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Poisson transform for the Heisenberg group and eigenfunctions of the sublaplacian

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POISSON TRANSFORM FOR THE HEISENBERG GROUP AND EIGENFUNCTIONS OF THE SUBLAPLACIAN

BY

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ABSTRACT. We define an analogue of Poisson transform on the Heisenberg group and use it to characterise joint eigenfunctions of the sublaplacian $\mathcal{L}$ and $T = i\partial_t$ in terms of certain analytic functionals.

1. INTRODUCTION

For the standard Laplacian Δ on $\mathbb{R}^n$ the functions $e_{\lambda,\omega}(x) = e^{i\lambda x\cdot\omega}$ are eigenfunctions with eigenvalues $-\lambda^2$. More general eigenfunctions can be obtained by considering the Poisson integrals

$$\mathcal{P}_\lambda f(x) = \int_{S^{n-1}} e^{i\lambda x\cdot\omega} f(\omega)\, d\omega.$$

As $\omega \to e_{\lambda,\omega}(x)$ is an analytic function the above Poisson transform makes sense even if f is replaced by certain analytic functionals. Thus the most general eigenfunctions of Δ with eigenvalues $-\lambda^2$, $\lambda \neq 0$ are of the form

$$u(x) = \int_{S^{n-1}} e^{i\lambda x\cdot\omega} d\mu(\omega).$$

A similar result is known for Riemannian symmetric spaces also.

In two papers [4] and [5] Helgason characterised all eigenfunctions of the Laplace-Beltrami operator Δ on Riemannian symmetric spaces of rank one. All such eigenfunctions are of the form

$$u(x) = \int_{K/M} e^{(i\lambda+\rho)H(x^{-1}k)}\, d\mu(k)$$

where μ is an 'analytic functional'. He conjectured that the same is true for all Riemannian symmetric spaces. This was proved by Kashiwara et al [6] in 1978. For some non Riemannian symmetric spaces see the works of Oshima-Sekiguchi [8] and other references given in Helgason [3]. Note that in both cases the functions so obtained are joint eigenfunctions of all differential operators invariant under the group of isometries of the symmetric spaces.

In a long paper published in 1989 [9] Strichartz has considered the problem of characterising all eigenfunctions which satisfy certain 'weak L^2 conditions'. He has treated the problem for hyperbolic spaces, semi-Riemannian spaces and Heisenberg groups.

Our interest in this paper is the Heisenberg group H^n. Denoting by $\mathcal{L}$ the sublaplacian on H^n which plays the role of Δ for the subelliptic geometry, Strichartz has obtained a characterisation for eigenfunctions of $\mathcal{L}$ satisfying certain conditions. The group of isometries for the Heisenberg geometry is provided by the Heisenberg motion group G_n which is the semi-direct product of H^n and $K = U(n)$, the unitary group. Therefore, it is natural to consider joint eigenfunctions of all differential operators

on H^n that commute with the action of G_n. Such operators are precisely the polynomials in $\mathcal{L}$ and $T = i\frac{\partial}{\partial t}$. Thus we are led to consider the joint eigenfunctions of $\mathcal{L}$ and T.

As $\mathcal{L}$ and T commute there is a well defined joint spectrum Σ_n which is called the Heisenberg fan. For each $k \in \mathbb{N}$ let

$$R_k = \{(\lambda, (2k+n)|\lambda|) : \lambda \in \mathbb{R}\backslash\{0\}\}$$

so that R_k is a union of two rays emanating from 0. We also define the limiting ray

$$R_\infty = \{(0, \tau) : \tau \geq 0\}.$$

Then Σ_n is the union of $R_k,\ k \in \mathbb{N}$ and R_∞. In this paper our main concern is to characterise all joint eigenfunctions of T and $\mathcal{L}$ having eigenvalues from $R_k,\ k \in \mathbb{N}$. When $k = \infty$ the eigenfunctions are precisely harmonic functions on $\mathbb{C}^n$ which are well known.

It would be nice to characterise all joint eigenfunctions with arbitrary eigenvalues. If $Tf = sf$ then clearly $f(z,t) = e^{-ist} f_s(z)$. This leads us to study eigenfunctions of the operator

$$L_s = -\Delta + \frac{s^2}{4}|z|^2 + is \sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right).$$

The spectrum of this operator is known when $s \in \mathbb{R}$ but not when s is complex. Even when s is purely imaginary, though we have some knowledge, we don't have enough information to characterise all eigenfunctions. In this paper we only treat the case when the eigenvalues are from R_k. The general situation will be treated elsewhere.

Note that both functions $e_{\lambda,\omega}(x)$ on $\mathbb{R}^n$ and $e^{(i\lambda+\rho)H(x^{-1}k)}$ on Riemannian spaces are related to the irreducible unitary representations of the underlying group of isometries. For the Heisenberg group H^n we start with the Schrödinger representations $\pi_\lambda(z,t)$. An expansion of $\pi_\lambda(z,t)$ in terms of the operator analogues of spherical harmonics lead us to define a Poisson kernel $\mathcal{P}_k^\lambda(z,\omega)$ on $\mathbb{C}^n \times S^{2n-1}$. We use this kernel to define Poisson transform on the Heisenberg group. We show that all joint eigenfunctions of $\mathcal{L}$ and T with eigenvalues from R_k are Poisson integrals of certain 'analytic functionals'.

The plan of the paper is as follows. We set up notation and recall all the results required for reading the paper in section 2. In section 3 we define the Poisson transform and characterise all eigenfunctions that are square integrable modulo the centre. Finally the case of general eigenfunctions is treated in section 4.

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2. Notations and preliminaries

In this section we set up notation and collect relevant results that are needed in later sections. We closely follow the notations used in [13] and [14].

The Heisenberg group H^n is $\mathbb{C}^n \times \mathbb{R}$ equipped with the group law $(z,t)(w,s) = (z+w, s+t+\frac{1}{2}Im(z\cdot\bar{w}))$. The Schrödinger representations π_λ parametrised by non-zero reals are realised on $L^2(\mathbb{R}^n)$ and given explicitly by

$$\pi_\lambda(z,t)\ \varphi(\xi) = e^{i\lambda t}\ e^{i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y)}\ \varphi(\xi + y)$$

for $\varphi \in L^2(\mathbb{R}^n)$. The group Fourier transform of a function $f \in L^1(H^n)$ is defined by

$$\hat{f}(\lambda) = \int_{H^n} f(z,t)\ \pi_\lambda(z,t)\ dz\ dt.$$

Then for suitable functions the inversion formula

$$f(z,t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} tr(\hat{f}(\lambda)\pi_\lambda(z,t)^*)\ |\lambda|^n d\lambda \tag{2.1}$$

holds.

For $f \in L^1(H^n)$ we denote by $f^\lambda(z)$ the inverse Fourier transform of f in the t-variable. We also write $\pi_\lambda(z) = \pi_\lambda(z,0)$ so that $\pi_\lambda(z,t) = e^{i\lambda t}\pi_\lambda(z)$ and

$$\hat{f}(\lambda) = \int_{\mathbb{C}^n} f^\lambda(z)\ \pi_\lambda(z)\ dz.$$

The operators of the form

$$W_\lambda(g) = \int_{\mathbb{C}^n} g(z)\ \pi_\lambda(z)\ dz$$

are called Weyl transforms. The convolution on H^n leads to a twisted convolution

$$g *_\lambda h(z) = \int_{\mathbb{C}^n} g(z-w)\ h(w)\ e^{i\frac{\lambda}{2}Im(z\cdot\bar{w})} dw$$

and Weyl transforms convert them into products

$$W_\lambda(g *_\lambda h) = W_\lambda(g)\ W_\lambda(h).$$

By taking an orthonormal basis for the Heisenberg Lie algebra which is identified with $\mathbb{R}^{2n} \times \mathbb{R}$ we define the sublaplacian as

$$\mathcal{L} = -\sum_{j=1}^{n}(X_j^2 + Y_j^2).$$

In terms of the coordinates (z,t), $\mathcal{L}$ takes the form

$$\mathcal{L} = -\Delta - \frac{1}{4}|z|^2\partial_t^2 + \sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right)\partial_t. \tag{2.2}$$

Note that $\mathcal{L}(e^{i\lambda t}g(z)) = e^{i\lambda t}L_\lambda g(z)$ where

$$L_\lambda = -\Delta + \frac{\lambda^2}{4}|z|^2 + i\lambda\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right). \tag{2.3}$$

These operators are called special Hermite operators.

We also consider the Hermite operators $H(\lambda) = -\Delta + \lambda^2|x|^2$ on $\mathbb{R}^n$ which has an explicit spectral decomposition. For $\alpha \in \mathbb{N}^n$ let $\Phi_\alpha(x)$ stand for the n-dimensional Hermite functions which satisfy $H(1)\Phi_\alpha = (2|\alpha| + n)\ \Phi_\alpha$. Given $\lambda \neq 0$ define $\Phi_\alpha^\lambda(x) = |\lambda|^{\frac{n}{4}}\Phi_\alpha(|\lambda|^{\frac{1}{2}}x)$ and let

$$P_k(\lambda)\ f = \sum_{|\alpha|=k}(f, \Phi_\alpha^\lambda)\ \Phi_\alpha^\lambda,\ f \in L^2(\mathbb{R}^n).$$

Then one has the spectral decomposition

$$H(\lambda) = \sum_{k=0}^{\infty}(2k+n)\ |\lambda|\ P_k(\lambda).$$

The Hermite functions Φ_α^λ are used to define the so called special Hermite functions. We define

$$\Phi_{\alpha\beta}^\lambda(z) = (2\pi)^{-\frac{n}{2}}|\lambda|^{\frac{n}{2}}(\pi_\lambda(z)\Phi_\alpha^\lambda, \Phi_\beta^\lambda)$$

and they form an orthonormal basis for $L^2(\mathbb{C}^n)$. Moreover,

$$L_\lambda \Phi^\lambda_{\alpha\beta} = (2|\alpha| + n)|\lambda|\ \Phi^\lambda_{\alpha\beta}.$$

Let $L^\alpha_k(s), \alpha > -1$ be Laguerre polynomials of type α. We define

$$\varphi^{n-1}_{k,\lambda}(z) = L^{n-1}_k\left(\frac{1}{2}|\lambda|\ |z|^2\right)\ e^{-\frac{1}{4}|\lambda||z|^2}.$$

Then the special Hermite expansion

$$g(z) = \sum_\alpha \sum_\beta (g, \Phi^\lambda_{\alpha\beta})\ \Phi^\lambda_{\alpha\beta}(z)$$

can be put in the compact form

$$g(z) = (2\pi)^{-n}|\lambda|^n \sum_{k=0}^\infty f *_{(-\lambda)}\ \ \varphi^{n-1}_{k,\lambda}(z).$$

The functions $\varphi^{n-1}_{k,\lambda}$ enjoy the orthogonality relation

$$\varphi^{n-1}_{k,\lambda} *_\lambda \varphi^{n-1}_{j,\lambda} = \delta_{kj}\ (2\pi)^n|\lambda|^{-n}\varphi^{n-1}_{k,\lambda}. \tag{2.4}$$

We also make use of the identity

$$\sum_\alpha (W_\lambda(f)\ \Phi^\lambda_\alpha, W_\lambda(g)\Phi^\lambda_\alpha) = (2\pi)^n|\lambda|^{-n}(f,g) \tag{2.5}$$

valid for all f and $g \in L^2(\mathbb{C}^n)$.

The unitary group $K = U(n)$ acts on functions on the sphere S^{2n-1}. Let $M = U(n-1)$ considered as a subgroup of K so that we can identify S^{2n-1} with K/M. The left regular representation of K can be decomposed in terms certain class one representations δ having unique M-fixed vectors. For each pair (p,q) of non-negative integers let $\mathcal{H}^{p,q}$ be the space of all harmonic polynomials of the form

$$P(z) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta}\ z^\alpha\ \bar{z}^\beta,\ z \in \mathbb{C}^n. \tag{2.6}$$

Then $\delta = \delta_{p,q}$ defined on $\mathcal{H}^{p,q}$ by $\delta(k)\ P(\omega) = P(k^{-1}\omega)$ are irreducible unitary and they account for all such representations. We denote this class of representations by $\hat{K}_0$. We also denote by $\hat{K}(m)$ the subcollection

$$\hat{K}(m) = \{\delta \in \hat{K}_0 : 0 \le p \le m,\ q \in \mathbb{N}\}.$$

The dimension of $\mathcal{H}^{p,q}$ is denoted by $d(\delta) = d(p,q)$.

For each $\delta = \delta_{p,q}$ we fix an orthonormal basis $P^\delta_j,\ 1 \le j \le d(\delta)$ for $\mathcal{H}^{p,q}$ so that

$$\{P^\delta_j(\omega) : 1 \le j \le d(\delta),\ \delta \in \hat{K}_0\}$$

forms an orthonormal basis for $L^2(S^{2n-1})$. We have an operator analogue of spherical harmonics which plays an important role in our analysis. We proceed to define them now.

For each $m \in \mathbb{N}$ and $\lambda \in \mathbb{R}\backslash\{0\}$ let E^λ_m be the finite dimensional subspace of $L^2(\mathbb{R}^n)$ spanned by $\Phi^\lambda_\alpha,\ |\alpha| = m$. Let $\mathcal{O}(E^\lambda_m)$ stand for the space of all bounded linear operators $S_\lambda : E^\lambda_m \to L^2(\mathbb{R}^n)$. We make $\mathcal{O}(E^\lambda_m)$ into a Hilbert space by defining

$$(S_\lambda, T_\lambda)_m = (2\pi)^{-n}|\lambda|^n \sum_{|\alpha|=m} (S_\lambda \Phi^\lambda_\alpha, T_\lambda \Phi^\lambda_\alpha).$$

These Hilbert spaces turn out to be the analogues of $L^2(S^{2n-1})$. This point of view initiated by Geller [2] turned out to be useful in analysing some problems on the Heisenberg group, see eg [2] and [14].

To each $P \in \mathcal{H}^{p,q}$ and $m \in \mathbb{N}$ Geller [2] associates an operator $G_\lambda(P) \in \mathcal{O}(E_k^\lambda)$. This is called the Weyl correspondence of P and is related to the Weyl transform by

$$G_\lambda(P)\ P_m(\lambda) = \lambda^n\ 2^{p+q}\ (-1)^p\ W_\lambda(P\varphi_{m-q,\lambda}^{n+p+q-1}) \tag{2.7}$$

for $\lambda > 0$ and for $\lambda < 0$

$$G_\lambda(P)\ P_m(\lambda) = (-\lambda)^n\ 2^{p+q}\ (-1)^q\ W_\lambda(P\varphi_{m-q,\lambda}^{n+p+q-1}).$$

It turns out that $G_\lambda(P_j^\delta)$ form an orthogonal system which suitably normalised is actually an orthonormal basis for $\mathcal{O}(E_m^\lambda)$. The norms of $G_\lambda(P_j^\delta)$ are easily calculated.

Lemma 2.1. *For each $P \in \mathcal{H}^{p,q}$ and $\lambda > 0$*

$$\|G_\lambda(P)\|_m^2 = \lambda^n \left(\frac{4}{\lambda}\right)^{p+q} 2^{n+p+q-1}\ \frac{\Gamma(m+n+q)}{\Gamma(m-p+1)} \int\limits_{S^{2n-1}} |P(\omega)|^2 p\omega.$$

When $\lambda < 0$ the roles of p and q are interchanged.

Proof: Assume $\lambda > 0$.

$$\sum_{|\alpha|=m} (G_\lambda(P)\ \Phi_\alpha^\lambda, G_\lambda(P)\ \Phi_\alpha^\lambda) = \sum_\alpha (G_\lambda(P)\ P_m(\lambda)\ \Phi_\alpha^\lambda, G_\lambda(P)\ P_m(\lambda)\ \Phi_\alpha^\lambda)$$

which, in view of (2.5) and (2.7), is equal to

$$\begin{aligned}
& (\lambda^n 2^{p+q})^2 \sum_\alpha (W_\lambda(P\varphi_{m-p,\lambda}^{n+p+q-1})\Phi_\alpha^\lambda, W_\lambda(P\varphi_{m-p,\lambda}^{n+p+q-1})\Phi_\alpha^\lambda) \\
= \ & (\lambda^n 2^{p+q})^2 (2\pi)^n \lambda^{-n} \int |P(z)\varphi_{m-p,\lambda}^{n+p+q-1}(z)|^2 dz \\
= \ & (2\pi)^n \left(\frac{4}{\lambda}\right)^{p+q} \int\limits_{S^{2n-1}} |P(\omega)|^2 d\omega \cdot \int\limits_0^\infty \left| L_{m-p}^{n+q+p-1}\left(\frac{1}{2}r^2\right)\right|^2 e^{-\frac{1}{2}r^2} r^{2(n+p+q)-1} dr.
\end{aligned}$$

The last integral can be calculated using the formula

$$\int\limits_0^\infty |L_k^\alpha(r)|^2 e^{-\frac{1}{2}r^2} r^\alpha dr = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}.$$

Recalling the definition of norm on $\mathcal{O}(E_m^\lambda)$ we get

$$\|G_\lambda(P)\|_m^2 = \lambda^n \left(\frac{4}{\lambda}\right)^{p+q} 2^{n+p+q-1}\ \frac{\Gamma(m+n+q)}{\Gamma(m-p+1)} \int\limits_{S^{2n-1}} |P(\omega)|^2 d\omega.$$

We define the constant $c_m^\lambda(\delta) = c_m^\lambda(p,q)$ by

$$c_m^\lambda(\delta)^2 = \left(\frac{4}{\lambda}\right)^{p+q} 2^{n+p+q-1}\ \frac{\Gamma(m+n+q)}{\Gamma(m-p+1)} \tag{2.8}$$

for $\lambda > 0$ with a similar definition for $\lambda < 0$. Then

$$\{|\lambda|^{-\frac{n}{2}} c_m^\lambda(\delta)^{-1} G_\lambda(P_j^\delta) : 1 \le j \le d(\delta),\ \delta \in \hat{K}(m)\}$$

forms an orthonormal basis for $\mathcal{O}(E_m^\lambda)$. Every operator $S_\lambda \in \mathcal{O}(E_m^\lambda)$ has the expansion

$$S_\lambda = |\lambda|^{-n} \sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} c_m^\lambda(\delta)^{-2} (S_\lambda, G_\lambda(P_j^\delta))_m\ G_\lambda(P_j^\delta). \tag{2.9}$$

Finally we recall the following analogue of Hecke-Bochner identity for the projection operators associated to L_λ.

Let $f \in L^1(\mathbb{C}^n)$ be of the form $f = Pg$ where g is radial and $P \in \mathcal{H}^{p,q}$. Then for $\lambda > 0$

$$f *_\lambda \varphi_{k,\lambda}^{n-1} = \left(\frac{\lambda}{2\pi}\right)^{p+q} P(z)\ g *_\lambda \varphi_{k-p,\lambda}^{n+p+q-1}(z) \tag{2.10}$$

where the convolution on the right hand side is taken on $\mathbb{C}^{n+p+q}$. When $\lambda < 0$ there is a similar formula with the roles of p and q interchanged. We will make use of this in the sequel. The formula (2.9) appears as Theorem 2.6.1 in [14] but with a wrong constant.

3. Poisson transform for H^n

The Fourier inversion formula for Schwartz class functions on H^n, namely

$$f(z,t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} tr(\pi_\lambda(z,t)^*\ \hat{f}(\lambda))\ |\lambda|^n d\lambda \tag{3.1}$$

can be rewritten as follows. Since

$$tr(\pi_\lambda(z,t)^* \hat{f}(\lambda)) = e^{-i\lambda t} \sum_{m=0}^{\infty} tr(\pi_\lambda(z)^* \hat{f}(\lambda)\ P_m(\lambda))$$

and

$$tr(\pi_\lambda(z)^* \hat{f}(\lambda)\ P_m(\lambda)) = \sum_{|\alpha|=m} (\hat{f}(\lambda)\Phi_\alpha^\lambda, \pi_\lambda(z)\Phi_\alpha^\lambda),$$

recalling the definition of the inner product on $\mathcal{O}(E_k^\lambda)$ we get

$$f(z,t) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i\lambda t} \sum_{m=0}^{\infty} (\hat{f}(\lambda), \pi_\lambda(z))_m d\lambda. \tag{3.2}$$

We show below that $e^{-i\lambda t}(\hat{f}(\lambda), \pi_\lambda(z))_m$ are joint eigenfunctions of T and $\mathcal{L}$. Thus (3.2) shows that f is a superposition of eigenfunctions of $\mathcal{L}$ and T.

The inner product $(S_\lambda, \pi_\lambda(z))_m$ is defined for any $S_\lambda \in \mathcal{O}(E_m^\lambda)$. Both S_λ and $\pi_\lambda(z)$ can be expanded in terms of the orthonormal basis

$$\{|\lambda|^{-\frac{n}{2}} c_m^\lambda(\delta)^{-1} G_\lambda(P_j^\delta) : 1 \le j \le d(\delta),\ \delta \in \hat{K}(m)\}$$

where $c_m^\lambda(\delta) = c_m^\lambda(p,q)$ are defined as in (2.8). By Parseval's formula

$$\begin{aligned} &(S_\lambda, \pi_\lambda(z))_m \\ &= |\lambda|^{-n} \sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} c_m^\lambda(\delta)^{-2}\ (S_\lambda, G_\lambda(P_j^\delta))_m\ (G_\lambda(P_j^\delta), \pi_\lambda(z))_m. \end{aligned} \tag{3.3}$$

The functions $(G_\lambda(P_j^\delta, \pi_\lambda(z))_m$ can be explicitly calculated.

Proposition 3.1. *Let $P \in \mathcal{H}^{p,q}$. Then for $\lambda > 0$*

$$(G_\lambda(P), \pi_\lambda(z))_m = (-1)^p\ 2^{p+q}\ P(z)\ \varphi_{m-p,\lambda}^{n+p+q-1}(z)$$

and for $\lambda < 0$

$$(G_\lambda(P), \pi_\lambda(z))_m = (-1)^q\ 2^{p+q}\ P(z)\ \varphi_{m-q,\lambda}^{n+p+q-1}(z).$$

Proof: We assume $\lambda > 0$, the proof being similar for $\lambda < 0$. We have the expansion

$$\pi_\lambda(z)\ \Phi_\alpha^\lambda = (2\pi)^{\frac{n}{2}} \lambda^{-\frac{n}{2}} \sum_\beta\ \Phi_{\alpha\beta}^\lambda(z)\ \Phi_\beta^\lambda$$

and therefore

$$(\pi_\lambda(z), G_\lambda(P))_m = (2\pi)^{-\frac{n}{2}} \lambda^{\frac{n}{2}} \sum_{|\alpha|=m} \sum_\beta\ \Phi_{\alpha\beta}^\lambda(z)(\Phi_\beta^\lambda, G_\lambda(P)\Phi_\alpha^\lambda).$$

Since we have the identity (see 2.7)

$$G_\lambda(P)\Phi_\alpha^\lambda = (-1)^p\ \lambda^n\ 2^{p+q}\ W_\lambda(P\varphi_{m-p,\lambda}^{n+p+q-1})\Phi_\alpha^\lambda$$

we have

$$\begin{aligned} & (\pi_\lambda(z), G_\lambda(P))_m \\ = \ & (2\pi)^{-\frac{n}{2}}\ \lambda^{\frac{3}{2}n}\ (-1)^p\ 2^{p+q} \sum_{|\alpha|=m} \sum_\beta\ \Phi_{\alpha\beta}^\lambda(z)\ (\Phi_\beta^\lambda, W_\lambda(P\varphi_{m-p,\lambda}^{n+p+q-1})\Phi_\alpha^\lambda). \end{aligned}$$

The sum on the right hand side is

$$(2\pi)^{\frac{n}{2}}\ \lambda^{-\frac{n}{2}} \sum_{|\alpha|=m} \sum_\beta \left(\int_{\mathbb{C}^n} \bar{P}(w)\varphi_{m-p,\lambda}^{n+p+q-1}(w)\ \overline{\Phi_{\alpha\beta}^\lambda(w)} dw \right)\ \Phi_{\alpha\beta}^\lambda(z)$$

which can be calculated using the identity

$$(2\pi)^n \sum_{|\alpha|=m} \sum_\beta\ (f, \Phi_{\alpha\beta}^\lambda)\ \Phi_{\alpha\beta}^\lambda(z) = f *_{(-\lambda)} \varphi_{k,\lambda}^{n-1}(z).$$

We obtain

$$(\pi_\lambda(z), G_\lambda(P))_m = \lambda^n\ (-1)^p\ 2^{p+q}\ (2\pi)^{-n}(\bar{P}\varphi_{m-p,\lambda}^{n+p+q-1}) *_{(-\lambda)} \varphi_{m,\lambda}^{n-1}(z).$$

Since $\bar{P} \in \mathcal{H}^{q,p}$ by Hecke-Bochner formula

$$\bar{P}\varphi_{m-p,\lambda}^{n+p+q-1} *_{(-\lambda)} \varphi_{m,\lambda}^{n-1}(z) = (2\pi)^n\ \lambda^{-n}\ \bar{P}(z)\ \varphi_{m-p,\lambda}^{n+p+q-1}(z) \tag{3.4}$$

and hence

$$(G_\lambda(P), \pi_\lambda(z))_m = (-1)^p\ 2^{p+q}\ P(z)\ \varphi_{m-p,\lambda}^{n+p+q-1}(z)$$

proving the proposition.

From equation (3.4) we infer that

$$L_{(-\lambda)}(P\varphi_{m-p,\lambda}^{n+p+q-1}) = (2m+n)\ \lambda\ (P\varphi_{m-p,\lambda}^{n+p+q-1})$$

so that the equation (3.3) is an expansion of $(S_\lambda, \pi_\lambda(z))_m$ in terms of certain eigenfunctions of $L_{-\lambda}$. In order to justify our claim that $(S_\lambda, \pi_\lambda(z))_m$ is an eigenfunction, we need to study the convergence of (3.3). The L^2 norms of $(G_\lambda(P), \pi_\lambda(z))_m$ are easily calculated.

Lemma 3.2. *For $\lambda > 0$ and $P \in \mathcal{H}^{p,q}$*

$$\int_{\mathbb{C}^n} |(\pi_\lambda(z), G_\lambda(P))_m|^2 dz = \lambda^{-n}\ c_m^\lambda(p,q)^2 \left(\int_{S^{2n-1}} |P(\omega)|^2 d\omega \right).$$

Proof: In view of Proposition 3.1 we have

$$\begin{aligned} & \int_{\mathbb{C}^n} |(\pi_\lambda(z), G_\lambda(P))_m|^2 dz \\ = \ & 2^{2(p+q)} \left(\int_{s^{2n-1}} |P(\omega)|^2 d\omega \right) \left(\int_0^\infty |\varphi_{m-p,\lambda}^{n+p+q-1}(r)|^2\ r^{2(p+q+n)-1}\ dr \right). \end{aligned}$$

The second integral on the right hand side is

$$\lambda^{-n-p-q} \int_0^\infty \left| L_{m-p}^{n+p+q-1}\left(\frac{1}{2}r^2\right)\right|^2 e^{-\frac{1}{2}r^2} r^{2(p+q+n)-1} dr$$

$$= \lambda^{-n-p-q}\ 2^{n+p+q-1}\ \frac{\Gamma(m+q+n)}{\Gamma(m-p+1)}.$$

This proves the lemma.

We also note that

$$\int_{\mathbb{C}^n} (\pi_\lambda(z), G_\lambda(P))_m (G_\lambda(Q), \pi_\lambda(z))_m\ dz = \lambda^{-n}\ c_m^\lambda(p,q)^2 \int_{S^{2n-1}} P(\omega)\ \bar{Q}(\omega)\ d\omega.$$

Hence (3.3) is an orthogonal expansion which allows us to calculate the L^2 norm of $(S_\lambda, \pi_\lambda(z))_m$. We are ready to state and prove the following result.

Theorem 3.3. *Let $\lambda > 0$ and $m \in \mathbb{N}$. For every $S_\lambda \in \mathcal{O}(E_m^\lambda)$ the function $f(z,t) = e^{-i\lambda t}(S_\lambda, \pi_\lambda(z))_m$ is a joint eigenfunction of T and $\mathcal{L}$ with eigenvalues λ and $(2m+n)\lambda$ and*

$$\int_{\mathbb{C}^n} |f(z,t)|^2 dz = \lambda^{-2n} ||S_\lambda||_m^2.$$

Conversely, every joint eigenfunction of $\mathcal{L}$ and T with eigenvalues $(2m+n)\lambda$ and λ and square integrable over $\mathbb{C}^n$ is of the form $e^{-i\lambda t}(S_\lambda, \pi_\lambda(z))_m$ for some $S_\lambda \in \mathcal{O}(E_m^\lambda)$.

Proof: From equation (3.3) we obtain using the result of Lemma 3.2

$$\int_{\mathbb{C}^n} |(S_\lambda, \pi_\lambda(z))_m|^2 dz = \lambda^{-3n} \sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} c_m^\lambda(\delta)^{-2}\ |(S_\lambda, G_\lambda(P_j^\delta))_m|^2.$$

Since

$$||S_\lambda||_m^2 = \lambda^{-n} \sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} c_m^\lambda(\delta)^{-2}\ |(S_\lambda, G_\lambda(P_j^\delta))_m|^2$$

we obtain

$$\int_{\mathbb{C}^n} |f(z,t)|^2 dz = \lambda^{-2n}\ ||S_\lambda||_m^2. \tag{3.5}$$

The series for $(S_\lambda, \pi_\lambda(z))_m$ thus converges in $L^2(\mathbb{C}^n)$. As the operator $L_{-\lambda}$ is closed we see that $f(z,t)$ is an eignefunction of $\mathcal{L}$.

To prove the converse we see that any joint eigenfunction of T and $\mathcal{L}$ is of the form $e^{-i\lambda t} f_\lambda(z)$ where $L_{-\lambda} f_\lambda = (2m+n)\lambda\ f_\lambda$. As $f_\lambda \in L^2(\mathbb{C}^n)$ it is easy to see that $W_\lambda(f_\lambda) P_m(\lambda) = W_\lambda(f_\lambda)$. Now the expansion

$$W_\lambda(f_\lambda) = \sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} a_j^\lambda(\delta)\ G_\lambda(P_j^\delta)$$

gives, upon using the formula

$$G_\lambda(P)\ P_m(\lambda) = \lambda^n\ 2^{p+q}\ (-1)^p\ W_\lambda\ (P\ \varphi_{m-p,\lambda}^{n+p+q-1}),$$

the result

$$W_\lambda(f_\lambda) = \sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} b_j^\lambda(\delta)\ W_\lambda\ (P_j^\delta\ \varphi_{m-p,\lambda}^{n+p+q-1}).$$

Hence f_λ has the expansion

$$f_\lambda(z) = \sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} b_j^\lambda(\delta)\ P_j^\delta(z)\ \varphi_{m-p,\lambda}^{n+p+q-1}(z).$$

From this and Proposition 3.1 it is clear that $f_\lambda(z) = (S_\lambda, \pi_\lambda(z))_m$ for a suitable operator $S_\lambda \in \mathcal{O}(E_m^\lambda)$. This completes the proof.

We can restate the above theorem as follows.

Theorem 3.4. *A function $f_\lambda \in L^2(\mathbb{C}^n)$ satisfies $L_{-\lambda} f_\lambda = (2m+n)\ \lambda\ f_\lambda$ if and only if $f_\lambda(z) = (S_\lambda, \pi_\lambda(z))_m$ for some $S_\lambda \in \mathcal{O}(E_m^\lambda)$.*

When $f \in L^2(H^n)$, $\hat{f}(\lambda) = W_\lambda(f^\lambda)$ is a Hilbert-Schmidt operator. Hence $(\hat{f}(\lambda), \pi_\lambda(z))_m$ is an eigenfunction of $L_{-\lambda}$. Thus (3.2) shows that f is a superposition of eigenfunctions of $\mathcal{L}$.

We can produce operators $S_\lambda \in \mathcal{O}(E_m^\lambda)$ by looking at Weyl transforms of compactly supported distributions. Indeed as shown in [12], $W_\lambda(u)$, when u is a compactly supported distribution on $\mathbb{C}^n$ defines a bounded operator from certain Sobolev spaces into $L^2(\mathbb{R}^n)$ (see Theorem 3.3 in [12]). They need not extend to the whole of $L^2(\mathbb{R}^n)$ as bounded operators but they certainly belong to $\mathcal{O}(E_m^\lambda)$ for every m.

The results of Theorem 3.3 and 3.4 motivate the following definitions. For each $\lambda > 0$ (which we assume for the sake of definiteness; the same can be done for $\lambda < 0$ as well) and $m \in \mathbb{N}$ we define a kernel

$$\mathcal{P}_m^\lambda(z,\omega) = \lambda^{-\frac{n}{2}} \sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} c_m^\lambda(\delta)^{-1} (G_\lambda(P_j^\delta), \pi_\lambda(z))_m\ \overline{P_j^\delta}(\omega) \tag{3.6}$$

for $z \in \mathbb{C}^n$ and $\omega \in S^{2n-1}$. We call this the Poisson kernel for the Heisenberg group associated to the point $(\lambda, (2m+n)\lambda)$ on the Heisenberg fan. The Poisson kernel $\mathcal{P}_m^\lambda(z,\cdot) \in L^2(S^{2n-1})$. This can be seen as follows. Since $\pi_\lambda(z)$ are unitary operators

$$||\pi_\lambda(z)||_m^2 = (2\pi)^{-n}\ \lambda^n\ \frac{(m+n-1)!}{m!(n-1)!}.$$

But

$$\pi_\lambda(z) = \lambda^{-n} \sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} c_m^\lambda(\delta)^{-2}\ (\pi_\lambda(z), G_\lambda(P_j^\delta))_m\ G_\lambda(P_j^\delta) \tag{3.7}$$

which gives

$$\lambda^{-n} \sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} c_m^\lambda(\delta)^{-2}\ |(G_\lambda(P_j^\delta), \pi_\lambda(z))_m|^2 = ||\pi_\lambda(z)||_m^2 = (2\pi)^{-n}\ \lambda^n\ \frac{(m+n-1)!}{m!(n-1)!}.$$

But then

$$\int_{S^{2n-1}} |\mathcal{P}_m^\lambda(z,\omega)|^2 d\omega = \lambda^{-n} \sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} c_m^j(\delta)^{-2}\ |(G_\lambda(P_j^\delta), \pi_\lambda(z))_m|^2 < \infty.$$

Given $g \in L^2(S^{2n-1})$ we can define its Poisson integral

$$\mathcal{P}_m^\lambda g(z) = \int_{S^{2n-1}} \mathcal{P}_m^\lambda(z,\omega)\ g(\omega)\ d\omega$$

which is given by the series

$$\lambda^{-\frac{n}{2}} \sum_{\delta\in\hat{K}(m)} \sum_{j=1}^{d(\delta)} c_m^\lambda(\delta)^{-1}\ (G_\lambda(P_j^\delta), \pi_\lambda(z))_m\ (g, P_j^\delta).$$

Defining

$$S_\lambda(g) = \lambda^{-\frac{n}{2}} \sum_{\delta\in\hat{K}(m)} \sum_{j=1}^{d(\delta)} c_m^\lambda(\delta)^{-1}\ (g, P_j^\delta)\ G_\lambda(P_j^\delta) \tag{3.8}$$

we see that $||S_\lambda(g)||_m^2 = ||g||_2^2$ and

$$\mathcal{P}_m^\lambda\ g(z) = (S_\lambda(g),\ \pi_\lambda(z))_m.$$

Therefore, we have yet another reformulation of Theorem 3.3.

Theorem 3.5. *A function $f_\lambda \in L^2(\mathbb{C}^n)$ satisfies $L_{-\lambda} f_\lambda = (2m+n)\lambda\ f_\lambda$ if and only if $f_\lambda = \mathcal{P}_m^\lambda g$ for some $g \in L^2(S^{2n-1})$.*

To prove the theorem we only need to establish a correspondence between $L^2(S^{2n-1})$ and $\mathcal{O}(E_m^\lambda)$. This is given by the equation (3.8). We remark that this correspondence is not one to one. In the next section we consider Poisson integrals of more general functionals on spaces of analytic functions on S^{2n-1}.

4. General eigenfunctions of $\mathcal{L}$ and T

In this section we characterise all joint eigenfunctions $f(z,t)$ of $\mathcal{L}$ and T that satisfy certain growth conditions. They are characterised by certain functionals on a space of analytic functions on S^{2n-1}.

For each $k \in \mathbb{N}$ let $\mathcal{B}_k$ be the subspace of functions φ of $L^2(S^{2n-1})$ for which

$$\sum_{\delta\in\hat{K}_0} \sum_{j=1}^{d(\delta)} |(\varphi, P_j^\delta)|^2\ 2^{-k(p+q)}\ \Gamma(n+p+q) < \infty. \tag{4.1}$$

Then $\mathcal{B}_k$ becomes a Hilbert space if we define $||\varphi||_{(k)}$ as the square root of the above expression. We take $\mathcal{B} = \bigcup_{k=0}^{\infty} \mathcal{B}_k$ and equip it with the inductive limit topology. We show below that the Poisson kernel $\mathcal{P}_m^\lambda(z,\cdot) \in \mathcal{B}$ so that we can talk about the Poisson transform

$$\int_{S^{2n-1}} \mathcal{P}_m^\lambda(z,\omega)\ d\mu(\omega)$$

whenever $\mu \in \mathcal{B}^*$, the dual of $\mathcal{B}$. Our main result is the following characterisation of joint eigenfunctions of $\mathcal{L}$ and T.

Theorem 4.1. *A function f on the Heisenberg group which satisfies the condition*

$$|f(z,t)| \le c\ e^{\frac{1}{4}|\lambda|(1-\epsilon)|z|^2}$$

for some $\epsilon > 0$ is a joint eigenfunction of T and $\mathcal{L}$ with eigenvalue $(\lambda, (2m+n)|\lambda|)$ if and only if

$$f(z,t) = e^{-i\lambda t} \int_{S^{2n-1}} \mathcal{P}_m^\lambda(z,\omega)\ d\mu(\omega)$$

for some functional $\mu \in \mathcal{B}^*$.

This theorem is proved in several steps. First we make the observation that $\mathcal{P}_m^\lambda(z,\cdot) \in \mathcal{B}$. Recall that

$$\mathcal{P}_m^\lambda(z,\omega) = \sum_{\delta\in\hat{K}(m)} \sum_{j=1}^{d(\delta)} a_{\delta,j}^\lambda(z)\ \overline{P_j^\delta}(\omega)$$

where, assuming $\lambda > 0$,

$$a_{\delta,j}^\lambda(z) = (-1)^p \left(\frac{1}{2}\lambda|z|^2\right)^{\frac{1}{2}(p+q)} \left(\frac{\Gamma(k-p+1)}{\Gamma(n+p+q)}\right)^{\frac{1}{2}} P_j^\delta(z')\ \varphi_{m-p,\lambda}^{n+p+q-1}(z)$$

with $z' = |z|^{-1}z$. Choose k so that $\frac{1}{2}\ \lambda\ |z|^2 \le 2^k$. Then

$$||\mathcal{P}_m^\lambda(z,\cdot)||_{(k+1)}^2 \le c \sum_{\delta\in\hat{K}(m)} \sum_{j=1}^{d(\delta)} \Gamma(k-p+1)\ \left(\frac{1}{2}\right)^{p+q} |P_j^\delta(z')|^2\ |\varphi_{m-p,\lambda}^{n+p+q-1}(z)|^2.$$

Since $|P_j^\delta(z')| \le c\ (p+q)^{n-1}$ and

$$|\varphi_{m-p,\lambda}^{n+p+q-1}(z)| \le c\ \frac{\Gamma(m-p+n+p+q)}{\Gamma(m-p+1)\Gamma(n+p+q)} \le c\ \frac{(n+p+q)^{m-p}}{\Gamma(m-p+1)}$$

the above series converges. We also have the estimate

$$||\mathcal{P}_m^\lambda(z,\cdot)||_{(k+1)}^2 \le c \sum_{\delta\in\hat{K}(m)} d(\delta)\ \left(\frac{1}{2}\right)^{p+q} (n+p+q)^{2m+2n} \tag{4.2}$$

whenever $\frac{1}{2}\ \lambda\ |z|^2 \le 2^k$.

The elements of the dual space $\mathcal{B}^*$ are characterised by the following property.

Lemma 4.2. *A linear functional μ on $\mathcal{B}$ belongs to $\mathcal{B}^*$ if and only if*

$$\sum_{\delta\in\hat{K}_0} \sum_{j=1}^{d(\delta)} |(\mu, P_j^\delta)|^2\ \frac{2^{m(p+q)}}{\Gamma(n+p+q)} < \infty$$

for every $m \in \mathbb{N}$.

Proof: Since the topology on $\mathcal{B}$ is the inductive limit topology, $\mu \in \mathcal{B}^*$ if and only if $\mu \in \mathcal{B}_m^*$ for every m. If $\mu \in \mathcal{B}_m^*$ then as $\mathcal{B}_m$ is a Hilbert space

$$(\mu,\varphi) = \sum_{\delta\in\hat{K}_0} \sum_{j=1}^{d(\delta)} b_{\delta,j}(\varphi, P_j^\delta)\ 2^{-m(p+q)}\ \Gamma(n+p+q)$$

for some $b_{\delta,j}$ satisfying

$$\sum_{\delta\in\hat{K}_0} \sum_{j=1}^{d(\delta)} |b_{\delta,j}|^2\ 2^{-m(p+q)}\ \Gamma(n+p+q) < \infty.$$

Taking $\varphi = P_j^\delta$ we get $(\mu, P_j^\delta) = b_{\delta,j}\ 2^{-m(p+q)}\ \Gamma(n+p+q)$ and hence

$$\sum_{\delta\in\hat{K}_0} \sum_{j=1}^{d(\delta)} |(\mu, P_j^\delta)|^2\ \frac{2^{m(p+q)}}{\Gamma(n+p+q)} < \infty.$$

The converse is trivial.

Proposition 4.3. *For every $\mu \in \mathcal{B}^*$ the function*

$$f(z,t) = e^{-i\lambda t} \int_{S^{2n-1}} \mathcal{P}_m^\lambda(z,\omega)\ d\mu(\omega)$$

is a joint eigenfunction of T and $\mathcal{L}$.

Proof: As $\mathcal{P}_m^\lambda(z,\cdot) \in \mathcal{B}$ the series defining $f(z,t)$ converges uniformly over compact subsets.

$$f(z,t) = e^{-i\lambda t} \sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} (\mu, P_j^\delta)\ a_{\delta,j}^\lambda(z)$$

shows that $f(z,t)$ is the superposition of the eigenfunctions $e^{-i\lambda t} a_{\delta,j}^\lambda(z)$. Since L_λ is a closed operator $f(z,t)$ is an eigenfunction of $\mathcal{L}$.

This proves one implication of Theorem 4.1. We now proceed to prove the converse.

Proposition 4.4. *Let $f_\lambda(z) = P(z)\ g_\lambda(z)$ where g_λ is radial and $P \in \mathcal{H}^{p,q}$ be an eigenfunction of $L_{-\lambda}$ with eigenvalue $(2m+n)\lambda$. Further assume that $|f_\lambda(z)| \leq c\ e^{\frac{1}{4}|\lambda|(1-\epsilon)|z|^2}$ for some $\epsilon > 0$. Then $f_\lambda(z) = c\ \lambda^{\frac{p+q}{2}} P(z)\ \varphi_{m-p,\lambda}^{n+p+q-1}(z)$ for some c.*

Proof: We can assume $\lambda = 1$ since $f_1(z) = f_\lambda\left(\frac{z}{\sqrt{\lambda}}\right)$ is an eigenfunction of L_{-1}. Consider the equation

$$\left(-\Delta + \frac{1}{4}|z|^2 - iN\right)\ (Pg) = (2m+n)\ Pg.$$

As g is radial and $P \in \mathcal{H}^{p,q}$, $N(Pg) = i(p-q)Pg$ and so the above equation becomes

$$\left(-\Delta + \frac{1}{4}|z|^2\right)\ (Pg) = (2(m-p)+n+p+q)\ Pg.$$

The expression od Δ in polar coordinates takes the form

$$\Delta = \frac{d^2}{dr^2} + \frac{2n-1}{r}\ \frac{d}{dr} + \frac{1}{r^2}\Delta_S$$

where Δ_S is the spherical Laplacian on S^{2n-1}. As $P(z) = |z|^{p+q}P(z')$ and $\Delta_S P(z') = -(p+q)(p+q+2n-2)\ P(z')$ we get

$$\left(-\frac{d^2}{dr^2} - \frac{2n-1}{r}\frac{d}{dr} + \frac{1}{4}r^2\right)\ (r^{p+q}g(r)) = (2(m-p)+n+p+q)\ (r^{p+q}g(r)).$$

The above equation further simplifies to

$$\left(-\frac{d^2}{dr^2} - \frac{(2d-1)}{r}\frac{d}{dr} + \frac{1}{4}r^2\right)\ g(r) = (2(m-p)+d)\ g(r)$$

where $d = n+p+q$. We are thus looking for radial eigenfunctions of special Hermite operator L on $\mathbb{C}^d$. The above equation can be reduced to the confluent hypergeometric equation

$$su''(s) + (d-s)\ u'(s) + (m-p)\ u(s) = 0.$$

This equation has two linearly independent solutions (see Olver [7]):

$$u_1(s) = L_{m-p}^{d-1}(s),\ u_2(s) \sim e^s(-s)^{m-p-d}.$$

The second solution leads to an eigenfunction $f(z)$ which has the growth $e^{\frac{1}{4}|z|^2}$ which is not allowed under our growth assumption. The solution $u_1(s)$ leads to $g(z) = c\ \varphi_{m-p}^{n+p+q-1}(z)$ and this proves the proposition.

We now prove an expansion result for eigenfunctions of $L_{-\lambda}$.

Proposition 4.5. *Every eigenfunction f_λ of $L_{-\lambda}$ with eigenvalue $(2m+n)\lambda$ which satisfies the growth condition $|f_\lambda(z)| \leq c\ e^{\frac{1}{4}\lambda(1-\epsilon)|z|^2}$ for some $\epsilon > 0$ can be expanded as*

$$f_\lambda(z) = \sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} c_{\delta,j}\ \lambda^{\frac{p+q}{2}} P_j^\delta(z)\ \varphi_{m-p,\lambda}^{n+p+q-1}(z).$$

Proof: As before we can assume $\lambda = 1$. Let then $f(z)$ be an eigenfunction of L_{-1} and consider $F(\sigma) = f(\sigma \cdot z), \sigma \in K$ which is right M-invariant. By Peter-Weyl theorem we have the expansion

$$F(\sigma) = \sum_{\delta \in \hat{K}_0} d(\delta) \int_K F(\sigma k^{-1})\ \chi_\delta(k)\ dk \tag{4.3}$$

where $\chi_\delta(k) = tr(\delta(k))$ is the character of δ. Defining

$$f_\delta(z) = d(\delta) \int_K f(k^{-1}z)\ \chi_\delta(k)\ dk$$

we obtain the expansion

$$f(z) = \sum_{\delta \in \hat{K}_0} f_\delta(z). \tag{4.4}$$

We now expand $f_\delta(z)$ in terms of the functions $P_j^{\delta'}(\omega),\ 1 \le j \le d(\delta'),\ \delta' \in \hat{K}_0$.

$$\begin{aligned}\int_{S^{2n-1}} f_\delta(z)\ \overline{P_j^{\delta'}(\omega)}\ d\omega &= d(\delta) \int_{S^{2n-1}} \int_K f(k^{-1}z)\ \overline{P_j^{\delta'}(\omega)}\ \chi_\delta(k)\ dk\ d\omega \\ &= d(\delta) \int_K \chi_\delta(k) \left(\int_{S^{2n-1}} f(|z|\omega)\ \overline{P_j^{\delta'}(k\omega)}\ d\omega \right) dk.\end{aligned}$$

But

$$P_j^{\delta'}(k\omega) = \delta'(k)\ P_j^{\delta'}(\omega) = \sum_{i=1}^{d(\delta)} (\delta'(k)\ P_j^{\delta'}, P_i^{\delta'})\ P_i^{\delta'}(\omega).$$

This gives us

$$\begin{aligned}&\int_{S^{2n-1}} f_\delta(|z|\omega)\ \overline{P_j^\delta(\omega)}\ d\omega \\ &= d(\delta) \int_K \chi_\delta(k) \left(\sum_{i=1}^{d(\delta')} \int_{S^{2n-1}} f(|z|\omega) \overline{P_i^{\delta'}}(\omega) d\omega \right) \overline{(\delta'(k) P_j^{\delta'}, P_i^{\delta'})}\ dk \\ &= \sum_{i=1}^{d(\delta')} \left(\int_{S^{2n-1}} f(|z|\omega)\ \overline{P_i^{\delta'}}(\omega) d\omega \right) d(\delta) \int_K \chi_\delta(k)\ \overline{(\delta'(k) P_j^{\delta'}, P_i^{\delta'})}\ dk.\end{aligned}$$

The last integral is zero unless $\delta = \delta'$ in which case we obtain

$$f_\delta(z) = \sum_{j=1}^{d(\delta)} \left(\int_{S^{2n-1}} f_\delta(|z|\omega)\ \overline{P_j^\delta(\omega)}\ d\omega \right)\ P_j^\delta(\omega).$$

This gives

$$\begin{aligned}f_\delta(\sigma z) &= \sum_{j=1}^{d(\delta)} \left(\int_{S^{2n-1}} f_\delta(|z|\omega)\ \overline{P_j^\delta(\omega)} d\omega \right)\ P_j^\delta(\sigma\omega) \\ &= \sum_{j=1}^{d(\delta)} \sum_{i=1}^{d(\delta)} \left(\int_{S^{2n-1}} f_\delta(|z|\omega)\ \overline{P_j^\delta(\omega)}\ d\omega \right)\ (\delta(\sigma) P_j^\delta, P_i^\delta)\ P_i^\delta(\omega). \qquad (4.5)\end{aligned}$$

On the other hand

$$\begin{aligned} f_\delta(\sigma z) &= d(\delta) \int_K f(k\sigma z)\ \chi_\delta(k^{-1})\ dk \\ &= d(\delta) \int_K f(kz)\chi_\delta(\sigma k^{-1})\ dk. \end{aligned}$$

But

$$\begin{aligned} \chi_\delta(\sigma k^{-1}) &= tr(\delta(\sigma)\delta(k^{-1})) \\ &= \sum_{i=1}^{d(\delta)} \sum_{j=1}^{d(\delta)} (\delta(\sigma)P_j^\delta, P_i^\delta)(\delta(k^{-1})P_i^\delta, P_j^\delta) \end{aligned}$$

which gives

$$f_\delta(\sigma z) = d(\delta) \sum_{j=1}^{d(\delta)} \sum_{i=1}^{d(\delta)} \left(\int_{S^{2n-1}} f(kz)(\delta(k^{-1})P_i^\delta, P_j^\delta) dk \right) (\delta(\sigma)P_j^\delta, P_i^\delta). \tag{4.6}$$

Comparing (4.5) and (4.6) we obtain

$$\left(\int_{S^{2n-1}} f_\delta(|z|\omega)\overline{P_j^\delta(\omega)} d\omega \right)\ P_j^\delta(\omega) = d(\sigma) \int_K f(kz)(\delta(k^{-1})P_j^\delta, P_j^\delta)\ dk. \tag{4.7}$$

As the right hand side of (4.7) is an eigenfunction of L_{-1} with eigenvalue $(2m+n)$ so is the left hand side. This means that $g(z)P_j^\delta(z)$ is an eigenfunction of L where

$$g(z) = |z|^{-p-q} \int_{S^{2n-1}} f(|z|\omega)\overline{P_j^\delta(\omega)} d\omega.$$

Using Proposition 4.4 we get

$$g(z) = c_{\delta,j}\ \varphi_{m-p}^{n+p+q-1}\ (z)$$

and therefore

$$\left(\int_{S^{2n-1}} f(|z|\omega)\ \overline{P_j^\delta}(\omega)\ d\omega \right) P_j^\delta(\omega) = c_{\delta_j}\ P_j^\delta(z)\ \varphi_{m-p}^{n+p+q-1}(z).$$

Combining this with (4.4) and (4.5) we complete the proof.

We are now in a position to prove the converse part of Theorem 4.1. In $f_\lambda(z)$ is an eigenfunction of $L_{-\lambda}$ with eigenvalue $(2m+n)\lambda$ then we have the expansion given in Proposition 4.5. Taking the L^2 norm of $f_\lambda(z)$ over S^{2n-1} we get

$$\sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} |c_{\delta,j}|^2\ 2^{p+q} \left(\frac{1}{2}\lambda|z|^2 \right)^{p+q} (\varphi_{m-p,\lambda}^{n+p+q-1}(z))^2 < \infty. \tag{4.8}$$

Define $b_{\delta,j} = \lambda^{n/2}\ (-1)^p\ 2^{-p-q}\ \lambda^{\frac{p+q}{2}}\ c_k^\lambda(p,q)\ c_{\delta,j}$ and consider

$$\mu = \sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} b_{\delta,j}\ P_j^\delta.$$

We only need to show that $\mu \in \mathcal{B}^*$. For then

$$\int_{S^{2n-1}} \mathcal{P}_m^\lambda(z,\omega)\ d\mu(\omega) = \sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} c_{\delta,j}\ \lambda^{\frac{p+q}{2}}\ P_j^\delta(z)\ \varphi_{m-p,\lambda}^{n+p+q-1}(z) = f_\lambda(z)$$

which will complete the proof of Theorem 4.1.

For each $\alpha > -1$ the Laguerre polynomials $L_k^\alpha(t)$ satisfy the estimates

$$|L_k^\alpha(t)| \leq L_k^\alpha(0) = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1)}.$$

We also have $\frac{d}{dt}L_k^\alpha(t) = -L_{k-1}^{\alpha+1}(t)$, see Szego [11]. Therefore,

$$\begin{aligned} |L_k^\alpha(t)| &\geq L_k^\alpha(0) - t\, L_{k-1}^{\alpha+1}(0) \\ &= \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1)} \left(1 - \frac{kt}{\alpha+1}\right) \end{aligned}$$

and so when $\alpha + 1 > 2kt$, we have

$$|L_k^\alpha(t)| \geq \frac{1}{2} \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1)}.$$

Using this estimate in (4.8) we see that

$$\sum_{q=Q}^{\infty} \sum_{j=1}^{d(\delta)} |c_{\delta,j}|^2 \, (2r)^{p+q} \, \frac{\Gamma(m+q+n)}{\Gamma(n+p+q)} < \infty$$

for any $r > 0$ if Q is chosen so that $n + p + Q > 2(m-p)r$. Therefore,

$$\sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} |c_{\delta,j}|^2 \, 2^{k(p+q)} \, \frac{\Gamma(m+q+n)}{\Gamma(np+q)} < \infty$$

for every $k \in \mathbb{N}$. Recalling the definition of $b_{\delta,j}$ we get

$$\sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} |b_{\delta,j}|^2 \, 2^{(k+2)(p+q)} \, \lambda^{-(p+q)} \, c_k^\lambda(p,q)^{-2} \, \frac{\Gamma(m+q+n)}{\Gamma(n+p+q)} < \infty.$$

Since

$$c_k^\lambda(p,q)^2 = \left(\frac{4}{\lambda}\right)^{p+q} \, 2^{n+p+q-1} \, \frac{\Gamma(m+n+q)}{\Gamma(m-p+1)}$$

we have

$$\sum_{\delta \in \hat{K}(m)} \sum_{j=1}^{d(\delta)} |b_{\delta,j}|^2 \, 2^{(k-1)(p+q)} \, \frac{\Gamma(m-p+q)}{\Gamma(n+p+q)} < \infty.$$

As this is true for every k we see that $\mu \in \mathcal{B}^*$.

References

[1] G.B. Folland, *Harmonic analysis in phase space*, Ann. Math. Stud. 122 (1989), Princeton Univ. press, Princeton.

[2] D. Geller, *Spherical harmonics, the Weyl transform and the Fourier transform on the Heisenberg group*, Canad. J. Math. 36 (1984), 615-684.

[3] S. Helgason, *Topics in harmonic analysis on homogeneous spaces*, Prog. Math. 13 (1981), Birkhauser, Boston.

[4] S. Helgason, *A duality for symmetric spaces with applications to group representations*, Adv. Math. 5 (1970), 1-154.

[5] S. Helgason, *Eigenspaces of the Laplacian, integral representations and irreducibility*, J. Funct. Anal. 17 (1974), 328-353.

[6] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima and M. Tanaka, *Eigenfunctions of invariant differential operators on a symmetric space*, Ann. Math. 107 (1978), 1-39.

[7] F.W.J. Olver, *Asymptotics and special functions*, Comp. Sci. and Appl. Math., Academic press, New York (1974).

[8] T. Oshima and J. Sekiguchi, *Eigenspaces of invariant differential operators on an affine symmetric space*, Invent. Math. 57 (1980), 1-81.

[9] R.S. Strichartz, *Harmonic analysis as spectral theory of Laplacians*, J. Funct. Anal. 87 (1989), 51-148.

[10] R.S. Strichartz, *L^p harmonic analysis and Radon transforms on the Heisenberg group*, J. Funct. Anal. 96 (1991), 350-406.

[11] G. Szego, *Orthogonal polynomials*, Amer. Math. Sco., Colloq. Publ., Providence, RI (1967).

[12] S. Thangavelu, *Mean periodic functions on phase space and the Pompeiu problem with a twist*, Ann. Inst. Fourier (Grenoble), 45 (1995), 1007-1035.

[13] S. Thangavelu, *Harmonic analysis on the Heisenberg group*, Prog. Math. 159, Birkhauser, Boston (1998).

[14] S. Thangavelu, *An introduction to the uncertainty principle*, Prog. Math. 217, Birkhauser, Boston (2004).

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