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Open manifolds, Ozsvath-Szabo invariants and Exotic $\mathbb{R}^4$'s

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OPEN MANIFOLDS, OZSVATH-SZABO INVARIANTS AND EXOTIC $\mathbb{R}^4$'S

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ABSTRACT. We construct an invariant of certain open four-manifolds using the Heegaard Floer theory of Ozsvath and Szabo. We show that there is a manifold X homeomorphic to $\mathbb{R}^4$ for which the invariant is non-trivial, showing that X is an exotic $\mathbb{R}^4$. This is the first invariant that detects exotic $\mathbb{R}^4$'s.

1. INTRODUCTION

In this paper, we construct invariants of certain open 4-manifolds using the Heegaard Floer theory of Ozsvath and Szabo, and show that our invariants can detect exotic $\mathbb{R}^4$s. Previous constructions of exotic $\mathbb{R}^4$s used indirect arguments to establish exoticity.

Given an $(n+1)$-dimensional field theory, a direct limit construction can be used to construct an invariant of open $(n+1)$-dimensional manifolds (which we see in detail later). The subtlety in the case of Ozsvath-Szabo invariants is that they do not give a field theory, but satisfy a more complicated composition law. However if we restrict to a class of cobordisms, which we call *admissible cobordisms*, we do get a field theory. Using this, we construct our invariants.

Recall that the Ozsvath-Szabo invariants of a smooth, oriented 3-manifold M associate homology groups to M equipped with a $Spin^c$ structure t. Further, given a smooth cobordisms W between 3-manifolds M_1 and M_2 and a $Spin^c$ structure s on W, we get an induced map on the groups associated to the restrictions of s to M_1 and M_2. To make this into a field theory, one needs a composition rule for a cobordism W_1 from M_1 to M_2 equipped with a $Spin^c$ structure s_1 and a cobordism W_2 from M_2 to M_3 equipped with a $Spin^c$ structure s_2 with $s_1|_{M_2} = s_2|_{M_2}$. However, such $Spin^c$ structures s_1 and s_2 do not in general uniquely determine a $Spin^c$ structure on the composition $W = W_1 \coprod_{M_2} W_2$ of W_1 and W_2. However we have a weaker composition law, where we sum over $Spin^c$structures on W restricting to s_1 and s_2.

We now find sufficient conditions under which s_1 and s_2 uniquely determine a $Spin^c$ structure s on W. The $Spin^c$ structures on a manifold X are a torseur of $H^2(X,\mathbb{Z})$. Consider the Mayer-Vietoris sequence for $W = W_1 \cup W_2$

$$\to H^1(W_1) \oplus H^1(W_2) \to H^1(M_2) \xrightarrow{\delta} H^2(W) \to H^2(W_1) \oplus H^2(W_2) \to H^2(M_2)$$

From this sequence, it follows that, given s_1 and s_2 as above, there is a unique $Spin^c$ structure s on W which restricts to s_1 and s_2 if and only if the coboundary map $\delta : H^1(M_2) \to H^2(W)$ is trivial. This is equivalent to the map induced by inclusions $H^1(W_1) \oplus H^1(W_2) \to H^1(M)$ being surjective. Motivated by this, we make the following definition.

Definition 1.1. A smooth 4-dimensional cobordism W from M_1 to M_2 is admissible if the map induced by inclusion $H^1(W) \to H^1(M_2)$ is surjective.

We shall see basic properties of such cobordisms in Section 2. We now turn to the corresponding notions for open manifolds. Let X be an open 4-manifold which we assume for simplicity has one end.

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Let $K_1 \subset K_2 \subset \dots$ be an exhaustion of X by compact manifolds and let $M_i = \partial K_i$. We assume here and henceforth (for all exhaustions) that $K_i \subset int(K_{i+1})$. For $i < j$, let $W_{ij} = K_j - int(K_i)$ be cobordisms from M_i to M_j.

Definition 1.2. The exhaustion $\{K_i\}$ of X is said to be admissible if each cobordism W_{ij}, $i, j \in \mathbb{N}$, $i < j$, is admissible. The manifold X is said to be admissible if it has an admissible exhaustion.

We shall need to consider the appropriate notion of $Spin^c$ structures for the ends of 4-manifolds.

Definition 1.3. An asymptotic $Spin^c$ structure s on X is a $Spin^c$ structure on $X - K$ for a compact subset $K \subset X$. Two asymptotic $Spin^c$ structures s_1 and s_2, defined on $X - K_1$ and $X - K_2$, are said to be equal if there is a compact set $K_0 \supset K_1, K_2$ with $s_1|_{M-K_0} = s_2|_{M-K_0}$.

Given an admissible open 4-manifold X and an asymptotic $Spin^c$ structure s, we can define invariants of X, which we call the *End Floer Homology*, using direct limits. We shall see in Section 3 that an admissible exhaustion gives a directed system.

Theorem 1.4. *There is an invariant $HE(X, s)$ which is the direct limits of the reduced Heegaard Floer homology groups $HF^+_{red}(M_i, s|_{M_i})$ under morphisms induced by the cobordisms W_{ij}. Furthermore this is independent of the admissible exhaustion of X.*

We shall also need a *twisted* version of these invariants. Let $K \subset X$ be a compact set, s a $Spin^c$-structure on $X - K$ and ω a 2-form on $X - K$. Then we consider the reduced Floer theory with ω-twisted coefficients (as in [10]). Once more we get a directed system whose limit gives an invariant $\underline{HE}(X, s)$.

By taking an exhaustion of $\mathbb{R}^4$ by balls, we have the following proposition.

Proposition 1.5. *For the unique asymptotic $Spin^c$ structure s on $\mathbb{R}^4$, we have $\underline{HE}(\mathbb{R}^4, s) = 0$.*

Our main result is that there are manifolds homeomorphic to $\mathbb{R}^4$ but with non-vanishing end Floer homology.

Theorem 1.6. *There is a 4-manifold X homeomorphic to $\mathbb{R}^4$ such that there is a compact set $K \subset X$, a $spin^{\mathbb{C}}$ structure s on $X - K$ and a closed 2-form ω on $X - K$ with $\underline{HE}(X, s) \neq 0$ with ω-twisted coefficients.*

Thus, X is an exotic $\mathbb{R}^4$. Previous constructions of exotic $\mathbb{R}^4$'s used indirect arguments to show that they are exotic. The *End Floer homology* is the first invariant that detects exotic $\mathbb{R}^4$'s.

2. Admissible cobordisms and admissible ends

We henceforth assume that all our manifolds are smooth and oriented and all cobordisms are compact and 4-dimensional. By $W : M_1 \to M_2$ we mean a smooth cobordism from the closed 3-manifold M_1 to the closed 3-manifold M_2. Given $W_1 : M_1 \to M_2$ and $W_2 : M_2 \to M_3$, $W_2 \circ W_1$ denotes the composition of the cobordisms W_1 and W_2.

In this section we prove some simple results concerning admissible cobordisms and admissible ends.

Lemma 2.1. *Suppose $W_1 : M_1 \to M_2$ and $W_2 : M_2 \to M_3$ are admissible cobordisms, then $W = W_2 \circ W_1$ is admissible.*

Proof. We need to show that the map $H^1(W) \to H^1(M_3)$ induced by inclusion is surjective. This is the composition of maps $H^1(W) \to H^1(W_2)$ and $H^1(W_2) \to H^1(M_3)$ induced by inclusion, with the latter surjective by hypothesis. We shall show that the map $H^1(W) \to H^1(W_2)$ is surjective.

Let $\alpha \in H^1(W_2)$ be a class. Let $i_j : M_2 \to W_j$, $j = 1, 2$, be inclusion maps. Consider the Mayer-Vietoris sequence

$$\cdots \to H^1(W) \to H^1(W_1) \oplus H^1(W_2) \overset{i_1^* + i_2^*}{\to} H^1(M_2) \to \dots$$

By admissibility of W_1, there is a class $\beta \in H^1(W_1)$ with $i_1^*(\beta) = i_2^*(\alpha)$. Hence the image of the class $(\alpha, -\beta) \in H^1(W_1) \oplus H^1(W_2)$ in $H^1(M_2)$ is zero, and so $(\alpha, -\beta)$ is the image of a class $\varphi \in H^1(W)$. In particular α is the image of φ under the map induced by inclusion. □

Lemma 2.2. *Suppose $W_1 : M_1 \to M_2$ and $W_2 : M_2 \to M_3$ are cobordisms with $W = W_2 \circ W_1$ admissible. Then W_2 is admissible.*

Proof. By hypothesis the map $H^1(W) \to H^1(M_3)$ is surjective. This factors through the map $H^1(W_2) \to H^1(M_3)$, which must also be surjective. □

We need criteria for when cobordisms corresponding to attaching handles are admissible.

Lemma 2.3. *Let $M = M_1$ be a 3-manifold, W the cobordism corresponding to a handle addition and M_2 the other boundary components of W. The following hold.*

1. *A product cobordism is admissible.*
2. *The cobordism corresponding to attaching a 1-handle to a closed 3-manifold M is admissible.*
3. *If K is a knot in a closed 3-manifold which represents a primitive, non-torsion element in $H_1(M)$, then the cobordism corresponding to attaching a 2-handle to M is admissible.*

Proof. We shall show that the map induced by the inclusion from $H_1(M_2)$ to $H_1(W)$ is an isomorphism in each case. As the map on cohomology is the adjoint of this map, it follows that it is a surjection.

The case of a product cobordism is immediate. In the second case we see that $H_1(M_2) = H_1(W) = H_1(M) \oplus \mathbb{Z}$ with the isomorphism induced by inclusion. In the third case we have $H_1(M) = H \oplus \mathbb{Z}$, with $[K]$ generating the $\mathbb{Z}$ component. It is easy to see that $H_1(W) = H_1(M_2) = H$. □

Now let X be an open manifold and let $K_1 \subset K_2 \subset \dots$ be an exhaustion of X and M_i and W_{ij} be as before.

Lemma 2.4. *The exhaustion $\{K_i\}$ is admissible if and only if each of the manifolds $K_{j+1} - int(K_j)$ is admissible.*

Proof. Each W_{ij} is the composition of cobordisms $K_{j+1} - int(K_j)$. The result follows by Lemma 2.1. □

Thus, if X is obtained from a compact manifold K by attaching handles as in Lemma 2.3 then X is admissible. Our examples of exotic R^4s will be of this form.

It is immediate from the definition that any refinement of an admissible cobordism is admissible. To show independence of our invariants under exhaustions, we need the following lemma.

Lemma 2.5. *Let $K_1 \subset L_1 \subset K_2 \subset L_2 \dots$ be an exhaustion of X with $K_1 \subset K_2 \subset \dots$ and $L_1 \subset L_2 \subset \dots$ admissible exhaustions. Then the exhaustion $L_1 \subset K_2 \subset L_2 \subset K_3 \dots$ is admissible.*

Proof. It suffices to show that the cobordisms $K_{j+1} - int(L_j)$, $j \geq 1$ and $L_j - int(K_j)$, $j \geq 2$ are admissible. This follows from Lemma 2.2 as the cobordisms $K_{j+1} - int(K_j)$ and $L_{j+1} - int(L_j)$ are admissible and we have $K_{j+1} - int(K_j) = (K_{j+1} - int(L_j)) \circ (L_j - int(K_j))$ and $L_{j+1} - int(L_j) = (L_{j+1} - int(K_j)) \circ (K_j - int(L_j))$. □

3. Invariants for admissible ends

We are now ready to define our invariants for an admissible open 4-manifold X. We shall construct invariants based on reduced Heegaard Floer theory HF^+_{red}. First we recall some facts about Ozsvath-Szabo theory.

Associated to each closed, oriented 3-manifold M and $Spin^c$ structure t on M we have abelian groups $HF^+_{red}(M,t)$. Further, given a cobordism $W : M_1 \to M_2$ with a $Spin^c$ structure s on W such that $t_i = s|_{M_i}$, we get an induced homomorphism on the abelian groups $F_{W,s} : HF^+_{red}(M_1,t_1) \to HF^+_{red}(M_2,t_2)$ induced by the corresponding homomorphism on HF^+. This homomorphism is well defined up to choice of sign. We shall denote the above cobordism with its $Spin^c$ structure by $(W,s) : (M_1,t_1) \to (M_2,t_2)$.

Further, if $(W_1,s_1) : (M_1,t_1) \to (M_2,t_2)$ and $(W_2,s_2) : (M_2,t_2) \to (M_3,t_3)$, with $W = W_2 \circ W_1$, we have the composition formula

$$F_{W_2,s_2} \circ F_{W_1,s_1} = \sum_{s|_{W_i}=s_i} \pm F_{W,s}$$

We shall consider the special case when W_1 is admissible.

Lemma 3.1. *If W_1 is admissible then there is a unique $Spin^c$ structure s on W with $s|_{W_i} = s_i$. For this $Spin^c$ structure $F_{W_2,s_2} \circ F_{W_1,s_1} = \pm F_{W,s}$*

Proof. Recall that $Spin^c$ structures are a torseur of $H^2(\cdot,\mathbb{Z})$. Consider the Mayer-Vietoris sequence for $W = W_1 \cup W_2$

$$\to H^1(W_1) \oplus H^1(W_2) \to H^1(M_2) \xrightarrow{\delta} H^2(W) \to H^2(W_1) \oplus H^2(W_2) \to H^2(M_2)$$

By admissibility the map $H^1(W_1) \oplus H^1(W_2) \to H^1(M_2)$ is a surjection, hence $H^2(W) \to H^2(W_1) \oplus H^2(W_2)$ is an injection. This shows uniqueness of the $Spin^c$ structure. As $s_1|_{M_2} = t_2 = s_2|_{M_2}$, existence follows from the same exact sequence.

The second statement follows from the first using the composition formula. □

For an admissible exhaustion, it follows that we get a directed system of abelian groups up to sign. It is easy to see that we can choose signs to get a directed system, and the direct limit of the system does not depend on the choice of signs.

Definition 3.2. The End Floer homology $HE(X,s)$ is the direct limit of the directed system constructed above.

Proposition 3.3. *This is independent of the admissible exhaustion chosen.*

Proof. By elementary properties of direct limits, the limit does not change on passing to a refinement. Given two admissible exhaustions $K_1 \subset K_2 \subset \dots$ and $L_1 \subset L_2 \subset \dots$, by passing to an exhaustion we can assume that $K_1 \subset L_1 \subset K_2 \subset L_2 \subset \dots$. By Lemma 2.5 the exhaustion $L_1 \subset K_2 \subset L_2 \subset K_3 \dots$ is admissible. As $L_1 \subset L_2 \subset \dots$ and $K_2 \subset K_3 \subset \dots$ are refinements of this exhaustion, the direct limits for the exhaustions $K_1 \subset K_2 \subset \dots$ and $L_1 \subset L_2 \subset \dots$ are the same (as they are both isomorphic to the direct limit corresponding to the exhaustion $L_1 \subset K_2 \subset L_2 \subset K_3 \dots$) . □

We consider the ω-twisted version of this as in [10]. Let $K \subset X$ be a compact manifold and ω a 2-form on $X - K$. We consider an admissible exhaustion with $K \subset K_1$. For this, we can define the twisted groups $\underline{HF}^+_{red}(M_i,t_i)$ and homomorphisms associated to W_{ij} which are well defined up to sign and multiplication by powers of T. For any composition $W = W_2 \circ W_1$ associated with the exhaustion as above, the coboundary map $\delta : H^1(M_2) \to H^2(W)$ is zero. It follows by the composition rule for ω-twisted coefficients that we have a directed system up to multiplication by powers of T and sign.

Once more, we can make choices for the homomorphisms to get a directed system and the direct limit is independent of the choices.

The direct limit is the End Floer homology $\underline{HE}(X, s)$ with ω-twisted coefficients. As in the untwisted case, this is well defined.

4. Exotic $\mathbb{R}^4$'s

We now construct a manifold X homeomorphic to $\mathbb{R}^4$ with $\underline{HE}(X) \neq 0$. This is done by first constructing a convex symplectic manifold W with one convex boundary component N_0 and one convex end and then gluing a compact manifold Y to W along N_0.

Let K be a non-trivial slice knot in S^3 and let N be obtained by 0-frame surgery about K. Then $N \times [0,1]$ admits a taut foliation by [4], and hence a symplectic structure with both ends convex by [2]. On attaching a 2-handle H to $N \times \{1\}$ corresponding to the surgery canceling the 0-frame surgery about K, we get a manifold P with boundary $S \cup N_0$ with $N_0 = N \times \{0\}$ and S a 3-sphere. In particular the end of $P - S$ is homeomorphic to the end of $\mathbb{R}^4$ by Freedman's theorem [3].

As in Theorem 3.1 of [5], we can attach a Casson handle in place of the 2-handle H to get a manifold W which is a convex symplectic manifold and with end homeomorphic to $\mathbb{R}^4$. Observe that W is simply-connected as the 2-handle is attached along the meridian of K, which normally generates $\pi_1(N)$. Also observe that in the proof of Theorem 3.1 of [5], the handles attached are as in Lemma 2.3, and hence the corresponding exhaustion is admissible.

Now, let Y' be obtained from B^4 by attaching a 2-handle along K with framing 0. Then $\partial Y' = N$. As K is slice, the generator of $H_2(Y) = \mathbb{Z}$ can be represented by an embedded sphere Σ. Let Y be obtained from Y' by performing surgery along Σ. Glue W to Y along $\partial Y = N = N_0$ to obtain X.

By a Mayer-Vietoris argument, X has the homology of $\mathbb{R}^4$. Further, as $\pi_1(Y)$ is normally generated by a meridian of K, to which a Casson handle is attached, $\pi_1(X) = 1$. Finally, as the end of X is homeomorphic to the end of $\mathbb{R}^4$, Y is simply-connected at infinity. Thus Y is homeomorphic to $\mathbb{R}^4$.

Finally, we show that the End Floer homology for X does not vanish. Consider the exhaustion of X with $K_1 = Y$, hence $M_1 = N$ and K_2, K_3, ... being the level sets after attaching successive handles as above. Note that $X - K$ is symplectic with symplectic form ω, and each of the cobordisms W_{1j} is a convex symplectic manifold with two boundary components M_1 and M_j. We consider ω-twisted coefficients and the $spin^c$ structure s associated to ω.

Let ξ be the contact structure on $N = M_1$. Let $c^+(\xi; [\omega]) \in \underline{HF}^+(M_1, s)$ be the contact element and $c^+_{red}(\xi; [\omega])$ its image in the reduced Floer homology. We shall show that the image x_j of the contact element $c^+_{red}(\xi; [\omega])$ in the reduced Floer homology group $\underline{HF}^+_{red}(M_i, t_i)$ is non-zero for each j. It follows that the direct limit, i.e., the End Floer homology, is non-zero.

As W_{1j} is a symplectic manifold with two convex boundary components, W_{1j} embeds in a symplectic 4-manifold M with both components of $M - K_j$ having $b_1^+ > 0$ by results of Eliashberg [1] and Kronheimer-Mrowka [6].

Let $j > 1$ be fixed. We proceed as in the proof of Theorem 4.2 of [10]. We attach a Giroux handle to $N = M_1$ and then attach a surface bundle over a surface with boundary. Let X_1 be the union of the Giroux handle and the surface bundle over a surface with boundary. Let B_1 be a ball in X_1. Similarly attach X_2, with $b_2^+(X_2) > 0$ to M_j to get a closed symplectic manifold and let B_2 be a ball in X_2. Let $W = W_{1j} \cup X$.

Let z_j be the image of the contact element $c^+(\xi; [\omega])$ in the Floer homology group $\underline{HF}^+(M_i, t_i)$. Observe that $x_j \neq 0$ if and only if z_j is not in the image of $\underline{HF}^\infty(M_i, t_i)$. Ozsvath and Szabo show that the image of $c^+(\xi; [\omega])$ in $HF^+(S^3, s_0)$ under the map induced by $W - B_2$ is non-zero. But by

Lemma 3.1, as W_{1j} is admissible, this factors through the map induced by W_{1j}, and hence the image of z_j in $HF^+(S^3, s_0)$ is non-zero. But as the cobordism $X_2 - int(B_2)$ has $b_2^+ > 0$, the induced map on $\underline{HF}^\infty$ is zero. It follows that z_j is not in the image of $\underline{HF}^\infty(M_i, t_i)$, i.e. $x_j \neq 0$, as claimed.

Thus, the End Floer homology of X does not vanish. We have seen that X is homeomorphic to $\mathbb{R}^4$. This completes the proof of Theorem 1.6. □

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