resp., from a white region into a black region) - where we think of an interval J - and a similar remark applies to colours of regions, as well - as being shaded black (resp., white) if $sh(J) = B$ (resp., $sh(J) = W$).

Case (2) $k = k(\sigma, \delta) = 0$:

In this case, there are four further possibilities, acording as whether (a) $*(\sigma)$ and $sh(\sigma)$ agree or disagree, and (ii) $sh(\sigma)$ is white or black. Specifically, in case $k = 0$, we define

$$col(\sigma, \delta) \ = \ \begin{cases} 0_+ & if \ *(\sigma) = sh(\sigma) = W \\ \overline{0_+} & if \ *(\sigma) \neq sh(\sigma) = W \\ 0_- & if \ *(\sigma) = sh(\sigma) = B \\ \overline{0_-} & if \ *(\sigma) \neq sh(\sigma) = B \end{cases} .$$

REMARK 2.2 We wish to make the fairly obvious observation here that the equivalence class of a decorated one-manifold (σ, δ) - where the underlying manifold σ is connected - is completely determined by its colour as defined above.

Let Obj denote a set, fixed once and for all, consisting of exactly one decorated 1-manifold from each equivalence class. Let $\mathcal{F}$ denote the set of functions $f : Col \to \mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ which are 'finitely supported' in the sense that $f^{-1}(\mathbb{Z}_+ \setminus \{0\})$ is finite; given an $f \in \mathcal{F}$, let X_f denote the element of Obj which has $f(k)$ connected components of colour k (in the sense of Remark 4) for each $k \in Col$. It is then seen that $f \leftrightarrow X_f$ is a bijection between $\mathcal{F}$ and Obj. Given $f \in \mathcal{F}$, let us define $\sigma(f)$ and $\delta(f)$ by demanding that $X_f = (\sigma(f), \delta(f))$.

If $k_0 \in Col$, we shall write $\mathbf{k_0}$ for the element of $\mathcal{F}$ given by

$$\mathbf{k_0}(k) = \begin{cases} 1 & \text{if } k = k_0 \\ 0 & otherwise \end{cases} \tag{2.1}$$

To be specific, we shall assume that $\sigma(\mathbf{k})$ is the unit circle in the plane - given by $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ - for every $\mathbf{k} \in \ Col$, oriented anti-clockwise. Further, writing

$$|k| = \begin{cases} m, & \text{if } k = m \in C \\ 0 & \text{if } k = 0_\pm \\ |m|, & \text{if } k = \bar{m}, m \in C \end{cases} ,$$

we define

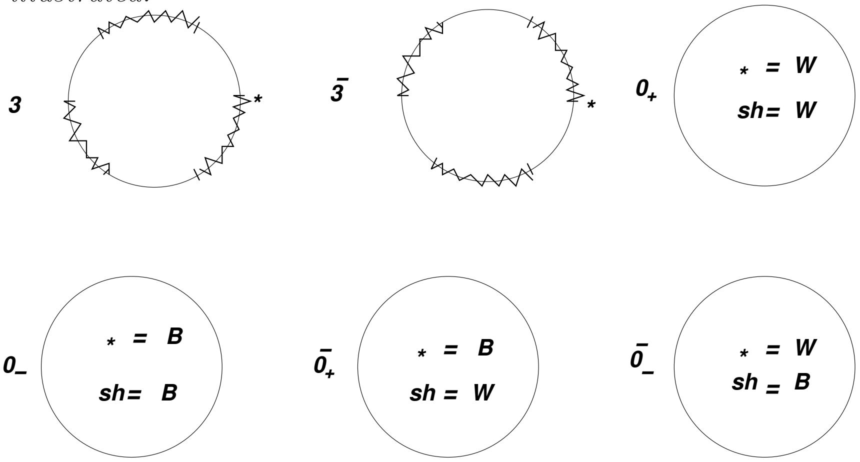
$$sh_{\delta(\mathbf{k})}(J) = \begin{cases} B & \text{if } k \in \{0_-, \bar{0}_-\} \\ W & \text{if } k \in \{0_+, \bar{0}_+\} \\ B & \text{if } J = \{(cos\theta, sin\theta) : \frac{(2m)2\pi}{2|k|}\} < \theta < \frac{(2m+1)2\pi}{2|k|}\} \ \&\ k \notin C \\ W & \text{if } J = \{(cos\theta, sin\theta) : \frac{(2m)2\pi}{2|k|}\} < \theta < \frac{(2m+1)2\pi}{2|k|}\} \ \&\ k \in C \end{cases} ,$$

$$P_{\delta(\mathbf{k})} = \begin{cases} \{(cos(\frac{m2\pi}{2|k|}), sin(\frac{m2\pi}{2|k|}) : 0 \leq m < 2|k|\} & \text{if } |k| \neq 0_\pm \\ \emptyset & \text{if } |k| = 0_\pm \end{cases} ,$$

and finally,

$$\{*_{\delta(\mathbf{k})}(\sigma(\mathbf{k}))\} = \begin{cases} \{(1,0)\} & \text{if } |k| \neq 0 \\ \{sh_{\delta(\mathbf{k})}(\sigma(\delta(\mathbf{k})))\} & \text{if } k = 0_\pm \\ \{B, W\} \setminus \{sh_{\delta(\mathbf{k})}(\sigma(\delta(\mathbf{k})))\} & \text{if } k = \bar{0}_\pm \end{cases} .$$

All this is seen best in the following diagrams - where the cases $\bar{\mathbf{3}}, \mathbf{3}, \mathbf{0}_+, \mathbf{0}_-, \bar{\mathbf{0}}_+, \bar{\mathbf{0}}_-$ *are illustrated:*



□

We 'transport' natural algebraic structures on $\mathcal{F}$ via the bijection described above, to define two operations, one binary and one unary, on the set Obj. To start with, note that $\mathcal{F}$ inherits a semigroup structure from $\mathbb{Z}_+$; we use this to define the **disjoint union** of elements of Obj by requiring that

$$X_f \coprod X_g \ = \ X_{f+g} \ . \tag{2.2}$$

It must be noticed that if 0 denotes the element of $\mathcal{F}$ corresponding to the identically zero function, then

$$X_f \coprod X_0 \ = \ X_f, \ \forall \ f \in \mathcal{F} \ .$$

In other words, the element X_0 of Obj - which is seen to be the empty set viewed as a '1-manifold' endowed with the only possible decoration - is the **empty object** and acts as identity for the binary operation of disjoint sum.

Next, there is clearly a unique involution '$-$' on the set Col given by

$$\bar{m} \;=\; \begin{cases} \bar{k} & \text{if } m = k \in C \\ k & \text{if } m = \bar{k}, \text{ for some } k \in C \end{cases}$$

This gives rise to an involution $\mathcal{F} \ni f \mapsto \bar{f} \in \mathcal{F}$ defined by

$$\bar{f} = f \circ - \;;$$

this, in turn, yields an involution on Obj defined by

$$\bar{X}_f = X_{\bar{f}} \;, \forall \; f \in \mathcal{F} \;. \tag{2.3}$$

It should be observed that if $X_f = (\sigma, \delta)$ and $\bar{X}_f = (\bar{\sigma}, \bar{\delta})$, then there is an orientation reversing diffeomorphism $\phi : \sigma \to \bar{\sigma}$ such that $\phi_*(\delta) = \bar{\delta}$. (This may be thought of as one justificaton for our definitions of (a) the transported decoration $\phi_*(\delta)$, in the case of orientation reversing diffeomorphisms, and (b) the colour, in the case of connected decorated one-manifolds.)

Definition 2.3 *A* **decoration** *on a 2-manifold* Σ *is a triple* $\Delta = (\ell, *, sh)$ *- where*

(i) ℓ *is a smooth compact 1-submanifold of* Σ *such that (a)* $\ell \cap \partial\Sigma = \partial\ell$*, and (b)* ℓ *meets* $\partial\Sigma$ *transversally.*

(ii) $* : \mathcal{C}(\partial\Sigma) \to (\partial\Sigma \cap \ell) \cup \{B, W\}$*; and*

(iii) $sh : \mathcal{C}(\Sigma \backslash \ell) \to \{B, W\}$ *-*

with these three ingredients being required to satisfy the following conditions:

(a) if $J \in \mathcal{C}(\partial\Sigma)$*, then* $|J \cap \ell| = |J \cap \partial\ell|$ *is even, with all intersections being transversal, and*

$$*(J) \in \begin{cases} J \cap \ell & if \; J \cap \ell \neq \emptyset \\ \{B, W\} & if \; J \cap \ell = \emptyset \end{cases} \;, \; and$$

(b) 'sh' is a 'checkerboard shading' of $\Sigma \setminus \ell$*.*

Remark 2.4 *Notice that every decoration* Δ *on a 2-manifold* Σ **induces** *a decoration* $\delta = \Delta|_\sigma$ *on every closed 1-submanifold* σ *of* $\partial\Sigma$ *by requiring that*

$$\begin{aligned} P_\delta &= \sigma \cap \partial\ell_\Delta \\ *_\delta &= (*_\Delta)|_{\mathcal{C}(\sigma)} \\ sh_\delta(J) &= sh_\Delta(\Omega) \;, \; if \;\; J \in \mathcal{C}(\sigma \setminus P_\delta), \;\; J \subset \overline{\Omega}, \;\; \Omega \in \mathcal{C}(\Sigma \setminus \ell) \;. \end{aligned}$$

For any integer $b \geq 0$, we shall write A_b for the (compact 2-manifold given by the) complement in the 2-sphere S^2 of the union of $(b+1)$ pairwise disjoint embedded discs. (Thus, A_0 is a disc, A_1 is an annulus, and A_2 is a 'pair of pants'.)

DEFINITION 2.5 *By a* **planar decomposition** *of a 2-manifold* Σ*, we shall mean a finite (possibly empty) collection* $\Pi = \{\gamma_i : i \in I\}$ *of pairwise disjoint closed 1-submanifolds of* $\Sigma \setminus \partial\Sigma$ *such that each component of the complement of a small tubular neighbourhood of* $(\cup_{i\in I} \gamma_i)$ *is diffeomorphic to some* $A_b, b \geq 0$.

We shall call a triple (Σ, Δ, Π) *a* **planar decorated** *2-manifold if* Δ *is a decoration of a 2-manifold* Σ*, and* Π *is a planar decomposition of* Σ *satisfying the following compatibility condition:*

if $\gamma \in \Pi$*, then* γ *meets* ℓ *transversely, and in at most finitely many points.*

Let $\mathcal{M}$ denote the collection of all tuples $(\Sigma, \Delta, \Pi, \phi_0, \phi_1)$, where:

(1) (Σ, Δ, Π) is a planar decorated 2-manifold;

(2) ϕ_0 (resp., ϕ_1) uniquely determines an $f_0 \in \mathcal{F}$ (resp., $f_1 \in \mathcal{F}$) such that ϕ_0 (resp., ϕ_1) is an orientation reversing (resp., preserving) diffeomorphism from $\sigma(f_0)$ (resp., $\sigma(f_1)$) to a closed submanifold of $\partial\Sigma$, satisfying:

(i)

$$\partial\Sigma = \phi_0(\sigma(f_0)) \coprod \phi_1(\sigma(f_1)) \ ;$$

and

(ii)

$$(\phi_i)_*(\delta(f_i)) \ = \ \Delta|_{\phi_i(\sigma(f_i))} \ , i = 1, 2. \tag{2.4}$$

(Here again, it is crucially important that the 'transport' $\phi_*(\delta)$ of a decoration on a closed 1-manifold be defined differently, and as we have done, in the cases when the diffeomorphism ϕ preserves or reverses the underlying orientations.)

A tuple $(\Sigma, \Delta, \Pi, \phi_0, \phi_1) \in \mathcal{M}$ as above, will be referred to as a **pre-morphism** from X_{f_0} to X_{f_1}. The reason for the prefix is that we will want to think of several different pre-morphisms as being the same.

Thus, a morphism from X_{f_0} to X_{f_1} will, for us, be an equivalence class of pre-morphisms, with respect to the smallest equivalence relation generated by three kinds of 'moves'. More precisely:

DEFINITION 2.6 *(a) Two pre-morphisms* $(\Sigma^{(i)}, \Delta^{(i)}, \Pi^{(i)}, \phi_0^{(i)}, \phi_1^{(i)}) \in \mathcal{M}, \ i = 1, 2,$ *with* $\Delta^{(i)} = (\ell^{(i)}, *^{(i)}, sh^{(i)})$*, say, are said to be related by a:*

(i) **Type I move** *if there exists an orientation-preserving diffeomorphism* $\phi : \Sigma^{(1)} \to \Sigma^{(2)}$ *such that*

$$\Delta^{(2)} = \phi_*(\Delta^{(1)}) \ - \ i.e., \ \ell^{(2)} = \phi(\ell^{(1)}), \ *^{(1)} = *^{(2)} \circ \phi, \ sh^{(1)} = sh^{(2)} \circ \phi,$$

$$\Pi^{(2)} = \phi_*(\Pi^{(1)}) \ (\ = \{\phi(\gamma)) : \gamma \in \Pi^{(1)}\}), \ \textit{and} \ \phi_j^{(2)} = \phi_j^{(1)} \circ \phi \ , \ j = 0, 1;$$

(ii) **Type II move** *if*

$$\Sigma^{(1)} = \Sigma^{(2)}, \ \Delta^{(1)} = \Delta^{(2)}, \ \phi_i^{(1)} = \phi_i^{(2)}$$

and there exists a (necessarily orientation preserving) diffeomorphism ϕ of $\Sigma^{(1)}$ onto itself which is isotopic via diffeomorphisms to id_Σ, such that $\Pi^{(2)} = \phi_*(\Pi^{(1)})$; and

(ii) **Type III move** *if*

$$\Sigma^{(1)} = \Sigma^{(2)}, \ \Delta^{(1)} = \Delta^{(2)}, \ \phi_i^{(1)} = \phi_i^{(2)}$$

and

$$\Pi^{(1)} \cup \Pi^{(2)} \in \{\Pi^{(1)}, \Pi^{(2)}\} \ .$$

(b) Two pre-morphism $(\Sigma^{(i)}, \Delta^{(i)}, \Pi^{(i)}, \phi_0^{(i)}, \phi_1^{(i)}) \in \mathcal{M}, \ i = 1, 2$, are said to be **equivalent** *if either can be obtained from the other by a finite sequence of moves of the above three types.*

(c) An equivalence class of pre-morphisms is called a **morphism**.

It must be observed that equivalent pre-morphisms have the same 'domain' and 'range', so that it makes sense - and is only natural - to say that the equivalence class of the pre-morphism $(\Sigma, \Delta, \Pi, \phi_0, \phi_1)$ defines a morphism from X_{f_0} to X_{f_1}.

Before verifying that we have a 'cobordism category' with objects given by Obj, and morphisms defined as above, it will help for us to define what is meant by disjoint unions, adjoints and boundaries of morphisms. As is to be expected, we shall first define these notions for pre-morphisms, verify that the definitions respect the three types of moves above, and conclude that the definitions 'descend' to the level of morphisms.

Define the **disjoint union of premorphisms** by the following completely natural prescription:

$$\begin{aligned}
(\Sigma^{(1)}, \Delta^{(1)}, \Pi^{(1)}, \phi_0^{(1)}, \phi_1^{(1)}) \ & \coprod \ (\Sigma^{(2)}, \Delta^{(2)}, \Pi^{(2)}, \phi_0^{(2)}, \phi_1^{(2)}) \\
& = \ (\Sigma, \Delta, \Pi, \phi_0, \phi_1) \ , \text{where} \\
\Sigma = \Sigma^{(1)} \coprod \Sigma^{(2)} \quad & , \quad \ell_\Delta = \ell_{\Delta^{(1)}} \coprod \ell_{\Delta^{(2)}} \\
*_\Delta|_{\mathcal{C}(\partial \Sigma^{(i)}} \ & = \ *_{\Delta^{(i)}} \ , \ \text{for } i = 1, 2 \\
sh_\Delta|_{\mathcal{C}(\Sigma^{(i)} \setminus \ell_{\Delta^{(i)}})} \ & = \ sh_{\Delta^{(i)}} \ , \ \text{for } i = 1, 2 \\
\Pi = \Pi^{(1)} \coprod \Pi^{(2)} \quad & , \quad \phi_j = \phi_j^{(1)} \coprod \phi_j^{(2)} \ , \ \text{for } j = 0, 1 \ .
\end{aligned}$$

Next, define the **adjoint of a pre-morphism** - which we shall denote using a 'bar' rather than a 'star' - by requiring that

$$\begin{aligned}
(\Sigma, \Delta, \Pi, \phi_0, \phi_1)^- &= (\overline{\Sigma}, \overline{\Delta}, \overline{\Pi}, \overline{\phi_0}, \overline{\phi_1}) \ , \ \text{where} \\
\ell_{\overline{\Delta}} &= \overline{\ell_\Delta} \\
\{*_{\overline{\Delta}}(J)\} &= \begin{cases} \{*_\Delta(J)\} & \text{if } J \cap \partial\ell_\Delta \neq \emptyset \\ \{B, W\} \setminus \{*_\Delta(J)\} & \text{if } J \cap \partial\ell_\Delta = \emptyset \end{cases} \\
sh_{\overline{\Delta}} &= sh_\Delta \\
\overline{\Pi} &= \Pi \\
\overline{\phi_0} = \phi_1 \quad &, \quad \overline{\phi_1} = \phi_0 \ ,
\end{aligned}$$

where we write $\bar{\Sigma}$ for the manifold Σ endowed with the opposite orientation; it is in this sense that equations such as $\overline{\Pi} = \Pi$ are interpreted.

Finally define the boundary of a pre-morphism by requiring that

$$\partial(\Sigma, \Delta, \Pi, \phi_0, \phi_1) = \bar{X}_{f0} \coprod X_{f_1} \ .$$

Some painstaking verification shows that, indeed, these definitions 'descend to the level of morphisms'. (For instance, to see this for disjoint unions, one verifies that if M_1' and M_2' are two pre-morphisms which are related by a move of type j for some $j \in \{I, II, III\}$ and if M'' is any other pre-morphism, then also $M_1' \coprod M''$ and $M_2' \coprod M''$ are related by a move of type j; and then argues that this is sufficient to 'make the descent'.) In particular, we wish to emphasise that if M is the morphism given by the equivalence class of the pre-morphism $(\Sigma, \Delta, \Pi, \phi_0, \phi_1)$, then M is a morphism from X_{f_0} to X_{f_1} - which we denote by $M \in Mor(X_{f_0}, X_{f_1})$ - while $\bar{M} \in Mor(X_{f_1}, X_{f_0})$.

Our next step is to define 'composition of morphisms'. This will be done in two stages:

Step 1: This consists of the 'Assertion' below, which asserts the existence of what may be called a *semi-normal form* of a pre-morphism; the prefix 'semi' is necessitated by this 'form' not quite being a canonical form, and the adjective 'normal' is chosen for obvious reasons. This assertion is really not much more than a re-statement of the fact - see condition (i) of Definition 2.3 - that ℓ_Δ meets $\partial\Sigma$ transversally, so we shall say nothing about the proof.

Assertion : Suppose $(\Sigma, \Delta, \Pi, \phi_0, \phi_1)$ is a pre-morphism. Let any enumerations $J_1^{(k,i)}, \cdots, J_{f_i(k)}^{(k,i)}$ be given, of all the components of $\phi_i(\sigma(f_i))$ of colour k, for each $i = 0, 1, k \in Col$. Then there exists an orientation-preserving diffeomorphism $\phi : \Sigma \to \Sigma_0$ such that:

(i) $\Sigma_0 \subset \mathbb{R}^2 \times [0, 1]$;

(ii) $\phi \circ \phi_i(\sigma(f_i))$ is $X_i \times \{i\}$, where X_i is the union of the circles with radius equal to $\frac{1}{4}$ and centres in the set $S^{(i)} = S_+^{(i)} \cup S_-^{(i)}$, for $i = 0, 1$, where

$$\begin{aligned} S_+^{(i)} &= \{(|k|, l) : 1 \leq l \leq f_{(i)}(k), k \in (C \setminus \{0_-\})\} \\ &\cup \{(-1, l) : 1 \leq l \leq f_{(i)}(0_-)\}, \text{ and} \\ S_-^{(i)} &= \{(|k|, -l) : 1 \leq l \leq f_{(i)}(\bar{k}), k \in (C \setminus \{0_-\})\} \\ &\cup \{(-1, -l) : 1 \leq l \leq f_{(i)}(\bar{0}_-))\} \ ; \end{aligned}$$

further, we may arrange matters such that $\phi(J_l^{(k,i)})$ is the circle in X_i with centre $(|k|, l), (|k|, -l), (-1, l)$ or $(-1, -l)$ according as $k \in C \setminus \{0_-\}, \bar{k} \in C \setminus \{0_-\}, k = 0_-$ or $k = \bar{0}_-$;

(iii) there exists an $\epsilon > 0$ such that, for $i = 0, 1$, $\{(x, y, z) \in \Sigma_0 : |z - i| < \epsilon\}$ is nothing but the family of cylinders given by $\{(x, y) : \exists (m, n) \in S^{(i)}$ such that $(x-m)^2 + (y-n)^2 = (\frac{1}{4})^2\} \times \{t \in [0, 1] : |t - i| < \epsilon\}$; and

(iv) For $i = 0, 1$, $\{(x, y, z) \in \phi(\ell_\Delta) : |z - i| < \epsilon\}$ is nothing but the set of verical lines given by $\{(k + \frac{1}{4}cos(\frac{j\pi}{k}), l + \frac{1}{4}sin(\frac{j\pi}{k}))\} \times \{t \in [0, 1] : |t - i| < \epsilon\}$, where $(k, l) \in S^{(i)}, k > 0, 1 \leq j \leq 2k$.

Finally, we shall refer to $(\Sigma_0, \phi_*(\Delta), \phi_*(\Pi), \phi \circ \phi_0, \phi \circ \phi_1)$ as a semi-normal form of $[(\Sigma, \Delta, \Pi, \phi_0, \phi_1)]$. $\square$

We now proceed to define **composition of morphisms**. Suppose $M' = [(\Sigma', \Delta', \Pi', \phi_0', \phi_1')] \in Mor(X_{f_0'}, X_{f_1'})$, and $M'' = [(\Sigma'', \Delta'', \Pi'', \phi_0'', \phi_1'')] \in Mor(X_{f_0''}, X_{f_1''})$, and suppose $f_1' = f_0''$. Then let $f = \phi_0'' \circ (\phi_1')^{-1}$, so that f is an orientation reversing diffeomorphism from the closed submanifold $\phi_1'(\sigma(f_1'))$ of $\partial\Sigma'$ to the closed submanifold $\phi_0''(\sigma(f_0''))$ of $\partial\Sigma''$.

Define

$$\begin{aligned} \Sigma &= \Sigma' \cup_f \Sigma'' \ , \\ \ell_\Delta &= \ell_{\Delta'} \cup_{f|_{\partial \ell_{\Delta'} \cap im(\phi_1')}} \ell_{\Delta''} \\ *_\Delta(J) &= \begin{cases} *_{\Delta'}(J) & \text{if } J \subset \Sigma' \\ *_{\Delta''}(J) & \text{if } J \subset \Sigma'' \end{cases} \\ sh_\Delta(\Omega) &= \begin{cases} sh_{\Delta'}(\Omega \cap \Sigma') & \text{if } \Omega \cap \Sigma' \neq \emptyset \\ sh_{\Delta''}(\Omega \cap \Sigma'') & \text{if } \Omega \cap \Sigma'' \neq \emptyset \end{cases} \end{aligned}$$

Strictly speaking, we should, for instance, have written *not* $sh_{\Delta''}(\Omega \cap \Sigma'')$, but instead $sh_{\Delta''}(\Omega'')$ where $(\Omega \cap \Sigma'') \supset \Omega'' \in \mathcal{C}(\Sigma'' \setminus \ell_{\Delta''})$. We need to verify that the above definition of the shading on Σ is unambiguous - for which we only need to observe that every component of $(\Sigma' \cap \Sigma'') \setminus \ell_\Delta$ inherits the same shading from Δ' and Δ''.[3] This will follow

[3]When dealing with such compositions, we shall identify Σ' and Σ'' with their images in Σ.

once we check that

$$f_*(\Delta'|_{(\Sigma'\cap\Sigma'')}) \;=\; \Delta''|_{(\Sigma'\cap\Sigma'')} \;;$$

but this is a consequence of two applications of equation (2.4) - to $(\phi_1')^{-1}$ and ϕ_0'', respectively.

Finally, define

$$\begin{aligned} \Pi &= \Pi' \cup \Pi'' \cup \mathcal{C}(\Sigma' \cap \Sigma'') \\ \phi_0 &= \phi_0' \;, \\ \phi_1 &= \phi_1'' \;. \end{aligned}$$

Step 2: Given two morphisms $M' = [(\Sigma', \Delta', \Pi', \phi_0', \phi_1')] \in Mor(X_{f_0'}, X_{f_1'})$, and $M'' = [(\Sigma'', \Delta'', \Pi'', \phi_0'', \phi_1'')] \in Mor(X_{f_0''}, X_{f_1''})$, we shall appeal to the assertions in Step 1 to assume that the pre-morphisms $(\Sigma', \Delta', \Pi', \phi_0', \phi_1')$ and $(\Sigma'', \Delta'', \Pi'', \phi_0'', \phi_1'')$ are in semi-normal form, with the enumerations of the boundary components of $\phi_1'(\sigma(f_1'))$ and of $\phi_0''(\sigma(f_0''))$ having been chosen in a compatible fashion. The definitions and the nature of 'semi-normal forms' show then that $(\Sigma, \Delta, \Pi, \phi_0, \phi_1)$, as defined in the paragraphs preceding Step 2, is indeed a pre-morphism (with ℓ_Δ being smooth (and without kinks at the 'glueing places') and transverse to Π). Finally, we shall define

$$M'' \circ M' \;=\; [(\Sigma, \Delta, \Pi, \phi_0, \phi_1)] \;.$$

What remains is to ensure that this rule for composition is an unambiguously defined operation, which is independent of the choices (of semi-normal representatives) involved. Suppose $(\Sigma^{(0)\prime}, \Delta^{(0)\prime}, \Pi^{(0)\prime}, \phi_0^{(0)\prime}, \phi_1^{(0)\prime})$ and $(\Sigma^{(0)\prime\prime}, \Delta^{(0)\prime\prime}, \Pi^{(0)\prime\prime}, \phi_0^{(0)\prime\prime}, \phi_1^{(0)\prime\prime})$ are also semi-normal forms of M' and M''. Then, by definition, there exists an orientation preserving diffeomorphism $\phi' : \Sigma' \to \Sigma^{(0)\prime}$ which 'transports' one pre morphism structure to the other. Next, by a judicious application of Lemma 5.2 - to a neighbourhood of the 'end' of Σ' to be glued (let us call this the 1-end) - we may find another orientation-preserving diffeomorphism $\psi' : \Sigma' \to \Sigma^{(0)\prime}$ which is 'identity on a small neighbourhood of the 1-end' and 'ϕ' outside a slightly larger neighbourhood of the 1-end'. It is clear, in view of the nature of semi-normal forms, that ψ' also transports the pre-morphism structure on Σ' to that on $\Sigma^{(0)\prime}$. In an entirely similar fashion, we can find an orientation-preserving diffeomorphism ψ'' which transports the pre-morphism structure on Σ'' to that on $\Sigma^{(0)\prime\prime}$ (and is the 'identity on a small neighbourhood of the 0-end' and 'ϕ'' outside a slightly larger neighbourhood of the 0-end'). Finally, it is a simple matter to see that $\psi' \coprod_{\phi_1'(\sigma(f_1'))} \psi''$ defines a smooth orientation-preserving diffeomorphism which transports the pre-morphism structure on $\Sigma' \coprod_{\phi_1'(\sigma(f_1'))} \Sigma''$ to that on $\Sigma^{(0)\prime} \coprod_{\phi_1^{(0)\prime}(\sigma(f_1^{(0)\prime}))} \Sigma^{(0)\prime\prime}$. This proves that our definition of the composition of two morphisms is indeed independent of the choice, in our definition, of semi-normal forms, as desired.

Only the definition of id_{X_f} remains before we can proceed to the verification that we have a 'cobordism category'. If $f \in \mathcal{F}$, we define

$$id_{X_f} \;=\; [(\Sigma, \Delta, \Pi, \phi_0, \phi_1)] \ ,$$

where

$$\begin{aligned}
\Sigma &= \sigma(f) \times [0,1] \\
\ell_\Delta &= P_{\delta(f)} \times [0,1] \\
*_{\delta(f)}(J) &= *_{\overline{\Delta}}(J \times \{0\}) \;=\; *_\Delta\,(J \times \{1\}) \\
sh_\Delta(J \times [0,1]) &= sh_{\delta(f)}(J) \\
\Pi &= \emptyset \\
f_0 = f_1 &= f \\
\phi_j(x) &= (x, j) \ , \text{ for } x \in \sigma(f), \ j = 0, 1 \ ,
\end{aligned}$$

where $\sigma(f) \times [0,1]$ is so oriented as to ensure that ϕ_0 (resp., ϕ_1) is orientation-reversing (resp., preserving).

PROPOSITION 2.7 *There exists a unique category $\mathcal{D}$ whose objects are given by members of the countable set Obj, such that, if $f_j \in \mathcal{F}, \ j = 0, 1,$ the collection of morphisms from X_{f_0} to X_{f_1} is given by $Mor(X_{f_0}, X_{f_1})$, and composition of morphisms is as defined earlier.*

(Note that the objects of $\mathcal{D}$ are equivalence classes of decorated one-manifolds, while morphisms between two such objects is an equivalence class of decorated cobordisms between them - and thus $\mathcal{D}$ is a 'cobordism category' in the sense of [BHMV].)

Proof: To verify the assertion that $\mathcal{D}$ is a category, we only need to verify that (i) composition of morphisms is associative; and that (ii) id_{X_f} is indeed the identity morphism of the object X_f.

The verification of (i) is straightforward, while (ii) is a direct consequence of the definition of a move of type III.

As for the remark about 'cobordism categories', observe that $\mathcal{D}$ comes equipped with:

(a) notions of 'disjoint unions' - for objects as well as morphisms; this yields a bi-functor from $\mathcal{D} \times \mathcal{D}$ to $\mathcal{D}$ that is invariant under the 'flip';

(b) an 'empty object' $\emptyset$ as well as an 'empty morphism' - *viz.* $id_\emptyset$ - which act as 'identity' for the operation of 'disjoint union';

(c) notions of adjoints, of objects as well as morphisms, such that

$$M \in Mor(X_{f_0}, X_{f_1}) \Rightarrow \bar{M} \in Mor(X_{f_1}, X_{f_0}) \ ;$$

and

(d) the notion of 'boundary' ∂ from morphisms to objects, which 'commutes' with disjoint unions as well as with adjoints.

Finally, if we let **2Cob** denote the category whose objects are compact oriented smooth 1-manifolds, and whose morphisms are cobordisms, then we have a 'forgetful functor' from $\mathcal{D}$ to **2Cob**. This is what we mean by a 'cobordism category in dimension 1+1'. $\square$

3 From subfactors to TQFTs on $\mathcal{D}$

We wish to show, in this section, how every extremal finite-index II_1 subfactor gives rise to a **unitary (1+1)-dimensional topological quantum field theory** - abbreviated throughout this paper to unitary TQFT - defined on the category $\mathcal{D}$ of the previous section. This involves using the subfactor to define a functor from $\mathcal{D}$ to the category of finite-dimensional Hilbert spaces, satisfying 'compatibility conditions' involving the various structures possessed by $\mathcal{D}$.

For this, we shall find it convenient to work with 'unordered tensor products' of vector spaces. Although this notion is discussed in [Tur], we shall say a few words here about such unordered tensor products for the reader's convenience as well as to set up the notation we shall use.

Given an ordered collection $\{V_i : 1 \leq i \leq n\}$ of vector spaces, and a permutation $\sigma \in S_n$, let us write $V_\sigma = V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(b)}$ and define the map $U_\sigma : V_\epsilon \to V_\sigma$ - where we write ϵ for the identity element of S_n - by the equation

$$U_\sigma(\otimes_{i=1}^b v_i) = \otimes_{i=1}^b v_{\sigma^{-1}(i)} \ .$$

We define the **unordered tensor product** of the spaces $\{V_i : 1 \leq i \leq n\}$ by the equation

$$\begin{aligned}&\bigotimes_{unord}\{V_i : i \in \{1, 2, \cdots, n\}\} \\ &= \ \{((x_\sigma)) \in \oplus_{\sigma \in S_n} V_\sigma : x_\sigma = U_\sigma x_\epsilon \ , \ \forall \sigma \in S_n\}\end{aligned}$$

(In case $x_\epsilon = \otimes_{i=1}^n v_i$ is a 'decomposable tensor', we shall write $\otimes_{unord}\{v_i\}_i\}$ for the element $((U_\sigma x_\epsilon))$.)

It is clear that the unordered tensor product is naturally isomorphic to the (usual, ordered) tensor product; in case each V_i is a Hilbert space, so is the unordered tensor product, and the natural isomorphism of the last sentence is unitary. In particular, for example, if we are given a collection $\{T_i \in L(V_i, W_i) : 1 \leq i \leq n\}$ of linear maps, there is a unique associated linear map

$$\otimes_{unord}\{T_i\}_i \ \in \ L(\bigotimes_{unord}\{V_i\}_i, \bigotimes_{unord}\{W_i\}_i)$$

such that

$$(\otimes_{unord}\{T_i\}_i)(\otimes_{unord}\{v_i\}_i) \ = \ (\otimes_{unord}\{Tv_i\}_i) \ , \ \forall i \in V_i, 1 \leq i \leq n \ .$$

In the interest of notational convenience, and in view of the isomorphism stated at the beginning of the last paragraph, we shall be sloppy and omit the subscript 'unord' in the sequel.

Suppose now that we have an extremal subfactor N of a II_1 factor M of finite index. Following the notation of [Jon], we write δ for the square root of the index $[M : N]$, and let

$$P_k \ = \ \begin{cases} \mathbb{C} & \text{if } \ k = 0_\pm \\ N' \cap M_{k-1} & \text{if } \ k = 1, 2, \cdots \end{cases} ,$$

where of course

$$(M_{-1} =)N \subset (M_0 =)M \subset M_1 \subset \cdots \subset M_k \subset \cdots$$

denotes the basic construction tower of Jones.

The sequence $P = \{P_k : k \in C\}$ of relative commutants has its natural **planar algebra structure**, as defined in [Jon]. (We shall find it convenient to primarily use the notation described in [KS1], which differs in a few minor details from [Jon]. We shall however consistently use the symbol $Z(T)$ - and **not** Z_T - for the multi-linear operator associated to a planar tangle T.) We shall further let

$$P_{\bar{k}} \ = \ P_k^* \ , \ \forall k \in C \ ,$$

where the superscript $*$ denotes the dual - equivalently, the complex conjugate - Hilbert space. Note that we have defined P_k for all $k \in \ Col$.

Let us write $\bigotimes\{m_i V_i : i \in I\}$ to denote the unordered tensor product of a collection containing exactly m_i vector spaces equal to V_i, for each $i \in I$.

We define

$$V(X_f) \ = \ \bigotimes\{f(k)P_k : k \in Col, f(k) \neq 0\} \ .$$

If $M = (\Sigma, \Delta, \Pi, \phi_0, \phi_1)$ is a pre-morphism, then $(\partial\Sigma, \Delta|_{\partial\Sigma})$ is a decorated 1-manifold, which we denote by $\partial(\Sigma, \Delta)$. It is to be noted that if the equivalence class of this decorated 1-maniold is given by the element X_f of Obj, then the Hilbert space

$$V(X_f) \ = \ \bigotimes\{P_{col(J,\Delta|J)} : J \in \mathcal{C}(\partial\Sigma)\}$$

depends only on (Σ, Δ), owing to the 'unordered nature' of our tensor products; and hence, we may write

$$V(\partial(\Sigma, \Delta)) \;=\; V(X_f) \;.$$

Next, to a morphism $M \in Mor(X_{f_0}, X_{f_1})$, we need to associate a linear map $Z_M \in L(V(X_{f_0}), V(X_{f_1}))$. To start with, we shall do the following: if $M = [(\Sigma, \Delta, \Pi, \phi_0, \phi_1)]$, we shall construct an element

$$\zeta_M \in V(\partial(\Sigma, \Delta)) \;,$$

and then verify that this vector ζ_M depends only on the equivalence class defining the morphism M. Finally, we shall appeal to natural identifications

$$V(\partial(\Sigma, \Delta)) \cong V(X_{f_0})^* \otimes V(X_{f_1}) \cong L(V(X_{f_0}), V(X_{f_1}))$$

to associate the desired operator Z_M to ζ_M, and hence to M.

All of the above will require some work. We start with a definition.

DEFINITION 3.1 *Given a planar decorated 2-manifold* (Σ, Δ, Π)*, and a component* $J \in \mathcal{C}(\partial\Sigma)$*, we shall say that* J *is* **good** *if* $col(J, \Delta|_J) \notin C$ *and* **bad**, *if it is not good.*

We will arrive at the definition of the desired association $M \mapsto Z_M$ by discussing a series of cases of increasing complexity. We assume, in what follows, that we are given a fixed planar decorated 2-manifold (Σ, Δ, Π).

Case 1: $\Pi = \emptyset$ *and* **all** *components* $J \in \mathcal{C}(\partial\Sigma)$ *are good.*

The assumption $\Pi = \emptyset$ implies that Σ is - diffeomorphic to, and may hence be identified with - A_b for some $b \geq 0$. The 'goodness' assumption says that the colour of each of the components of $\partial\Sigma$ belongs to the set $\{\bar{k} : k \in C\}$; suppose $\{col(J, \Delta|_J) : J \in \mathcal{C}(\partial\Sigma)\} \;=\; \{\overline{k_i} : 0 \leq i \leq b\}$ where $k_i \in C \; \forall \; i$. For a point $x \in \Sigma \setminus (\partial\Sigma \cup \ell_\Delta)$, let $\mathcal{N}_x$ denote the result of stereographically projecting Σ onto the plane, with x thought of as the north pole. The assumption that all the J's are good has the consequence that $\mathcal{N}_x$ is a *planar network* in the sense of [Jon] (with unbounded component positively or negatively oriented according as the component of the point x is shaded white or black according to Σ). The partition function of $\mathcal{N}_x$ - obtained from the planar algebra of the subfactor $N \subset M$ - yields a linear functional η_x of $\otimes\{P_{k_i} : 0 \leq i \leq b\}$, and hence an element $\zeta_\Delta \in \otimes\{P_{\overline{k_i}} : 0 \leq i \leq b\}$.

Let $f \in \mathcal{F}$ be defined, as above, by requiring that $V(X_f) = V(\partial(\Sigma, \Delta))$, and let ϕ denote the identification of $\sigma(f)$ with $\partial\Sigma$. Consider the pre-morphism $(\Sigma, \Delta, \emptyset, \phi_\emptyset, \phi)$

from X_0 to X_f - where $\phi_\emptyset$ is the unique map from $\sigma(0)$ to $\emptyset$. If we let $M = [(\Sigma, \Delta, \emptyset, \phi_\emptyset, \phi)]$ denote the associated morphism, then notice that $\partial M = X_f$, and that

$$\begin{aligned} V(X_f) &= V(\partial(\Sigma, \Delta)) \\ &= \bigotimes \{P_{\overline{k_i}} : 0 \leq i \leq b\} \ . \end{aligned}$$

Thus, at least in this case, we have been able to associate an element ζ_Δ of $V(\partial(\Sigma, \Delta))$ to the pre-morphism $(\Sigma, \Delta, \emptyset, \phi_\emptyset, \phi)$.

The following two observations ensure that ζ_Δ is independent of various choices.

(a) If also $x' \in \Sigma \backslash (\partial\Sigma \cup \ell_\Delta)$, then the networks $\mathcal{N}_x$ and $\mathcal{N}_{x'}$ are related by an diffeotopy of S^2 (corresponding to a rotation which maps x to x'); and since the partition function obtained from an extremal subfactor is an invariant of networks on the 2-sphere - see [Jon] -, we find that $\eta_x = \eta_{x'}$.

(b) Two different identifications of Σ with A_b give the same ζ_Δ since (i) any diffeomorphism of A_b to itself preserving the boundary is isotopic to the identity - by virtue of triviality of the mapping class group of the sphere; and (ii) the partition function of a tangle is 'well-behaved with respect to re-numbering its internal discs' - see eqn. (2.3) in [KS1].
(The fact that the mapping class group of a compact surface is trivial only for genus zero is one of the main reasons for our seemingly complicated definition - involving planar decompositions - of the category $\mathcal{D}$.)

For each $k \in C$, define a map $\beta_k : P_k \to P_{\bar{k}}^*$ by the equation

$$\beta_k(x)(y) = \tau_k(xy) \ , \tag{3.5}$$

where τ_k denotes the normalised trace on P_k defined by $\tau_k(z) = tr_{M_k}(z)$ - so $\delta^k \tau_k$ agrees with the result of applying the 'trace-tangle' (followed by the identification of P_{0_+} with $\mathbb{C}$).

The non-degeneracy of the trace implies that β_k is an isomorphism, and hence we also have the isomorphism $\beta_{\bar{k}}^{-1} : P_{\bar{k}} \to P_k$ for $k \in C$. For later use, we observe here that if $\{e_i\}$ is a basis for P_k with corresponding dual basis $\{e^i\}$ for $P_{\bar{k}}^*$, then

$$\sum_i \tau_k(e_i z) \beta_{\bar{k}}^{-1}(e^i) = z \ , \ \forall z \in P_k \ . \tag{3.6}$$

Case 2: $\Pi = \emptyset$ *and not all components* $J \in \mathcal{C}(\partial\Sigma)$ *are necessarily good.*

Define $\mathcal{C}_{g,\Delta}(\partial\Sigma) = \{J \in \mathcal{C}(\partial\Sigma) : J \text{ is good for } (\Sigma, \Delta)\}$ and $\mathcal{C}_{b,\Delta}(\partial\Sigma) = \{J \in \mathcal{C}(\partial\Sigma) : J \text{ is bad for } (\Sigma, \Delta)\}$. Also, suppose $\{col(J, \Delta|_J) : J \in \mathcal{C}_{g,\Delta}(\partial\Sigma)\} = \{\overline{k_i} : 0 \leq i \leq b_g\}$ and $\{col(J, \Delta|_J) : J \in \mathcal{C}_{b,\Delta}(\partial\Sigma)\} = \{k_i : b_g + 1 \leq i \leq b\}$ where $k_i \in C, \ 0 \leq i \leq b$.

The following bit of notation will be handy: if $\Delta = (\ell, *, sh)$ is a decoration of a 2-manifold Σ, let us define the 'rotated $*$s', denoted $(*+1)_\Delta$ (resp., $(*-1)_\Delta$) by demanding that (a) if $\ell \cap J = \emptyset$, then $\{(* \pm 1)_\Delta\}(J)\} = \{B, W\} \setminus \{*_\Delta\}(J)\}$, and (b) if $\ell \cap J \neq \emptyset$, then $(*+1)_\Delta(J)$ (resp., $(*-1)_\Delta(J)$) is the 'first point immediately after (resp., before) $*_\Delta(J)$' as one traverses J in the orientation induced by Σ.

Now define the 'improved' decoration $\widetilde{\Delta}$ on Σ by

$$\begin{aligned} \ell_{\widetilde{\Delta}} &= \ell_\Delta \\ sh_{\widetilde{\Delta}} &= sh_\Delta \\ *_{\widetilde{\Delta}}(J) &= \begin{cases} *_\Delta(J) & \text{if } J \in \mathcal{C}_{g,\Delta}(\partial\Sigma) \\ (*+1)_\Delta(J) & \text{if } J \in \mathcal{C}_{b,\Delta}(\partial\Sigma) \end{cases} . \end{aligned}$$

The point is that all components $J \in \mathcal{C}(\partial\Sigma)$ are good for $(\Sigma, \widetilde{\Delta})$. So, the analysis of of Case 1 applies, and we we can construct the element

$$\zeta_{\widetilde{\Delta}} \in \bigotimes\{P_{\overline{k_i}} : 0 \leq i \leq b\}.$$

Finally, we define

$$\zeta_\Delta = \Big(\otimes\{\{id_{P_{\overline{k_i}}} : 0 \leq i \leq b_g\} \cup \{\beta_{k_i}^{-1} : b_g + 1 \leq i \leq b\}\}\Big)\, \zeta_{\widetilde{\Delta}} \tag{3.7}$$

Observe that $\zeta_\Delta \in V(\partial M)$ in this case too, if M, f, ϕ are defined as in Case 1.

Remark 3.2 *Suppose (Σ, Δ) is as in Case 2 above, suppose $\mathcal{C}_{b,\Delta}(\partial\Sigma) = \{J_0\}$ and $\mathcal{C}_{g,\Delta}(\partial\Sigma) = \{J_1, \cdots, J_b\}$, and suppose*

$$col(J_i, \Delta|_{J_i}) = \begin{cases} k_i & i = 0 \\ \overline{k_i} & 1 \leq j \leq b \end{cases}$$

As $\Pi = \emptyset$, we may, and do, assume that $\Sigma = A_b \subset S^2$. Suppose now that x is a point on S^2 which lies in that component of $S^2 \setminus J_0$ which does not meet Σ. Then we wish to note that the result of stereographically projecting (Σ, Δ), with x viewed as the north pole, is a planar tangle, say T, in the sense of Jones, and that $\delta^{k_0} Z(T) : \otimes\{P_{k_i} : 1 \leq i \leq b\} \to P_{k_0}$ and $\zeta_\Delta \in \otimes\{P_{\overline{k_i}} : 1 \leq i \leq b\} \coprod \{P_{k_0}\}$ correspond via the natural isomorphism between $L(\otimes\{P_{k_i} : 1 \leq i \leq b\}, P_{k_0})$ and $\otimes\big\{\{P_{k_i}^* : 1 \leq i \leq b\} \coprod \{P_{k_0}\}\big\}$.

(Reason: In order to compute ζ_Δ, we first 'make it good' which involves replacing $*_\Delta(J_0)$ by $(*+1)_\Delta(J_0)$, then stereographically projecting the result from some point on the surface to obtain a network, say $\mathcal{N}$ - which can be seen to be $tr_{k_0} \circ (M_{k_0} \circ_{D_2} T)$. Hence,

$$(Z(\mathcal{N}))(x_0 \otimes \cdots \otimes x_b) = \delta^{k_0} \tau_{k_0}(x_0(Z(T)(x_1 \otimes \cdots \otimes x_b)) \;;$$

this means that

$$\zeta_{\tilde{\Delta}} = \sum \ \delta^{k_0}\tau_{k_0}(e^{(0)}_{i_0}(Z(T)(e^{(1)}_{i_1} \otimes \cdots \otimes e^{(b)}_{i_b})) \ e^{i_0}_{(0)} \otimes \cdots \otimes e^{i_b}_{(b)} \ ,$$

where $\{e^{(t)}_{i_t}\}$ *denotes a basis for* P_{k_t} *and* $\{e^{i_t}_{(t)}\}$ *is the dual basis for* $P^*_{k_t}$.

Hence,

$$\begin{aligned} &\zeta_\Delta \\ &= \sum \ \delta^{k_0}\tau_{k_0}(e^{(0)}_{i_0}(Z(T))(e^{(1)}_{i_1} \otimes \cdots \otimes e^{(b)}_{i_b})) \quad \beta^{-1}_{k_0}(e^{i_0}_{(0)}) \otimes e^{i_1}_{(1)} \otimes \cdots \otimes e^{i_b}_{(b)} \\ &= \delta^{k_0} \ \sum (Z(T))(e^{(1)}_{i_1} \otimes \cdots \otimes e^{(b)}_{i_b}) \otimes e^{i_1}_{(1)} \otimes \cdots \otimes e^{i_b}_{(b)} \quad \textit{by eq.}(3.6) \ , \end{aligned}$$

as desired.)

Case 3: $\Pi \neq \emptyset$.

Fix a sufficiently small tubular neighbourhood U_Π of $\cup\{\gamma : \gamma \in \Pi\})$ whose boundary meets ℓ_Δ transversely. (The 'sufficiently small' requirement will ensure that our construction below will be independent of the choice of U_Π.) Then to each component $\Omega \in \mathcal{C}(\Sigma \setminus U_\Pi)$ - which is (diffeomorphic to) an A_b - we wish to specify a decoration $\Delta(\Omega)$. Let us write $\mathcal{C}_{new}$ for the set of those $J \in \mathcal{C}(\partial\Omega)$ for which $J \not\subset \partial\Sigma$, where $\Omega \in \mathcal{C}(\Sigma \setminus U_\Pi)$.

First note that, by 'restriction', the decoration Δ naturally specifies all ingredients of Δ_Ω with the exception of $*_{\Delta(\Omega)}(J)$ when $J \in \mathcal{C}_{new}$ (and, of course, $\Omega \in \mathcal{C}(\Sigma \setminus U_\Pi)$ is such that $J \in \mathcal{C}(\partial\Omega)$). Choose the family

$$\{*_{\Delta(\Omega)}(J) : J \in \mathcal{C}_{new}\}$$

subject only to the following conditions, but otherwise arbitrarily:

For each $\gamma \in \Pi$, let U_γ denote the component of U_Π which contains γ. Then, $\mathcal{C}(\partial(\Sigma \setminus U_\gamma)) \cap \mathcal{C}_{new} = \{J_1(\gamma), J_2(\gamma)\}$ (say). Suppose $J_i(\gamma) \in \mathcal{C}(\partial\Omega_i(\gamma))$ where $\Omega_i(\gamma) \in \mathcal{C}(\Sigma \setminus U_\Pi)$, for $i = 1, 2$. (Notice that $J_1(\gamma) \neq J_2(\gamma)$, although the $\Omega_i(\gamma)$ need not necessarily be distinct.) The conditions we demand are:

(i) if $\gamma \cap \ell_\Delta \neq \emptyset$, then the points $*_{\Delta(\Omega_i(\gamma))}(J_i(\gamma))$, $i = 1, 2$ must lie in the same connected component of $\overline{U_\gamma} \cap \ell_\Delta$; and

(ii) if $\gamma \cap \ell_\Delta = \emptyset$, then $\{*_{\Delta(\Omega_i(\gamma))}(J_i(\gamma)) : i = 1, 2\} = \{B, W\}$.

Let us write $V(\partial(\Omega, \Delta(\Omega))$ to denote the Hilbert space corresponding to (the element of *Obj* in the equivalence class) $[(\partial\Omega, \Delta(\Omega)|_{\partial\Omega})]$. Each $(\Omega, \Delta(\Omega))$ is a decorated 2-manifold to which we may apply the analysis of Case 2, to obtain a vector

$$\zeta_{\Delta(\Omega)} \ \in \ V(\partial(\Omega, \Delta(\Omega)) \ = \ \bigotimes\{P_{col(J,\Delta(\Omega)|_J)} : J \in \mathcal{C}(\partial\Omega)\} \ . \qquad (3.8)$$

Notice that

$$\begin{aligned}
&\bigotimes\{V(\partial(\Omega,\Delta(\Omega)) : \Omega \in \mathcal{C}(\Sigma \setminus U_\Pi)\} && (3.9)\\
&= \bigotimes\{P_{col(J,\Delta(\Omega)|_J)} : J \in \mathcal{C}(\partial\Omega),\ \Omega \in \mathcal{C}(\Sigma \setminus U_\Pi)\}\\
&= \bigotimes\{P_{col(J,\Delta(\Omega)|_J)} : J \in \mathcal{C}(\partial\Sigma) \coprod \mathcal{C}_{new}\}\ . && (3.10)
\end{aligned}$$

Our two conditions above imply that

$$\bigotimes\{P_{col(J,\Delta(\Omega)|_J)} : J \in \mathcal{C}_{new}\}$$

is an unordered tensor product with an even number of terms which naturally split off into pairs of the form $\{P_k, P_k^*\}$ for some $k \in C$; then the obvious 'contractions' result in a natural linear surjection of $\bigotimes\{V(\partial(\Omega,\Delta(\Omega)) : \Omega \in \mathcal{C}(\Sigma \setminus U_\Pi)\}$ onto $V(\partial(\Sigma,\Delta))$. Finally, define ζ_Δ to be the image, under this contraction, of

$$\kappa(\Delta,\Pi)\ \bigotimes\{\zeta_{\Delta(\Omega)} : \Omega \in \mathcal{C}(\Sigma \setminus U_\Pi)\}\ ,$$

where

$$\kappa(\Delta,\Pi)\ =\ \delta^{-\frac{1}{2}\ |\Pi\cap\ell_\Delta|}\ .$$

We need, now, to verify that the definition of ζ_Δ is independent of the choices available in the definitions of the $\Delta(\Omega)$'s. It should be clear that the constant $\kappa(\Delta,\Pi)$ is independent of the choices under discussion. There are two components to this verification:

(a) For a fixed $J(0) \in \mathcal{C}(\partial\Sigma)$, define the decoration $\widetilde{\Delta}_{J(0)}$ of Σ by demanding that

$$\begin{aligned}
\ell_{\widetilde{\Delta}_{J(0)}} &= \ell_\Delta\\
sh_{\widetilde{\Delta}_{J(0)}} &= sh_\Delta\\
\{*_{\widetilde{\Delta}_{J(0)}}(J)\} &= \begin{cases} \{*_\Delta(J)\} & \text{if } J \neq J(0)\\ (*+1)_\Delta(J(0)) & \text{if } J = J(0)\end{cases}\ .
\end{aligned}$$

The first of the two components above is the observation - which follows from equation 3.7 - that

$$\zeta_{\widetilde{\Delta}_{J(0)}}\ =\ \bigotimes\left(\{id_{P_{col(J,\Delta|_J)}} : J \in \mathcal{C}(\partial\Sigma \setminus \{J(0)\})\} \cup \beta_{col(J(0),\Delta|_{J(0)})}\right)\ \zeta_\Delta\ . \qquad (3.11)$$

(b) The second component is the fact that the following diagram commutes, for all $k \in Col$:

$$\begin{tikzcd}
P_k \otimes P_k^* \arrow[d, "\beta_k \otimes \beta_k^{-1}"'] \arrow[dr] & \\
P_k^* \otimes P_k \arrow[r] & \mathbb{C}
\end{tikzcd}$$

(This is nothing but a re-statement of the fact that τ_k is a trace - i.e., $\tau_k(xy) = \tau_k(yx) \; \forall \; x, y \in P_k$.)

Now, in order to verify that our definition of ζ_Δ is indeed independent of the choices present in the definition of the $\Delta(\Omega)$'s, it is sufficient to verify that two possible choices $\{\Delta_j(\Omega) : \Omega \in \mathcal{C}(\Sigma \setminus U_\Pi)\}, \; j = 1, 2$ yield the same ζ_Δ *provided* they are related in the following special manner:

there exists a $\gamma \in \Pi$ such that

$$*_{\Delta_1(\Omega)}(J) \;=\; \begin{cases} *_{\Delta_2(\Omega)}(J) & \text{if } J \neq J_i(\gamma), \; i = 1, 2 \\ (*+1)_{\Delta_2(\Omega_1(\gamma))}(J) & \text{if } J = J_1(\gamma) \\ (*-1)_{\Delta_2(\Omega_2(\gamma))}(J) & \text{if } J = J_2(\gamma) \end{cases} .$$

(Basically, we are saying here that it is enough to tackle one γ at a time and to move the $*$ point one step at a time.)

If the $\{\Delta_i(\Omega)\}$ are so related, notice that if $J \in \mathcal{C}(\partial\Omega), \; \Omega \in \mathcal{C}(\Sigma \setminus U_\Pi)$, then

$$col(J, \Delta_1(\Omega)|J) \;=\; \begin{cases} \overline{col(J, \Delta_2(\Omega)|J)} & \text{if } J \in \{J_i(\gamma) : i = 1, 2\} \\ col(J, \Delta_1(\Omega)|J) & \text{otherwise} \end{cases} .$$

If we define - see equation (3.9) -

$$\begin{aligned} V_j &= \bigotimes \{V(\partial(\Omega, \Delta_j(\Omega))) : \Omega \in \mathcal{C}(\Sigma \setminus U_\Pi)\} \\ &= \bigotimes \{P_{col(J, \Delta_j(\Omega)|_J)} : J \in \mathcal{C}(\partial\Sigma) \coprod \mathcal{C}_{new}\} . \end{aligned}$$

and the operator $A : V_1 \to V_2$ by $A = \otimes A_J$ where

$$A_J = \begin{cases} \beta_{col(J_j(\gamma), \Delta_1(\Omega)|_{J_j(\gamma)})} & \text{if } J = J_j(\gamma), j = 1, 2 \\ id_{P_{col(J, \Delta_1(\Omega)|_J)}} & \text{otherwise} \end{cases} ,$$

the definitions[4] are seen to imply, by equation (3.7), that

$$\otimes\{\zeta_{\Delta_2(\Omega)} : \Omega \in \mathcal{C}(\Sigma \setminus U_\Pi)\} \;=\; A(\otimes\{\zeta_{\Delta_1(\Omega)} : \Omega \in \mathcal{C}(\Sigma \setminus U_\Pi)\}) .$$

It is seen from our 'second component (b)' above that the images under the surjections (induced by the 'natural contractions') from the V_j's to $V(\partial(\Sigma, \Delta))$ of the vectors $\otimes\{\zeta_{\Delta_j(\Omega)} : \Omega \in \mathcal{C}(\Sigma \setminus U_\Pi)\}$ are the same; in other words, both the choices $\{\Delta_j(\Omega) : \Omega \in \mathcal{C}(\Sigma \setminus \Pi)\}$ give rise to the same vector ζ_Δ, as desired.

We emphasise that if $M = [(\Sigma, \Delta, \Pi, \phi_0, \phi_1)]$ is a morphism, then

(a) the Hilbert space $V(\partial M)$ depends only on $\partial(\Sigma, \Delta)$;

(b) the associated vector, which we have chosen to call ζ_Δ above, depends *à priori* on the planar decorated 2-manifold (Σ, Δ, Π), and is independent of the ϕ_j's.

Our next step is to prove the following proposition.

[4] We adopt the convention here that $\beta_{\bar{k}} = \beta_k^{-1}$ for $k \in C$.

Proposition 3.3 *If* $(\Sigma^{(i)}, \Delta^{(i)}, \Pi^{(i)}, \phi_0^{(i)}, \phi_1^{(i)}),\ i = 1,2$ *are pre-morphisms, if we let* $\zeta_i = \zeta_{\Delta_i}$ *denote the vectors associated, as above, to the triples* $(\Sigma^{(i)}, \Delta^{(i)}, \Pi^{(i)})$*, and if the above pre-morphisms are related by a move of type* $J,\ J \in \{I, II, III\}$*, then* $\zeta_1 = \zeta_2$*.*

Proof: We shall argue case by case.

Case (I): $J = I$

Let ϕ define the Type I move between the two pre-morphisms as in Definition 2.6 (i).

First consider the subcase where $\Pi_1 = \emptyset$. If $\phi = id$, there is nothing to prove; but in view of the observation (b) in the discussion of Case 1, we may choose the identification of Σ_1 with an A_b to be $\psi \circ \phi$, where $\psi : \Sigma_2 \to A_b$ is the chosen identification for Σ_1 , and hence reduce to the case $\phi = id$.

If $\Pi \neq \emptyset$, then, in the notation of Case 3, first choose tubular neighbourhoods U_{Π_j} so that $U_{\Pi_2} = \phi(U_{\Pi_1})$, then make choices to ensure that $\Delta_2(\phi(\Omega)) = \phi_*(\Delta_1(\Omega)),\ \forall \Omega \in \mathcal{C}(\Sigma_1 \setminus U_{\Pi_1})$, and then observe that the analysis of the last paragraph applies to each pair $(\Omega, \phi(\Omega)), \Omega \in \mathcal{C}(\Sigma_1 \setminus U_{\Pi_1})$, and finally contract to obtain the desired conclusion.

Case (III): $J = III$

It clearly suffices to treat the case when $\Pi_2 = \Pi_1 \coprod \{\gamma_0\}$, and of course $\Sigma^{(1)} = \Sigma^{(2)},\ \Delta^{(1)} = \Delta^{(2)},\ \phi_i^{(1)} = \phi_i^{(2)}$. To start with, we may assume that the tubular neighbourhoods $U_{\Pi_j}, j = 1,2$ are such that $U_{\Pi_2} = U_{\Pi_1} \coprod U_0$, where U_0 is a tubular neighbourhood of γ_0. Then there exists a unique component $\Omega_0 \in \mathcal{C}(\Sigma_1 \setminus U_{\Pi_1})$ such that $\gamma_0 \subset \Omega_0$. Next, if we choose $\Delta_1(\Omega) = \Delta_2(\Omega)\ \forall \Omega \neq \Omega_0$, it is clear that also $\zeta_{\Delta_1(\Omega)} = \zeta_{\Delta_2(\Omega)}\ \forall \Omega \neq \Omega_0$. So we only need to worry about Ω_0; equivalently, we may as well assume that $\Pi_1 = \emptyset,\ \Pi_2 = \{\gamma_0\}$.

In other words, we may assume that $\Sigma_1 = A_b$ for some b. Consider the subcase where all components $J \in \mathcal{C}(\partial\Sigma_1)$ are good, and are of colours, say, $\overline{k_0}, \overline{k_2}, \cdots, \overline{k_b}$. Choose any point $x \in \Sigma_1 \setminus (\partial\Sigma_1 \cup \ell_{\Delta_1} \cup \gamma_0)$ and let $\mathcal{N}$ be the planar network obtained by stereographically projecting Σ_1 onto the plane with x as the north pole. The partition function of $\mathcal{N}$ specifies a map $\otimes\{P_{k_i} : 0 \leq i \leq b\} \to \mathbb{C}$ and hence an element of $\otimes\{P_{\overline{k_i}} : 0 \leq i \leq b\}$ which is, by definition, ζ_{Δ_1}.

In order to compute ζ_{Δ_2}, we may assume that the boundary of U_0 meets ℓ_{Δ_2} transversally, and only at smooth points. Note that $\Sigma_2 \setminus U_0$ has exactly two components - one of which contains x and will be denoted by Ω_1 and the other by Ω_2. Choose decorations for Ω_1 and Ω_2 - by appropriately choosing $*_{\Delta(\Omega_1)}$ and $*_{\Delta(\Omega_2)}$ - such that all boundary components of Ω_1 are good while exactly one boundary component of Ω_2, namely the one - call it γ_0'' - which meets ∂U_0, is bad. Suppose that $col(\gamma_0', \Delta(\Omega_1)|_{\gamma_0'}) = \overline{k}$ - where, of course γ_0' denotes the boundary component of Ω_1 which meets ∂U_0. Then $col(\gamma_0'', \Delta(\Omega_2)|_{\gamma_0''}) = k$. Also suppose that Ω_1 contains the boundary components of Σ_2

with colours $\overline{k_{i_0}}, \cdots, \overline{k_{i_a}}$ while Ω_2 contains the boundary components of Σ_2 with colours $\overline{k_{i_{a+1}}}, \cdots, \overline{k_{i_b}}$ where $\{i_0, \cdots, i_b\} = \{1, \cdots, b\}$.

To compute $\zeta_{\Delta(\Omega_1)}$, stereographically project Ω_1 from x and call the planar network so obtained as $\mathcal{N}_1$. The partition function of $\mathcal{N}_1$ gives a map $Z(\mathcal{N}_1) : \otimes\{P_k\} \coprod \{P_{k_{i_t}} : 0 \leq t \leq a\} \to \mathbb{C}$ or equivalently the element $\zeta_{\Delta(\Omega_1)} \in \otimes\{P_{\overline{k}}\} \coprod \{P_{\overline{k_{i_t}}} : 0 \leq t \leq a\}$.

To compute $\zeta_{\Delta(\Omega_2)}$, stereographically project Ω_2 from x and observe that as in Remark 3.2, the result is a planar tangle say, T, and by that remark, $\zeta_{\Delta(\Omega_2)} \in \otimes\{P_k\} \coprod \{P_{\overline{k_{i_t}}} : a+1 \leq t \leq b\}$ and $\delta^k \; Z(T) : \otimes\{P_{k_{i_t}} : a+1 \leq t \leq b\} \to P_k$ are related by canonical isomorphisms between the spaces in which they live. Observe, on the other hand, that $\kappa(\Delta_2, \{\gamma_0\}) = \delta^{-k}$.

Now note that $\mathcal{N} = \mathcal{N}_1 \circ_{\gamma_0} T$; the basic property of a planar algebra then ensures that $Z(\mathcal{N}) = Z(\mathcal{N}_1) \circ (\otimes\{Z(T)\} \coprod \{id_{P_{k_{i_t}}} : 0 \leq t \leq a\})$. Finally chasing the three isomorphisms above - which relate the vectors ζ_{Δ_1}, $\zeta_{\Delta(\Omega_1)}$ and $\zeta_{\Delta(\Omega_2)}$ to the operators $Z(\mathcal{N})$, $Z(\mathcal{N}_1)$ and $\delta^k \; Z(\mathcal{T})$ respectively - shows that ζ_{Δ_2} which is $\kappa(\Delta_2, \{\gamma_0\})$ times the contraction of $\zeta_{\Delta(\Omega_1)}$ and $\zeta_{\Delta(\Omega_2)}$ is indeed equal to ζ_{Δ_1}; this finishes the proof in this subcase.

The case that not all boundary components of Σ_1 are good follows, on applying the conclusion in above subcase to the improved decoration $\widetilde{\Delta_1}$.

Case (II): $J = II$

First consider the case when $\Pi^{(1)} \cap \Pi^{(2)} = \emptyset$. In this case, put $\Pi = \Pi^{(1)} \cup \Pi^{(2)}$. Then by the already proved 'invariance of ζ under type III moves' we see that both $\zeta_i, i = 1, 2$ are equal to the ζ associated with the pre-morphism given by $(\Sigma^{(1)}, \Delta^{(1)}, \Pi, \phi_0^{(1)}, \phi_1^{(1)})$.

So, only the case when $\Pi^{(1)} \cap \Pi^{(2)} \neq \emptyset$ needs to be handled. For this case, we will need a couple of facts about transversality - namely Corollary 5.10 and Proposition 5.8 - both statements and proofs of which have been relegated to §5.

For notational simplicity, let us write $(\Sigma, \Delta) = (\Sigma^{(i)}, \Delta^{(i)}), i = 1, 2$ and $\Pi_0 = \Pi^{(1)}, \Pi_1 = \Pi^{(2)}$. Let us write $B_t = \cup\{\gamma : \gamma \in \Pi_t\}, t = 0, 1$ and $A = \ell_\Delta$. Thus, what we are given is that there exists a diffeotopy, say F, of Σ such that (i) $F_t(B_0) = B_t$, and (ii) B_t and A meet transversally, for $t = 0, 1$. Let us define $B_t = F_t(B_0)\ \forall t \in [0, 1]$.

Thanks to Proposition 5.8, we may even assume that B_t meets A transversally for $t \in D$, where D is a dense set in $[0, 1]$. For each $t \in D$, if we let $\Pi_t = \mathcal{C}(B_t)$, then it follows that $(\Sigma, \Delta, \Pi_t, \phi_0^{(i)}, \phi_1^{(i)})$ may be regarded as a pre-morphism. Let us write ζ_t for the vector associated to the premorphism $(\Sigma, \Delta, \Pi_t, \phi_0^{(i)}, \phi_1^{(i)})$.

First, choose a small tubular neighbourhood U of B_0. By definition, there is a diffeomorphism H of $B_0 \times [-1, 1]$ onto the closure $\bar{U}$ of U, such that $H(x, 0) = x,\ \forall x \in B_0$. Let $B' = H(B_0 \times \{1\})$. We assume that U has been chosen 'sufficiently small' as to ensure that B' meets A transversally.

We assert next that if d denotes any metric on Σ (which yields its topology), there exists an $\epsilon > 0$ such that

$$d(x,y) \geq \epsilon, \ \forall \ x \in F_t(B_0), \ y \in F_t(B'), \forall t \in [0,1].$$

(*Reason:* If not, we can find a sequence $(t_n, x_n, y_n) \in [0,1] \times B_0 \times B'$ such that $d(F_{t_n}(x_n), F_{t_n}(y_n)) \leq \frac{1}{n}$ for all n. In view of the compactness present, we may - pass to a subsequence, if necessary, and - assume that there exists $(t,x,y) \in [0,1] \times B_0 \times B'$ such that $(t_n, x_n, y_n) \to (t,x,y)$; but this implies that $B_0 \cap B' \neq \emptyset$, thus arriving at the contradiction which proves the assertion.)

By arguing in a very similar manner to the reasoning of the last paragraph, we find that there exists $\eta > 0$ so that

$$|t_1 - t_2| < \eta \Rightarrow d(F_{t_1}(x), F_{t_2}(x)) < \epsilon/2 \ \forall x \in B.$$

Next, we may choose points $0 = t_0 < t_1 < t_2 < ... < t_k = 1$ so that (i) $|t_i - t_{i+1}| < \eta \forall i$, and (ii) each t_i belongs to the dense set D described a few paragraphs earlier.

Notice that our construction ensures that $F_{t_i}(B')$ does not intersect either B_{t_i} or $B_{t_{i+1}}$, for each i. Now it may be the case that $F_{t_i}(B')$ does not meet A transversally; in that case, we may appeal to Corollary 5.10 to deduce that there is a nearby curve, say B'_t - within $\epsilon/2$ - that is isotopic to $F_{t_i}(B')$ and intersects A transversally. Now, the curve B'_t gives rise to a premorphism and the associated vector, say ζ'_t agrees with both ζ_{t_i} and $\zeta_{t_{i+1}}$ by the reasoning of the first paragraph in the discussion of this case. Finally, we conclude that $\zeta_0 = \zeta_1$, as desired. □

We have thus associated a vector ζ_Δ to a pre-morphism $(\Sigma, \Delta, \Pi, \phi_0, \phi_1)$ which depends only on the morphism defined by that pre-morphism. Hence, if M denotes the morphism $[(\Sigma, \Delta, \Pi, \phi_0, \phi_1)]$, we may unambiguously write ζ_M for this vector ζ_Δ; by definition, we have

$$\zeta_M \in V(\partial M) \ . \tag{3.12}$$

Lemma 3.4 *For any morphism M, we have*

$$\zeta_{\bar{M}}(\xi) \ = \ \langle \xi, \zeta_M \rangle \ , \ \forall \ \xi \in V(\partial M) \ . \tag{3.13}$$

Proof: Assume that $M = [(\Sigma, \Delta, \Pi, \phi_0, \phi_1)]$, so that $\bar{M} = [(\overline{\Sigma}, \overline{\Delta}, \overline{\Pi}, \phi_1, \phi_0)]$.

We consider two cases.

Case 1: $\Pi = \emptyset$.

We may assume that $\mathcal{C}_{g,\Delta}(\partial\Sigma) = \{J_0, \cdots, J_a\}$ and $\mathcal{C}_{b,\Delta}(\partial\Sigma) = \{J_{a+1}, \cdots, J_b\}$. Let us write $\tilde{\Delta}$ for the 'improved decoration' as before. Then the components of $\partial\Sigma$ have colours $\bar{k}_0, \cdots, \bar{k}_b$ (say) according to $\tilde{\Delta}$.

We pause to make some notational conventions regarding orthonormal bases. Choose an orthonormal basis $\{e^{(t)}_{i_t} : 1 \leq i_t \leq \ dim\ P_{k_t}\}$ for P_{k_t} and let $\{e^{i_t}_{(t)} : 1 \leq i_t \leq \ dim\ P_{k_t}\}$ be the dual (also orthonormal) for ${P_{k_t}}^*$; thus,

$$e^{i_t}_{(t)}(\cdot) \ = \ \langle \cdot, e^{(t)}_{i_t} \rangle \ = \ \beta({e^{(t)}_{i_t}}^*) \ . \tag{3.14}$$

Note that also $\{{e^{(t)}_{i_t}}^* : 1 \leq i_t \leq \ dim\ P_{k_t}\}$ is an orthonormal basis for P_{k_t}; let $\{\phi^{i_t}_{(t)} : 1 \leq i_t \leq \ dim\ P_{k_t}\}$ be the dual (orthonormal) for ${P_{k_t}}^*$, and note, as in eq. (3.14) that

$$\phi^{i_t}_{(t)} \ = \ \beta(e^{(t)}_{i_t}) \ . \tag{3.15}$$

Finally, given multi-indices $\mathbf{i} = (i_0, \cdots, i_a), \mathbf{j} = (j_{a+1}, \cdots, j_b)$, we shall write $e_{\mathbf{i}} = \otimes^a_{t=0} e^{(t)}_{i_t}$, $\phi^{\mathbf{i}} = \otimes^a_{t=0} \phi^{i_t}_{(t)}$, $e^*_{\mathbf{i}} = \otimes^a_{t=0} {e^{(t)}_{i_t}}^*$, $e_{\mathbf{j}} = \otimes^b_{t=a+1} e^{(t)}_{j_t}$, $e^{\mathbf{j}} = \otimes^b_{t=a+1} e^{j_t}_{(t)}$, and $e^{\mathbf{j}*} = \otimes^b_{t=a+1} {e^{j_t}_{(t)}}^*$

By definition, in order to compute ζ_M, we need to first compute $\zeta_{\tilde{\Delta}}$; and for this, we need to stereographically project $(\Sigma, \tilde{\Delta})$ from a point x (in $\Sigma \setminus (\partial\Sigma \cup \ell_\Delta)$) to obtain a planar network, call it $\mathcal{N}$; then

$$\zeta_{\tilde{\Delta}} \ = \ \sum_{\mathbf{i},\mathbf{j}} Z(\mathcal{N})(e^*_{\mathbf{i}} \otimes e_{\mathbf{j}}) \ \phi^{\mathbf{i}} \otimes e^{\mathbf{j}} \ ,$$

and hence, by equations (3.14) and (3.15)

$$\zeta_M = \zeta_\Delta = \sum_{\mathbf{i},\mathbf{j}} Z(\mathcal{N})(e^*_{\mathbf{i}} \otimes e_{\mathbf{j}}) \ \phi^{\mathbf{i}} \otimes e^*_{\mathbf{j}} \ .$$

In order to compute $\zeta_{\bar{M}}$, we need to compute what we had earlier called $\zeta_{\bar{\Delta}}$, for which we first need to compute $\zeta_{\tilde{\bar{\Delta}}}$. For this, we observe that if we project $(\Sigma, \tilde{\bar{\Delta}})$ from the same point x, we obtain the planar network $\mathcal{N}^*$ (which is the adjoint of the network $\mathcal{N}$, in the sense of [Jon]); hence, as before,

$$\zeta_{\tilde{\bar{\Delta}}} \ = \ \sum_{\mathbf{i},\mathbf{j}} Z(\mathcal{N}^*)(e_{\mathbf{i}} \otimes e^*_{\mathbf{j}}) \ e^{\mathbf{i}} \otimes \phi^{\mathbf{j}} \ ,$$

whence

$$\zeta_{\bar{\Delta}} \ = \ \sum_{\mathbf{i},\mathbf{j}} Z(\mathcal{N}^*)(e_{\mathbf{i}} \otimes e^*_{\mathbf{j}}) \ e^*_{\mathbf{i}} \otimes \phi^{\mathbf{j}} \ .$$

Hence,

$$\begin{aligned} \langle \phi^{\mathbf{i}} \otimes e^*_{\mathbf{j}}, \zeta_M \rangle \ &= \ \overline{Z(\mathcal{N})(e^*_{\mathbf{i}} \otimes e_{\mathbf{j}})} \\ &= \ Z(\mathcal{N}^*)(e_{\mathbf{i}} \otimes e^*_{\mathbf{j}}) \\ &= \ \zeta_{\bar{\Delta}}(\phi^{\mathbf{i}} \otimes e^*_{\mathbf{j}}) \ , \end{aligned}$$

where, in the third line above, we have used the fact that if T is a planar tangle, then

$$Z(T)(\otimes x_i)^* \;=\; Z(T^*)(\otimes x_i^*)\ .$$

As $\phi^{\mathbf{i}} \otimes e^*_{\mathbf{j}}$ ranges over a basis for $(\otimes_{i=0}^{a} P^*_{k_i}) \otimes (\otimes_{i=a}^{b} P_{k_i}) = V(\partial M)$, the proof of the Lemma, in this case, is complete.

The proof in the other case will appeal to the following easily proved fact.

Assertion: Let $\mathcal{H}$ and $\mathcal{K}$ be finite dimensional Hilbert spaces. Let $C_{\mathcal{K}} : \mathcal{H}\otimes\mathcal{K}\otimes\mathcal{K}^* \to \mathcal{H}$ and $C_{\mathcal{K}^*} : \mathcal{H}^* \otimes \mathcal{K}^* \otimes \mathcal{K} \to \mathcal{H}^*$ be the natural contraction maps. Then, for any $\zeta \in \mathcal{H}\otimes\mathcal{K}\otimes\mathcal{K}^*$, the equality $C_{\mathcal{K}^*}(\ \langle\cdot,\zeta\rangle\ ) = \langle\cdot, C_{\mathcal{K}}(\zeta)\rangle$ holds.

Case 2: $\Pi \neq \emptyset$.

In this case, let a tubular neighbourhood U_Π of $\cup\{\gamma : \gamma \in \Pi$ be chosen. For each component Ω of $\Sigma \setminus U_\Pi$, we may choose $\bar{\Delta}(\Omega) = \overline{\Delta(\Omega)}$; an application of Case 1 to this piece results in the equality

$$\zeta_{\bar{\Delta}(\Omega)}(\cdot) \;=\; \langle\cdot, \zeta_{\Delta(\Omega)}\rangle\ . \tag{3.16}$$

For each $\gamma \in \Pi$, (as before) let $\{J_1(\gamma), J_2(\gamma)\} = \{J \in \mathcal{C}(\partial(\Sigma \setminus U_\gamma)) : J \not\subset \partial\Sigma\}$ Now choose $\mathcal{H} = V(\partial(\Sigma,\Delta))$ and $\mathcal{K} = \bigotimes\{P_{col(J_1(\gamma),\Delta(\Omega)|_{J_1(\gamma)})} : \gamma \in \Pi\}$. Then $\mathcal{K}^*$ is naturally identified with $\bigotimes\{P_{col(J_2(\gamma),\Delta(\Omega)|_{J_2(\gamma)})} : \gamma \in \Pi\}$.

Let $\zeta \in \mathcal{H}\otimes\mathcal{K}\otimes\mathcal{K}^*$ be $\bigotimes\{\zeta_{\Delta(\Omega)} : \Omega \in \mathcal{C}(\Sigma\setminus U_\Pi)\})$. Then, by definition, $\zeta_M \;=\; \kappa(\Delta,\Pi)\, C_{\mathcal{K}}(\zeta)$ while equation 3.16 shows that $\zeta_{\bar{M}} \;=\; \kappa(\bar{\Delta},\bar{\Pi})\ C_{K^*}(\ \langle\cdot,\zeta\rangle\ )$. As $\kappa(\Delta,\Pi) \;=\; \kappa(\bar{\Delta},\bar{\Pi})$, an appeal to the foregoing 'Assertion' finishes the proof in this case, and hence of the Lemma. □

DEFINITION 3.5 *Given a morphism $M = [(\Sigma,\Delta,\Pi,\phi_0,\phi_1)]$, let $Z_0(M)$ be the operator from $V(X_{f_0})$ to $V(X_{f_1})$ which corresponds, under the natural isomorphism of $L(V(X_{f_0}),V(X_{f_1}))$ with $V(X_{f_0})^*\otimes V(X_{f_1})(\;=\;V(\partial M))$, to ζ_M; finally define*

$$Z_M \;=\; \delta^{-\frac{1}{4}|\partial\Sigma\cap\ell_\Delta|}\ Z_0(M)\ .$$

REMARK 3.6 *If $M = [(\Sigma,\Delta,\Pi,\phi_0,\phi_1)]$, the operator Z_M defined above is independent of ϕ_0,ϕ_1 in the sense that if $M' = [(\Sigma,\Delta,\Pi,\phi'_0,\phi'_1)]$ (corresponds to another possible splitting up of $\partial\Sigma$), then the operators Z_M and $Z_{M'}$ correspond under the natural identification*

$$L(V(X_{f_0}),V(X_{f_1})) \;=\; V(\partial M)) \;=\; V(\partial M')) \;=\; L(V(X_{f'_0}),V(X_{f'_1}))\ .$$

This is true because of two observations: (i) this statement is true for $Z_0(M)$ and $Z_0(M')$ by virtue of the remarks made in the paragraph - see (b) - preceding Proposition 3.3; and (ii) the powers of δ appearing in the definition of $Z(M)$ and $Z(M')$ are the same.

THEOREM 3.7 *The foregoing prescription defines a* **unitary TQFT** *on* $\mathcal{D}$ *-by which we mean that:*

(a) The association given by

$$\begin{aligned} Obj(\mathcal{D}) \ni X_f &\mapsto V(X_f) \\ Mor(\mathcal{D}) \ni M &\mapsto Z_M \end{aligned}$$

defines a functor V *from the category* $\mathcal{D}$ *to the category* $\mathcal{H}$ *of finite-dimensional Hilbert spaces.*

(b) The functor V *carries 'disjoint unions' to 'unordered tensor products'.*

(c) The functor V *is 'unitary' in the sense that it is 'adjoint-preserving'.*

Proof: The verification of (b) is straightforward.

(a) For verifying the identity requirement of a functor, we only need, in view of (b), to verify that $Z_{id_{X_{\mathbf{k}}}} = id_{V(X_{\mathbf{k}})}$ for all $k \in Col$, where $\mathbf{k}$ is as defined in the next section.

Consider first the case of $k \in C$. For this, begin by observing that $id_{X_{\mathbf{k}}}$ is (see the paragraph preceding Proposition 2.7) the class of the morphism, with $f_0 = f_1 = k$, given by what is called the 'identity tangle' in [Jon] and denoted by I_k^k in [KS1]. It is then seen from Definition 3.5 and Remark 3.2 that

$$Z_{id_{X_{\mathbf{k}}}} = \delta^{-k} Z_0(id_{X_{\mathbf{k}}}) = Z(I_k^k) = id_{V(X_{\mathbf{k}})} \tag{3.17}$$

as desired.

For the case when $\bar{k} \in C$, notice that $id_{X_{\mathbf{k}}}$ is the class of the morphism, with $f_0 = f_1 = \bar{k}$, given by I_k^k. An appeal to Remark 3.6 and the already proved equation (3.17) proves that $Z_{id_{X_{\mathbf{k}}}} = id_{V(X_{\mathbf{k}})}$.

To complete the proof of (a), we needs to check that the functor is well-behaved with respect to compositions. So, suppose $M' = [(\Sigma', \Delta', \Pi', \phi_0', \phi_1')]$ and $M'' = [(\Sigma'', \Delta'', \Pi'', \phi_0'', \phi_1'')]$, and that $\phi_1' = \phi_0''$. The definitions show that $\zeta_{M'' \circ M'}$ is equal to a scalar multiple - $\delta^{-\frac{1}{2}|\ell_{\Delta'} \cap im(\phi_1')|}$ - of the contraction of $\zeta_{M''} \otimes \zeta_{M'}$ along $V(X_{f_1'}) \otimes V(X_{f_0''})^*$. In other words,

$$Z_0(M'' \circ M') = \delta^{-\frac{1}{2}|\ell_{\Delta'} \cap im(\phi_1')|} Z_0(M'') \circ Z_0(M').$$

We hence deduce that

$$\begin{aligned} Z_{M'' \circ M'} &= \delta^{-\frac{1}{4}(|\ell_{\Delta'} \cap im(\phi_0')| + |\ell_{\Delta''} \cap im(\phi_1'')|)} Z_0(M'' \circ M') \\ &= \delta^{-\frac{1}{4}(|\ell_{\Delta'} \cap im(\phi_0')| + |\ell_{\Delta'} \cap im(\phi_1')|) + |\ell_{\Delta''} \cap im(\phi_0'')|) + |\ell_{\Delta''} \cap im(\phi_1'')|)} Z_0(M'') \circ Z_0(M') \\ &= Z(M'') \circ Z(M') \end{aligned}$$

thereby completing the proof of (a).

As for (c), if $M = [(\Sigma, \Delta, \Pi, \phi_0, \phi_1)]$, then

$$
\begin{aligned}
Z_M : V(X_{f_0}) \to V(X_{f_1}) \quad &, \quad \zeta_M \in V(X_{f_0})^* \otimes V(X_{f_1}) \\
Z_{\bar{M}} : V(X_{f_1}) \to V(X_{f_0}) \quad &, \quad \zeta_{\bar{M}} \in V(X_{f_0}) \otimes V(X_{f_1})^* \ .
\end{aligned}
$$

Let $\{e_i\}_i$ and $\{f_j\}_j$ denote orthonormal bases for $V(X_{f_0})$ and $V(X_{f_1})$ respectively, and let $\{e^i\}_i$ and $\{f^j\}_j$ denote their dual orthonormal bases for $V(X_{f_0})^*$ and $V(X_{f_1})^*$ respectively.

If we write $d = \delta^{-\frac{1}{4}|\ell_\Delta \cap \partial\Sigma|}$, then we see, thanks to Lemma 3.4, that for arbitrary indices k, l,

$$
\begin{aligned}
\langle f_k, Z_M(e_l) \rangle &= \langle e^l \otimes f_k, \sum_i e^i \otimes Z_M(e_i) \rangle \\
&= \langle e^l \otimes f_k, d\zeta_M \rangle \\
&= d\zeta_{\bar{M}}(e^l \otimes f_k) \\
&= \left(\sum_j Z_{\bar{M}}(f_j) \otimes f^j \right) (e^l \otimes f_k) \\
&= e^l(Z_{\bar{M}}(f_k)) \\
&= \langle Z_{\bar{M}}(f_k), e_l \rangle \ ,
\end{aligned}
$$

thereby ending the proof of (c). □

4 From TQFTs on $\mathcal{D}$ to subfactors

This section is devoted to an 'almost' converse to Theorem 3.7. Suppose, then, that we have a 'unitary TQFT' defined on $\mathcal{D}$. In the notation of Remark , let us write $P_k = V(X_{\mathbf{k}})$ for $k \in Col$.

The aim of this section is to prove the following result:

Theorem 4.1 *If V is a unitary TQFT defined on $\mathcal{D}$, then V arises from a subfactor planar algebra P as in Theorem 3.7 - with P_k as above, for $k \in C$ - if and only if the following conditions are met:*

$$P_{0_\pm} = \mathbb{C} \ \ \textit{and} \ \ P_1 \neq \{0\} \ .$$

Further, the TQFT determines the subfactor planar algebra uniquely.

We shall prove this theorem by making/establishing a series of observations/assertions.

(0) If V is constructed from of a subfactor planar algebra as in Theorem 3.7, then the conditions displayed above are indeed met.

(1) If Σ is any object - in a cobordism category on which a TQFT V has been defined - then $V(\bar{\Sigma})$ is naturally identified with (the dual space) $V(\Sigma)^*$ in such a way that if M is a morphism with $\partial M = \bar{\Sigma}_1 \coprod \Sigma_2 = \bar{\Sigma}_3 \coprod \Sigma_4$, then the associated linear maps in $Hom(V(\Sigma_1), V(\Sigma_2))$ and $Hom(V(\Sigma_3), V(\Sigma_4))$ correspond via the isomorphism

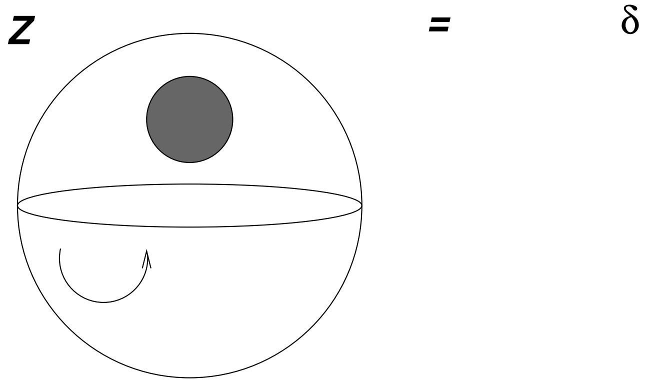
$$Hom(V(\Sigma_1), V(\Sigma_2)) \cong V(\partial M) \cong Hom(V(\Sigma_3), V(\Sigma_4)) \tag{4.18}$$

(This is a consequence of the *self-duality* theorem in [Tur].)

(2) $P_{\bar{k}} = P_k^* \; \forall k \in Col$.

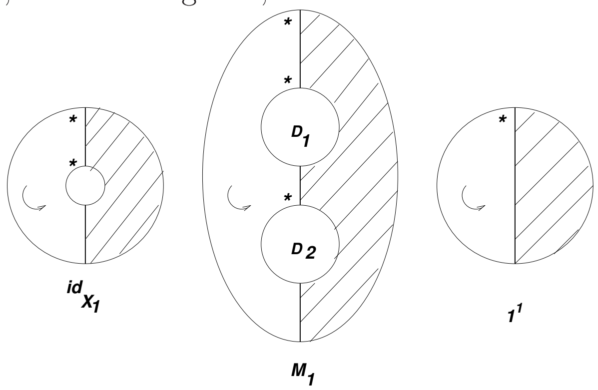
(This follows immediately from (1).)

(3) There exists a positive number δ such that



where the picture on the left corresponds to the morphism given by $M_0 = [(\Sigma_0, \Delta, \Pi, \phi_0, \phi_1)]$, with Σ being the 2-sphere with the orientation indicated in the picture, ℓ_Δ consisting of one circle with interior shaded black, Π consisting of one circle which may be taken as the equator, $\phi_0, \phi_1 : \emptyset \to \Sigma_0$.

Reason : Observe first that the identity morphism id_{X_1}, the 'multiplication tangle' M_1, and the tangle $1^{\mathbf{1}}$, which are illustrated in the following picture



satisfy the relation:

$$M_1 \circ_{D_1} 1^{\mathbf{1}} = id_{X_{\mathbf{1}}}$$

This implies that if we write $A = Z_{1^{\mathbf{1}}}$, then $A \neq 0$ (since $Z_{id_{X_{\mathbf{1}}}} = id_{P_1} \neq 0$).

By definition, $A : \mathbb{C}(= V(\emptyset)) \to P_1$; therefore, we may deduce that, with $\delta = \|A\|^2$, we have

$$\begin{aligned}
\delta\ id_{\mathbb{C}} &= A^*A \\
&= (Z_{1^{\mathbf{1}}})^* Z_{1^{\mathbf{1}}} \\
&= Z_{\overline{1^{\mathbf{1}}}} Z_{1^{\mathbf{1}}} \\
&= Z_{\overline{1^{\mathbf{1}}} \circ 1^{\mathbf{1}}} \\
&= Z_{M_0}\ .
\end{aligned}$$

(3') Assertion (3) remains valid, even when the shading in the figure illustrated in its statement is reversed so that the interior of the small disc is shaded white and the exterior black.

Reason: This is because we may use a diffeotopy so that the small circle with black interior is bloated up so as to fill up the exterior of a small circle antipodal to the given circle, and a subsequent rotation would change the resulting picture to the one where the interior of the circle is shaded white.

(4) For $k \in \ Col$, define

$$|k| = \begin{cases} 0 & \text{if } k = 0_\pm \\ k & \text{if } k \in C \setminus \{0_+, 0_-\} \\ m & \text{if } k = \bar{m}.m \in C \end{cases}$$

and for $f \in \mathcal{F}$, $\ f \in \mathcal{F}$, define

$$|X_f| = \sum_{k \in \ Col} f(k)|k|\ ;$$

and, finally, for any morphism $M \in Hom(X_{f_0}, X_{f_1})$, define

$$Z(M) = \delta^{\frac{(|X_{f_0}| - |X_{f_1}|)}{2}}\ Z_M.$$

(5) Each planar tangle T - as in Remark 3.2 - may be viewed naturally as a morphism from $X_{\mathbf{k_1}} \coprod \cdots \coprod X_{\mathbf{k_b}}$ to $X_{\mathbf{k_0}}$. (Here and in the sequel, when we regard a planar tangle as a morphism (with $\Pi = \emptyset$), we shall always assume that the orientation of the underlying planar surface is the usual - anti-clockwise - one.) Observe, then, that

$$Z(T) = \delta^{\frac{\sum_{i=1}^{b} k_i - k_0}{2}} Z_{T_1} \;\in\; Hom(\otimes_{i=1}^{b} P_{k_i}, P_{k_0}) \; . \tag{4.19}$$

Then the collection $P = \{P_k : k \in C\}$ has the structure of a planar algebra (in the sense of the definition in [KS1]) if the multilinear operator associated to a planar tangle T is defined as $Z(T)$ (as above). This planar algebra is connected and has modulus δ (in the terminology of [KS1]). In particular, each $P_k, k \in C$ is a unital associative algebra.

Reason : Since our tensor products are unordered, it is fairly clear that the association of operator to planar tangle is well-behaved with respect to 're-numbering of the internal discs' of the tangle. It will be convenient to adopt the convention of using a 'subscript 1' to indicate the pre-morphism asssociated to a planar tangle; so the morphism associated to the planar tangle T is denoted by T_1.

We need to check that the association of operator to planar tangle is well-behaved with respect to composition. So suppose T (resp. S) is a planar tangle with b (resp. m) internal discs $D_1, \cdots, D_b$ (resp. $C_1, \cdots, C_m$) of colours $k_1, \cdots, k_b$ (resp. $l_1, \cdots, l_m$) respectively, and with external disc of colour k_0 (resp. k_i), for some $1 \leq i \leq b$. Then the 'composition' $T \circ_{D_i} S$ is a tangle with internal discs $D_1, \cdots, D_{i-1}, C_1, \cdots, C_m, D_{i+1}, \cdots, D_b$, which is obtained by 'sticking S into the i-th disc of T'.

Let S' denote the pre-morphism given by

$$S' = \left(\coprod_{j=1}^{i-1} id_{X_{\mathrm{j}}}\right) \coprod S_1 \coprod \left(\coprod_{j=i+1}^{b} id_{X_{\mathrm{j}}}\right) \; .$$

Then, the pre-morphism $(T \circ_{D_i} S)_1$ corresponding to the tangle $T \circ_{D_i} S$ is equivalent to the pre-morphism given by

$$(T \circ_{D_i} S)_1 = T_1 \circ S' \; .$$

Note that we need 'equivalent' in the precedinmg sentence, since the pre-morphism given by the composition on the right has b circles in its planar decomposition while the one on the left side has none, but since both sides describe planar pieces, all these extra circles may be ignored using 'Type III moves'.) Hence,

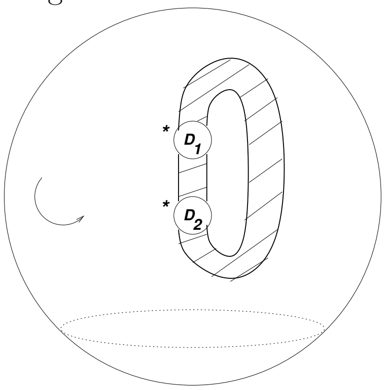
$$\begin{aligned} Z(T \circ_{D_i} S) &= \delta^{\frac{\sum_{j=1}^{i-1} k_j + \sum_{p=1}^{m} l_p + \sum_{q=i+1}^{b} k_q - k_0}{2}} Z_{(T \circ_{D_i} S)_1} \\ &= \delta^{\frac{\sum_{j=1}^{i-1} k_j + \sum_{p=1}^{m} l_p + \sum_{q=i+1}^{b} k_q - k_0}{2}} Z_{T_1} \circ Z_{S'} \\ &= \left(\delta^{\frac{\sum_{j=1}^{b} k_j - k_0}{2}} Z_{T_1}\right) \circ \left(\delta^{\frac{\sum_{p=1}^{m} l_p - k_i}{2}} Z_{S_1}\right) \\ &= Z(T) \circ \otimes(\{id_{P_{k_j}} : j \neq i\} \cup \{Z_S\}) \; , \end{aligned}$$

thus establishing that P is indeed a planar algebra with respect to the specified structure. The 'connected'-ness of this algebra is the statement that $P_{0_\pm} = \mathbb{C}$, while the assertion about 'modulus δ' is the content of assertions (3) and (3') above.

(6) (This assertion has a version for each $k \in C$, but for convenience of illustration and exposition, we only describe the case $k = 2$.)

Let tr_2 denote the pre-morphism, with $\Pi = \emptyset$, given by the illustration below - with $\Sigma = A_1$, ℓ_Δ consisting of four curves each connecting a point on D_1 to a point on D_2, and the shading as illustrated below:

tr_2 =

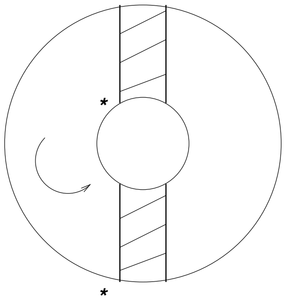


Then Z_{tr_2} is a non-degenerate normalised trace τ_2 on P_2.

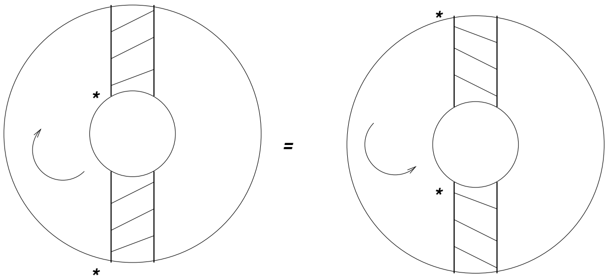
(For general k, there will be $2k$ strings joining D_1 and D_2, with the region immediately to the north-east of the *'s being black as in the picture. In the case of 0_+ (resp., $)_-$), the entire A_1 is shaded white (resp., black).

Reason : Consider the pre-morphism S_2 given by the decorated 2-manifold illustrated below, with $\Pi = \emptyset$, with the ϕ_j so chosen that $Z_{S_2} : P_2 \to P_{\bar{2}}$:

S_2 =



It is then seen that the adjoint pre-morphism is given as below, also with $\Pi = \emptyset$, so $Z_{\bar{S}_2} : P_{\bar{2}} \to P_2$:

$\bar{S}_2 =$ 

It is seen from the diagrams that

$$\bar{S}_2 \circ S_2 = id_{X_{\mathbf{2}}} \ , \ S_2 \circ \bar{S}_2 = id_{X_{\bar{\mathbf{2}}}} \ .$$

It follows that Z_{S_2} is unitary, and in particular invertible. However, it is a consequence of observation (1) above that

$$\left(Z_{S_2}(X)\right)(y) = Z_{tr_2}(xy) = \tau_2(xy) \ .$$

Non-degeneracy of τ_2 is a consequence of the invertibility of Z_{S_2}. The fact that τ_2 is a trace is easily verified.

(7) The inner-product and the non-degeneracy of τ_k - in (5) above - imply the existence of an invertible, conjugate-linear mapping $P_k \ni x \mapsto x^* \in P_k$ via the equation

$$\tau_k(xy) = \langle y, x^* \rangle \ \forall x, y \in P_k \ .$$

(8) For any planar tangle T, as in (4) above, and all $x_i \in P_{k_i}, 1 \leq i \leq b$, we have:

$$(Z_T(\otimes\{x_i : 1 \leq i \leq b\})^* \ = \ Z_{T^*}(\otimes\{x_i^* : 1 \leq i \leq b\}) \ ,$$

where the adjoint tangle T^* is defined as in [Jon] or [KS1].

In particular, we also have

$$(Z(T)(\otimes\{x_i : 1 \leq i \leq b\})^* \ = \ Z(T^*)(\otimes\{x_i^* : 1 \leq i \leq b\}) \ ,$$

Reason It clearly suffices yto prove that

$$\langle x_0, (Z_T(\otimes\{x_i : 1 \leq i \leq b\})^* \rangle \ = \ \langle x_0, Z_{T^*}(\otimes\{x_i^* : 1 \leq i \leq b\}) \rangle \ , \ \forall x_0 \in P_{k_0}$$

or equivalently that

$$\tau_k(x_0 Z_T(\otimes\{x_i : 1 \leq i \leq b\}) \ = \ \langle Z_{\overline{T^*}}(x_0), \otimes\{x_i : 1 \leq i \leq b\} \rangle \ . \tag{4.20}$$

To start with, we need to observe that

$$\overline{T^*} = S_{k_0} \circ T \circ \coprod_{j=1} \overline{S}_{k_j} \tag{4.21}$$

(This is because: T^* is obtained from the planar tangle T by rotating the $*$ on the boundary of the internal discs anti-clockwise to the next point, the $*$ on the boundary of the external disc clockwise to the next point, and then applying an orientation reversing map to it; while we need to only apply an orientation reversal to form the 'bar' of a morphism; so that the left side of equation 4.21 is obtained by just rotating the $*$'s in the manner indicated above. On the other hand, the result of 'pre-multiplying' by an S serves merely to 'rotate the external $*$ anticlockwise by one', while 'post-multiplying' by a disjoint union of the $\overline{S}$ serves merely to 'rotate the internal $*$'s anti-clockwise by one'.)

If the $x_i, 0 \leq i \leq b$, are as in equation (4.20), let us define

$$f_i = Z_S(x_i) = \langle \cdot, x_i^* \rangle \ ;$$

since $\overline{S_i}$ is 'inverse' to S_i, this means $x_i = Z_{\overline{S}}(f_i)$. Next, we may appeal to equations (4.18) and (4.21) to deduce that

$$\begin{aligned}
\langle Z_{\overline{T^*}}(x_0), \otimes\{x_i : 1 \leq i \leq b\} \rangle &= Z_{\overline{T^*}}(\otimes_{i=1}^b f_i)(x_0) \\
&= (Z_S \circ Z_T(\otimes_{i=1}^b x_i))(x_0) \\
&= \tau_k(Z_T(\otimes_{i=1}^b x_i) x_0) \ ,
\end{aligned}$$

as desired.

As for the final statement, it follows from the already established assertion and the fact that the tangles T and T^* have the same $k_1, \cdots, k_b; k_0$ data.

(9) The following special case of (8) above is worth singling out:

$$(xy)^* = y^* x^* \ , \forall x, y \in P_k \ ;$$

and hence, $1^* = 1$, where we simply write 1 for the identity 1^k of P_k.

Reason : $M_k^* = M_k^{op}$; and the identity in an algebra is unique.

(10) $\tau_k(x^*) = \overline{\tau_k(x)}$, $\forall\ x \in P_k$.

Reason :

$$\begin{aligned}
\tau_k(x) &= \langle 1, x^* \rangle \\
&= \overline{\langle x^*, 1 \rangle} \\
&= \overline{\langle x^*, 1^* \rangle} \\
&= \overline{\tau_k(x^*)} \ .
\end{aligned}$$

(11) $x^{**} = x \ , \forall x \in P_k$.

Reason :

$$\begin{array}{rcl} \tau_k(x^{**}y^*) & = & \tau_k((yx^*)^*) \quad \text{by (9)} \\ & = & \overline{\tau_k(yx^*)} \quad \text{by (10)} \\ & = & \overline{\tau_k(x^*y)} \\ & = & \overline{\langle x^*, y^* \rangle} \\ & = & \langle y^*, x^* \rangle \\ & = & \tau_k(xy^*) \ , \end{array}$$

and the non-degeneracy of τ_k completes the proof.

(12) The left-regular representation λ of the (unital) algebra is a (faithful) $*$-homomorphism from P_k into $\mathcal{L}(P_k)$.

Reason : For all $a, x, y \in P_k$, we have:

$$\begin{array}{rcl} \langle \lambda(a)x, y \rangle & = & \langle ax, y \rangle \\ & = & \tau_k(y^*(ax)) \\ & = & \tau_k((a^*y)^*x) \\ & = & \langle x, a^*y \rangle \\ & = & \langle x, \lambda(a^*)y \rangle \ , \end{array}$$

thereby establishing (by the non-degeneracy of the inner-product) that $\lambda(a)^* = \lambda(a^*)$.

(13) P_k is a C^*-algebra (with respect to $*$ being given by (7)), and tau_k is a faithful tracial state on P_k; further, P_k is in *standard form* with respect to τ_k.

(14) P is a subfactor planar algebra, and the TQFT associated to it by Theorem 3.7 is nothing but V.

(15) Only the uniqueness of the subfactor planar algebra remains in order to complete that proof of Theorem 4.1. Suppose a subfactor planar algebra P gives rise to a TQFT V as in Theorem 3.7. Then, note that $P_k = V(X_{\mathbf{k}})$, $\forall\ k \in C$, that the index δ^2 of the subfactor is determined by the TQFT (as seen by step (3)), and that the operator $Z(T)$ associated to a planar tangle is determined by δ and Z_{T_1} - see equation (4.19). □

Remark 4.2 *It is true - and a consequence of the main result of [KS2] - that a TQFT which arises, as in §3, from a subfactor planar algebra is determined uniquely by the numerical invariant it asociates to 'closed cobordisms'. This is in spite of the fact that* *these TQFTs are, in general* **not**[5] *cobordism-generated; so the truth of the last sentence is not a consequence of a similar result - see [Tur] or [BHMV], for instance - which is applicable to 'cobordism generated TQFTs'. In fact, the methods of [KS2] can be used to show that, under some minimal conditions on the cobordism category where it is defined, any unitary TQFT is determined by the numerical invariant it associates to 'closed cobordisms'.*

5 Topological Appendix

This section is devoted to the proof of some facts which are needed in earlier proofs. We have relegated these proofs to this 'Appendix' so as to not interrupt the flow of the treatment in the body of the paper.

5.1 Glueing 'classes'

This subsection is devoted to establishing a fact - Lemma 5.2 - which is needed in what we termed 'Step 2' in the process of defining composition of morphisms.

LEMMA 5.1 *Let $\delta > 0$, and $0 < 2\epsilon < \frac{1}{2}$ be given. Then there exists a smooth function:*

$$\mu : [\delta, 1] \times [0, 1] \to [0, \infty)$$

satisfying:

(i) $\mu|_{[\delta,1]\times[\frac{1}{2},1]} \equiv 1.$

(ii) $\mu|_{[\delta,1]\times[0,\epsilon]} \equiv 0.$

(iii) $\mu|_{[\delta,1]\times(\epsilon,1]} > 0.$

(iv) $\int_0^{1/2} \mu(a, x)dx = a.$

Proof: Choose a smooth function $\lambda : [0, 1] \to [0, 1]$ such that $\lambda \equiv 0$ on $[0, \epsilon]$, $\lambda \equiv 1$ on $[1/2, 1]$, $\lambda > 0$ on $(\epsilon, 1]$, and $\int_0^{1/2} \lambda(x)d(x) = \delta$, where δ is as in the hypothesis.

Now choose a smooth function $\rho : [0, 1] \to [0, \infty)$ such that $\operatorname{supp} \rho \subset [\epsilon, 1/2]$, and $\int_0^{1/2} \rho(x)dx = 1$. Consider the function:

$$\begin{array}{rcl} \mu : [\delta, 1] \times [0, 1] & \to & [0, \infty) \\ (t, x) & \mapsto & \lambda(x) + (t - \delta)\rho(x) \end{array}$$

[5] For instance, in the case of the subfactor of fixed points under the outer action of a finite group G, the ζ_M's, for M in $Mor(\emptyset, X_{\mathbf{2}})$ turn out to be elements of $\mathbb{C}G$ which are fixed by all inner automorphisms of G, and hence do not span all of $P_2 = \mathbb{C}G$, in case G is non-abelian.

That μ is smooth is clear, as are the assertions (i),(ii) (iii) of the lemma. For the fourth, note that

$$\int_0^{1/2} \mu(a,x)dx = \int_0^{1/2} \lambda(x)dx + (a-\delta)\int_0^{1/2} \rho(x)dx = \delta + (a-\delta) = a$$

and the proof of the lemma is complete. $\square$

LEMMA 5.2 *Let $\phi : S^1 \times [0,1] \to S^1 \times [0,1]$ be a diffeomorphism which preserves orientation as well as the ends $S^1 \times \{0\}$ and $S^1 \times \{1\}$. Let $\{e^{ia_j}\}_{j=1}^m$ be a finite set of marked points on S^1, where $0 \leq a_j \leq 2\pi$. Assume that $\phi(\{e^{ia_j}\} \times [0,1])$ is contained in (and hence, equal to) $\{e^{ia_j}\} \times [0,1]$ for all j. Then there exists an $\epsilon > 0$ and an orientation preserving diffeomorphism $\psi : S^1 \times [0,1] \to S^1 \times [0,1]$ satisfying:*

(i) $\psi(\omega,t) = (\omega,t)$ *for all* $t \in [1/2,1]$ *and* $\omega \in S^1$.

(ii) $\psi(\omega,t) \equiv \phi(\omega,t)$ *for all* $t \in [0,\epsilon]$ *and all* $\omega \in S^1$.

(iii) $\psi(\{e^{ia_j}\} \times [0,1]) \subset \{e^{ia_j}\} \times [0,1]$ *for all* $j = 1,..,m$.

Proof: Write the diffeomorphism ϕ in terms of its components as:

$$\phi(\omega,t) = (\rho(\omega,t), \sigma(\omega,t))$$

Note that $\omega \mapsto \rho(\omega,0)$ is an orientation preserving diffeomorphism of S^1, which fixes the points e^{ia_j} for all $j = 1,\cdots,m$. Also $\sigma(\omega,0) = 0$ and $\partial_t\sigma(\omega,0) > 0$ for all $\omega \in S^1$ and $t \in [0,1]$. We may therefore choose $\epsilon > 0$ so small as to ensure the validity of (a)-(c) below:

(a) $0 < 2\epsilon < 1/2$.

(b) For $t \in [0,2\epsilon]$, the first projection map $\omega \mapsto \rho(\omega,t)$ is an orientation preserving diffeomorphism of S^1 which fixes the points e^{ia_j} for all $j = 1,..,m$. (This is because of the hypothesis on ϕ and because the set of diffeomorphisms is open in $C^\infty_{str}(S^1,S^1) = C^\infty_w(S^1,S^1)$.)

(c) $\sigma(\omega,t) < 1/2$ and $\partial_t\sigma(\omega,t) > 0$ for all $t \in [0,2\epsilon]$ and all $\omega \in S^1$.

Let $\lambda : [0,1] \to [0,1]$ be a smooth function such that $\lambda \equiv 1$ on $[0,\epsilon]$ and $\lambda \equiv 0$ on $[2\epsilon,1]$. Consider the smooth function:

$$a(\omega) := \frac{1}{2} - \int_0^{1/2} \lambda(t)\partial_t\sigma(\omega,t)dt, \quad \omega \in S^1$$

Note that $a(\omega) < \frac{1}{2}$ for all ω; also since $\lambda(t)\partial_t\sigma(\omega,t) \le \partial_t\sigma(\omega,t)$, and $\lambda \equiv 0$ on $[2\epsilon, 1]$, we have, for all $\omega \in S^1$,

$$\begin{aligned} a(\omega) &= \frac{1}{2} - \int_0^{1/2} \lambda(t)\partial_t\sigma(\omega,t)dt = \frac{1}{2} - \int_0^{2\epsilon} \lambda(t)\partial_t\sigma(\omega,t)dt \\ &\ge \frac{1}{2} - \int_0^{2\epsilon} \partial_t\sigma(\omega,t)dt = \frac{1}{2} - \sigma(\omega,2\epsilon) > 0 \end{aligned}$$

by (c) above. Since S^1 is compact and $a(\omega)$ is a smooth function of ω, there exists a $\delta > 0$ such that $a(\omega) > \delta$ for all $\omega \in S^1$.

To sum up, we find that $\omega \mapsto a(\omega)$ is a smooth function from S^1 to $[\delta, 1/2]$. Now consider the function:

$$\begin{aligned} S : S^1 \times [0,1] &\to [0,\infty) \\ (\omega, s) &\mapsto \int_0^s \left(\lambda(t)\partial_t\sigma(\omega,t) + \mu(a(\omega),t)\right) dt \end{aligned}$$

where μ is the smooth function obtained as in Lemma 5.1 - with δ, ϵ as in this proof. We have the following facts about the map S:

(d) S is smooth, and $S(\omega,s)$ is strictly monotonically increasing in s for all $\omega \in S^1$.

The smoothness is clear from the definition of S. Furthermore, for all $\omega \in S^1$ and $t \in [0,\epsilon]$ the integrand is identically $\partial_t\sigma(\omega,t)$ (by (ii) of Lemma 5.1 above) which is strictly positive (by item (c) above). For all $\omega \in S^1$ and $t \in (\epsilon, 1]$, the integrand is $\ge \mu(a(\omega),t)$, which is again strictly positive on $(\epsilon,1]$ (by (iii) of Lemma 5.1 above). Hence $S(\omega,s)$ is strictly increasing in s for all $\omega \in S^1$.

(e) $S(\omega,0) \equiv 0$ for all $\omega \in S^1$. Also $S(\omega,s) \equiv \sigma(\omega,s)$ for $s \in [0,\epsilon]$ and all $\omega \in S^1$.

The definition of S implies $S(\omega,0) \equiv 0$ for all ω. Since $\lambda(s) \equiv 1$ and $\mu(a(\omega),s) \equiv 0$ for $s \in [0,\epsilon]$ (by (ii) of the Lemma 5.1), we have $S(\omega,s) = \int_0^s \partial_t\sigma(\omega,t)dt = \sigma(\omega,s)$ for all $s \in [0,\epsilon]$ and all $\omega \in S^1$, and the second assertion follows.

(f) $S(\omega,s) \equiv s$ for $s \in [1/2,1]$ and all $\omega \in S^1$. In particular, $S(\omega,1) \equiv 1$ for all $\omega \in S^1$.

For this assertion, first note that:

$$\begin{aligned} S(\omega,1/2) &= \int_0^{1/2} \lambda(t)\partial_t\sigma(\omega,t)dt + \int_0^{1/2} \mu(a(\omega),t)dt \\ &= \int_0^{1/2} \lambda(t)\partial_t\sigma(\omega,t)dt + a(\omega) \quad \text{(by (iv) of lemma 5.1)} \\ &= 1/2 \quad \text{(by the definition of } a(\omega)\text{)} \end{aligned}$$

while for $t \geq 1/2$, we have $\lambda(t) \equiv 0$ and $\mu(a(\omega), t) \equiv 1$ (by (i) of Lemma 5.1), so that

$$\begin{aligned} S(\omega, s) &= S(\omega, 1/2) + \int_{1/2}^{s} \partial_t S(\omega, t) dt \\ &= \frac{1}{2} + \int_{1/2}^{s} \mu(a(\omega), t) dt \\ &= \frac{1}{2} + \int_{1/2}^{s} dt \\ &= s \quad \text{for} \quad s \in [1/2, 1] \end{aligned}$$

(g) $S(0) = 0$, $S(1) = 1$, and $S(\omega, -)$ maps $[0, 1]$ diffeomorphically to $[0, 1]$ for all $\omega \in S^1$.

This last assertion is clear from (d), (e), and (f).

Next, the (restricted) map $\rho : S^1 \times [0, 2\epsilon] \to S^1$ may be lifted to a map (of universal covers)

$$\widetilde{\rho} : \mathbb{R} \times [0, 2\epsilon] \to \mathbb{R}$$

such that

(h) each $\widetilde{\rho}(-, s)$ is a diffeomorphism of $\mathbb{R}$ to itself satisfying:

$$\widetilde{\rho}(x + 2n\pi, s) = \widetilde{\rho}(x, s) + 2n\pi \quad \text{for all} \quad x \in \mathbb{R}, \quad s \in [0, 2\epsilon] \ ;$$

and

(i) $\widetilde{\rho}(a_j, s) = a_j$ for all $j = 1, \cdots, m$.

Both these assertions follow from item (b) above.

In terms of the maps $\lambda, \widetilde{\rho}$ defined above, now define a mapping as follows:

$$\begin{aligned} \widetilde{R} : \mathbb{R} \times [0, 1] &\to \mathbb{R} \\ (x, s) &\mapsto \lambda(s)\widetilde{\rho}(x, s) + (1 - \lambda(s))x \end{aligned}$$

and check that:

(j) $\widetilde{R}(x + 2n\pi, s) = \widetilde{R}(x, s) + 2n\pi$ for all $s \in [0, 1]$ and all $x \in \mathbb{R}$.

Since $\widetilde{\rho}(-, s)$ is an orientation preserving diffeomorphism of $\mathbb{R}$, we may deduce that $\partial_x \widetilde{\rho}(x, s) > 0$ for all s and all x. Hence

(k) For all $s \in [0,1]$ and $x \in \mathbb{R}$,

$$\partial_x \widetilde{R}(x,s) = \lambda(s)\partial_x\widetilde{\rho}(x,s) + (1-\lambda(s)) > 0$$

(l) $\widetilde{R}(a_j, s) = \lambda(s)a_j + (1-\lambda(s))a_j = a_j$ for all $j = 1, \cdots, m$ and all $s \in [0,1]$.

(m) Since $\lambda(s) \equiv 1$ for $s \in [0,\epsilon]$, we have $\widetilde{R}(x,s) = \widetilde{\rho}(x,s)$ for $x \in [0,\epsilon]$ and all $x \in \mathbb{R}$.

(n) Since $\lambda(s) \equiv 0$ for $s \in [2\epsilon, 1]$, we have $\widetilde{R}(x,s) = x$ for $s \in [2\epsilon, 1]$ and all $x \in \mathbb{R}$.

It follows from (j) above that the map $\widetilde{R}$ descends to a map:

$$\begin{aligned} R : S^1 \times [0,1] &\rightarrow S^1 \\ (e^{ix}, s) &\mapsto (e^{i\widetilde{R}(x,s)}) \end{aligned}$$

Furthermore

(o) $R(e^{ia_j}, s) = e^{ia_j}$ for $j = 1, .., m$.

This follows from item (l) above.

(p) $R(\omega, s) \equiv \rho(\omega, s)$ for all $s \in [0,\epsilon]$.

This follows from item (m) above.

(q) $R(\omega, s) = \omega$ for all $s \in [2\epsilon, 1]$.

This follows from item (n) above.

(r) $R(-, s)$ is an orientation preserving diffeomorphism of S^1 for all $s \in [0,1]$.

This is clear from the items (b) and (p) above for $s \leq \epsilon$, and from item (q) above for $s \geq 2\epsilon$. For $s \in [\epsilon, 2\epsilon]$, it follows from item (k) above, and noting that $\widetilde{\rho}(-,s)$ and $1_{\mathbb{R}}$ both map the fundamental interval $[0, 2\pi)$ diffeomorphically to itself, and hence so does their convex combination $\widetilde{R}(-,s)$.

Finally we define the map:

$$\begin{aligned} \psi : S^1 \times [0,1] &\rightarrow S^1 \times [0,1] \\ (\omega, s) &\mapsto (R(\omega,s), S(\omega,s)) \end{aligned}$$

That ψ is an orientation diffeomorphism follows from items (g) and (r) above. The assertion (i) of the lemma follows from items (f) and (q) above since $2\epsilon < 1/2$. The assertion (ii) of the lemma follows from items (e) and (p) above. The assertion (iii) of the lemma follows from items (o) and (g) above. The lemma is proved. $\square$

5.2 On transversality

This subsection is devoted to proving some facts concerning transversality - especially Proposition 5.8 and Corollary 5.10 - which are needed in verifying - in §3 (see the proof of Case (II) of Proposition 3.4) - that the association $M \to \zeta_M$, of vector to morphism, is unambiguous.

DEFINITION 5.3 *Let M be a smooth manifold, possibly with boundary ∂M, and $I = [0,1]$. Let B be a submanifold of M, with $\partial B = B \cap \partial M$ if B has a boundary (i.e. B is a "neat" submanifold). Let $i_B : B \hookrightarrow M$ denote the inclusion. A smooth map $f : B \times I \to M$ is called an* isotopy *of i_B if each $f_t := f(\cdot, t) : B \to M$ is a closed embedding and if $f_0 = i_B$.*

In case $B = M$, and f is an isotopy of $f_0 = i_B = Id_M$, we call f a diffeotopy *of M.*

If M is non-compact, we say a diffeotopy f is compactly supported *if there exists a compact subset $K \subset M$ such that $f_t(x) \equiv x$ for all $x \in M \setminus K$ and all $t \in [0,1]$.*

LEMMA 5.4 *(*Transversality Lemma*) Let M° be a manifold without boundary and let A° be a submanifold which is a closed subset, also without boundary (both are allowed to be non-compact). Let N be a smooth manifold, possibly having boundary ∂N. Let $f : N \to M$ be a smooth map. Suppose*

$$\partial f := f|_{\partial N} : \partial N \to M^\circ$$

is transverse to A°. Then there exists an open ball S around the origin in some Euclidean space, and a map:

$$G : N \times S \to M^\circ$$

such that:

(i) *G is a submersion.*

(ii) *Writing $G(\cdot, s) = G_s$, we have $\partial G_s := G_s|_{\partial N}$ is identically equal to ∂f for all s.*

(iii) *$G_0 = f$ on N.*

Proof: See the proof of the Extension Theorem on pp. 72, 73 of [GuPo], and substitute $Y = M^\circ$, $X = N$, $C = \partial N$, and $Z = A^\circ$. The G they construct is the G of this lemma. $\square$.

PROPOSITION 5.5 *(Modifying an isotopy of a submanifold keeping ends fixed)*

Let M° be a smooth manifold without boundary (possibly non-compact) , and A° (also possibly non-compact) a smooth submanifold of M° which is a closed subset. Let B be *any compact manifold* without boundary*, and let* $f : B \times I \to M^\circ$ *be a smooth map. Assume:*

$$\partial f := f_0 \cup f_1 : (B \times \{0\}) \cup (B \times \{1\}) = \partial(B \times I) \to M^\circ$$

is transverse to A° *(This is equivalent to saying* $f_t(B) \pitchfork A^\circ$ *for* $t = 0, 1$*). Then there exists an open ball* S *around the origin in some Euclidean space, and a smooth map* $G : B \times I \times S \to M^\circ$ *such that:*

(i) G *is a submersion.*

(ii) *Write* $G_s := G(\cdot, \cdot, s)$*, and let* ∂G_s *denote the restriction of* G_s *to* $\partial(B \times I) = (B \times \{0\}) \cup (B \times \{1\})$*. Then* $\partial G_s = \partial f$ *for all* $s \in S$*.*

(iii) $G_0 \equiv f$ *on* $B \times I$*.*

(iv) *If* B *is a compact boundaryless submanifold of* M°*, and* $f : B \times I \to M^\circ$ *an isotopy of the inclusion map* i_B *of* B *in* M° *(see Definition 5.3), then by* shrinking S to a smaller open ball if necessary*, we have* $G_s : B \times I \to M^\circ$ *is also an isotopy for all* $s \in S$*, with* $G_s|_{B\times\{0\}} = f_0 = i_B$ *and* $G_s|_{B\times\{1\}} = f_1$ *for all* $s \in S$*.*

Proof: In the previous Lemma 5.4, take $N = B \times I$. Then the hypotheses here imply that ∂f on ∂N is transverse to A°, and (i), (ii) and (iii) follow from parts (i) (ii) and (iii) of the said Lemma 5.4.

We need to prove the assertion (iv). To show it, we need to show that $G_{s|B\times\{t\}}$ is an embedding for all $t \in [0, 1]$ and all s in a possibly smaller open ball S around 0. First define the map:

$$\begin{aligned} H : I \times S &\to C^\infty_{str}(B, M^\circ) \\ (t, s) &\mapsto G((\cdot, t), s) \end{aligned}$$

where the right side is the complete metric space of smooth maps from B to M°, with the strong topology[6]. (See Theorem 4.4 on p. 62 and the last para of p. 35 of [Hir].) (The topology implies $g_n \to g$ iff derivatives of all orders of the sequence g_n converge uniformly to the corresponding derivatives of g on B). Using the fact that B is compact, and that there are Lipschitz constants available for each derivative $D^\alpha G$ over all of the compact set $B \times I \times \overline{S}$ from the smoothness of G, it is easy to check that H defined above is continuous.

By Theorem 1.4 on p. 37 of [Hir], the subspace $\mathrm{Emb}(B, M^\circ)$ of smooth embeddings of B into M° is an open subset of $C^\infty_{str}(B, M^\circ)$. Hence $U := H^{-1}(\mathrm{Emb}(B, M^\circ))$ is an

[6] The strong and weak topologies coincide since B is compact

open subset of $I \times S$. Since $H(t, 0) = G((\cdot, t), 0) = f_t$ is an embedding for each t by the hypothesis that f is an isotopy, it follows that $I \times \{0\} \subset U$. By the compactness of I, there exists a smaller open ball $S' \subset S$ such that $I \times S' \subset U$. It follows that $H(I \times S') \subset \mathrm{Emb}(B, M^\circ)$, i.e. that $G((\cdot, t), s)$ is an embedding for all $t \in I$ and all $s \in S'$. This means $G_s : B \times I \to M^\circ$ is an isotopy for each $s \in S'$. Since $G_s(x, 0) \equiv f_0(x) = i_B$, and $G_s(x, 1) \equiv f_1(x)$ for all $s \in S$ and all $x \in B$ by (ii) above, (iv) follows and the proposition is proved. $\square$.

Corollary 5.6 *Let M° be a manifold without boundary, A^0 a boundaryless submanifold which is a closed subset, and $B \subset M$ a compact submanifold without boundary. Let an isotopy*

$$f : B \times I \to M^\circ$$

of $i_B : B \hookrightarrow M^\circ$ be given. Assume that $\partial f :=: B \times \{0\} \cup B \times \{1\} \to M^\circ$ is transverse to A° (viz. $f_t(B) \pitchfork A^\circ$ for $t = 0, 1$). Then there exists another isotopy $\widetilde{f} : B \times I \to M^\circ$ such that:

(i) *$\partial \widetilde{f} = \partial f$, (viz. $\widetilde{f}_0 = f_0 = i_B$ and $\widetilde{f}_1 = f_1$, i.e. the ends of the isotopy are left unchanged).*

(ii) *$\widetilde{f} : B \times I \to M^\circ$ is transverse to A°.*

(iii) *The map $\tilde{f}_t : B \to M^\circ$ is transverse to A° for almost all $t \in I$ (in particular for t in a dense subset of I).*

Proof: By (ii), (iii) and (iv) of the previous proposition 5.5, there is an open ball S in some Euclidean space, and a smooth map $G : B \times I \times S \to M^\circ$ such that ∂G_s is identically ∂f for all s, each $G_s : B \times I \to M^\circ$ is an isotopy, and $G_0 : B \times I \to M^\circ$ is the given isotopy f.

Since by (i) of proposition 5.5, G is a submersion, G is transversal to A°. Since $\partial G_s = \partial f$ for each $s \in S$, and ∂f is transverse to A° by hypothesis, it follows that

$$\partial G_s : \partial(B \times I) = B \times \{0\} \cup B \times \{1\} \to M^\circ$$

is already transverse to A° for each $s \in S$, so *a fortiori*

$$\partial G : (B \times \{0\} \cup B \times \{1\}) \times S \to M^\circ$$

is transverse to A°. By the Transversality Theorem on P. 68 of [GuPo] (this time substitute $Y = M^\circ$, $X = B \times I$, $Z = A^\circ$ and $F = G$ in said theorem), for a dense set of $s \in S$ the map $G_s : B \times I \to M^\circ$ is transverse to A°. Choose one such s, and define $\widetilde{f} := G_s$.

Hence $\widetilde{f} = G_s : B \times I \to M^\circ$ is transverse to A°. $\widetilde{f}$ is an isotopy by the first para, and $\partial\widetilde{f} = \partial G_s = \partial f$. This shows (i) and (ii).

Now again apply the aforementioned Transversality theorem on p. 68 of [Gu-Po] to $\widetilde{f}$ (with $(0,1)$ substituted for S, M° for Y, $\widetilde{f}$ for F and A° for Z) to conclude (iii). This proves the corollary. $\square$

REMARK 5.7 *We note that since S is convex, each G_s is homotopic to G_0, so the map $\widetilde{f}$ constructed above is actually homotopic to the given isotopy f (rel $B_0 \cup B_1$). We do not need this fact, however.*

PROPOSITION 5.8 *Let M be a compact manifold, with possible boundary ∂M. Let $B \subset M$ be a compact boundaryless submanifold which is a closed subset of M and disjoint from ∂M, and A a submanifold of M which is neat (i.e. with $\partial A = A \cap \partial M$). Let $F : M \times I \to M$ be a diffeotopy of M with $F_0(B) = B$ meeting A transversally, and $F_1(B) \pitchfork A$. Then there exists another diffeotopy $\widetilde{F} : M \times I \to M$, and a compact subset $K \supset B$ with $K \cap \partial M = \phi$ such that:*

(i) *$\widetilde{F}(x,t) \equiv x$ for all t and all $x \in M \setminus K$.*

(ii) *$\widetilde{F}_{0|B} = F_{0|B} = i_B$ and $\widetilde{F}_{1|B}(x,1) = F_{1|B}$ (i.e. the starting and finishing maps of the original diffeotopy remain unchanged on B).*

(iii) *$\widetilde{F}_t(B) \pitchfork A$ for almost all $t \in I$ (in particular for t in a dense subset of I).*

Proof: Note that each F_t is a diffeomorphism of M, and hence $F_t(\partial M) \subset (\partial M)$ for all t. Thus $B \cap \partial M = \phi$ implies that $F_t(B) \cap \partial M = \phi$ for all $t \in I$. Thus $F(B \times I) \subset M \setminus \partial M$.

Let us denote $M^\circ := M \setminus \partial M$, a non-compact manifold without boundary, and $A^\circ := A \setminus \partial A = A \cap M^\circ$, which is a submanifold of M° and a closed subset of it. Let $f : B \times I \to M^\circ$ denote the restriction of F to $B \times I$. Then, by the hypotheses on F, we have f is an isotopy of $i_B : B \hookrightarrow M^\circ$, and

$$\partial f := f_{|\partial(B \times I)} : B \times \{0\} \cup B \times \{1\} \to M^\circ$$

is transverse to A°. Now we apply the Corollary 5.6 to get a *new* isotopy:

$$\widetilde{f} : B \times I \to M^\circ$$

such that $\partial\widetilde{f} \equiv \partial f$, that is $\widetilde{f}_0 = f_0 = i_B$ and $\widetilde{f}_1 = f_1$ and $\widetilde{f}_t \pitchfork A^\circ$ for almost all $t \in I$.

By the Isotopy Extension Theorem (see Theorem 1.3 on p. 180 of [Hir]), there exists a diffeotopy:

$$\widetilde{F} : M^\circ \times I \to M^\circ$$

such that (i) $\widetilde{F}$ agrees with $\widetilde{f}$ on $B\times I$ (substitute $M^\circ = M$ and B for V in that theorem), and (ii) $\widetilde{F}$ is *compactly supported*, viz., there is a compact subset $K \subset M^\circ$ containing B such that $\widetilde{F}_t(x) \equiv x$ for all $x \in M^\circ \setminus K$ and all t (see the Definition 5.3).

Since $\widetilde{F}_t$ is stationary for all times outside the compact set K, we may define $\widetilde{F}(x,t) \equiv x$ for all $x \in \partial M$, and this extends $\widetilde{F}$ to M smoothly. (i) follows since $\widetilde{F}$ is supported in K. (ii) and (iii) follow because $\widetilde{F} = \widetilde{f}$ on $B \times I$. $\square$

PROPOSITION 5.9 *Let M° be a (possibly non-compact) manifold without boundary, and $A^\circ \subset M$ be a smooth submanifold which is a closed subset. Let $B \subset M^\circ$ be a* compact *smooth submanifold of M° without boundary, and let $i_B : B \hookrightarrow M^\circ$ denote the inclusion. Then there exists an isotopy $f : B \times [0,1] \to M^\circ$ such that:*

(i) $f_0 = i_B$.

(ii) $f_t : B \to M^\circ$ *is an embedding for each* t.

(iii) $f_1(B) \pitchfork A^\circ$.

(iv) *Letting d denote a Riemannian distance in M°, and given $\epsilon > 0$ any positive number, we can arrange that f_1 is an ϵ-approximation to i_B, that is:*

$$\sup_{x\in B} d(f_1(x), x) < \epsilon \ .$$

Proof: Substituting $N = B$ in the Lemma 5.2 above, we have an open ball S in some Euclidean space and a smooth map:

$$G : B \times S \to M^\circ$$

with $G_0 = i_B$ and G a submersion. Since $G_0 = i_B$ is an embedding, we may consider (as in the proof of (iv) of Prop. 5.3 above) the continuous map:

$$\begin{aligned} H : S &\to C^\infty_{str}(B, M^\circ) \\ s &\mapsto G_s \end{aligned}$$

Using the compactness of B, the consequent fact that the strong and weak topologies on $C^\infty(B, M^\circ)$ coincide, and the fact that $\mathrm{Emb}(B, M^\circ)$ is an open subset of $C^\infty_{str}(B, M^\circ)$, we can again shrink S if necessary to guarantee that $G_s : B \to M^\circ$ is an embedding for all $s \in S$ (as we did in the proof of Prop. 5.3 above). Indeed, given $\epsilon > 0$, we can take S to be a δ-ball such that the distance (in the metric on $C^\infty_{str}(B, M^\circ)$, see (a) of Theorem 4.4 on p. 62 of [Hir] and the fact that the weak and strong topologies coincide since B

is compact) between $H(s) = G_s$ and $G_0 = i_B$ is less than ϵ for $s \in S$. By the definition of this strong (=weak) topology it will follow that:

$$\sup_{x \in B}(G_s(x), x) < \epsilon \quad \text{for all} \quad s \in S$$

By the Transversality Theorem on p.68 of [Gu-Po], there exists a $\mu \in S$ such that $G_\mu : B \to M^\circ$ is an embedding transversal to A°. Define:

$$\begin{aligned} f : B \times [0,1] \quad &\to \quad M^\circ \\ (x,t) \quad &\mapsto \quad G(x, t\mu) \end{aligned}$$

(That is, we are defining f to be the restriction of G to the radial ray joining $0 \in S$ to $\mu \in S$.) Then clearly $f_0 = G_0 = i_B$ and $f_1(B) = G_\mu(B)$ meets A° transversally, and (i) and (iii) follow. Since G_s is an embedding for all $s \in S$ by the last para, we have each f_t is an embedding, and (ii) follows. The statement (iv) follows from the last line of the previous paragraph. So f is the required isotopy. $\square$

Corollary 5.10 *Let M be a manifold with boundary ∂M, and $A \subset M$ a neat submanifold with $\partial A = A \cap \partial M$. Let B be a compact boundaryless submanifold of M lying inside $M^\circ := M \setminus \partial M$. Then there exists a diffeotopy $F : M \times I \to M$ such that:*

(i) $F_0 = Id_M$.

(ii) *There exists a compact $K \subset M^\circ$, with $K \supset B$ such that $F_t(x) \equiv x$ for all $x \in M \setminus K$.*

(iii) $F_1(B) \pitchfork A$.

(iv) *For a fixed Riemannian metric d on M, and given $\epsilon > 0$,*

$$\sup_{x \in B} d(F_1(x), x) < \epsilon$$

Proof: Consider the noncompact manifold without boundary $M^\circ = M \setminus \partial M$, and set $A^\circ := A \setminus \partial A$.

By the Proposition 5.7 above, there is an isotopy:

$$f : B \times I \to M^\circ$$

with $f_0 = i_B$, and $f_1(B) \pitchfork A^\circ$. By the Isotopy Extension Theorem (Theorem 1.3 on p. 80 of [Hir]), there exists a compactly supported diffeotopy $\widetilde{F} : M^\circ \times I \to M^\circ$ such that $\widetilde{F}_0 = Id_{M^\circ}$ and $\widetilde{F}_t(x) \equiv x$ for all $x \in M^\circ \setminus K$ and all $t \in I$ (for some compact neighbourhood K of B in M°, and such that $\widetilde{F}_{|B \times I} = f$.

Since $K \subset M^\circ$, we may clearly extend $\widetilde{F}$ to $M \times I$ by setting $F_t(x) \equiv x$ for all t and all $x \in \partial M$ (as we did in the proof of Prop. 5.6 above), and this is the required diffeotopy. Since $F_1 = f_1$ on B, (iv) follows from (iv) of Proposition 5.9 above. $\square$

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