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Degree-one maps, surgery and four-manifolds

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DEGREE-ONE MAPS, SURGERY AND FOUR-MANIFOLDS

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Abstract. We give a description of degree-one maps between closed, oriented 3-manifolds in terms of surgery. Namely, we show that there is a degree-one map from a closed, oriented 3-manifold M to a closed, oriented 3-manifold N if and only if M can be obtained from N by surgery about a link in N each of whose components is an unknot.

We use this to interpret the existence of degree-one maps between closed 3-manifolds in terms of smooth 4-manifolds. More precisely, we show that there is a degree-one map from M to N if and only if there is a smooth embedding of M in $W = (N \times I)\#_n\overline{\mathbb{C}P^2}\#_m\mathbb{C}P^2$, for some $m \geq 0$, $n \geq 0$ which separates the boundary components of W. This is motivated by the relation to topological field theories, in particular the invariants of Ozsvath and Szabo.

1. Introduction

We assume that all manifolds are connected and that all 3-manifolds are smooth. For closed, oriented 3-manifolds M and N, we say that M *dominates* N (or M 1-dominates N) if there is a degree-one map from M to N. This gives a transitive relation on closed, oriented 3-manifolds which has been extensively studied by several authors (for instance, see [1], [5], [10], [11], [12], [13], [15]). Note that every manifold dominates S^3 and that if M dominates N then there is a surjection from $\pi_1(M)$ to $\pi_1(N)$.

In this paper, we characterise dominance in terms of Dehn surgery. We use this to interpret dominance in terms of smooth 4-manifolds. The latter is motivated by the relation to topological field theories, in particular the invariants of Ozsvath and Szabo [8][9].

Suppose N is a closed 3-manifold and M is obtained from N by surgery about a link in N all of whose components are homotopically trivial, then it is easy to see that there is a degree-one map from M to N. Our first result is the converse, namely that if there is a degree-one map from M to N, then M can be obtained from N by surgery about a link $L \subset N$ all of whose components are homotopically trivial. In fact we can find L each of whose components is an unknots.

Theorem 1.1. *For closed oriented 3-manifolds M and N, there is a degree-one map from M to N if and only if M can be obtained from N by surgery about a link in N each of whose components is an unknot in N.*

We next interpret dominance of 3-manifolds in terms of 4-manifolds. Observe that a partial ordering on closed orientable 3-manifolds can be defined by saying that M *strongly dominates* N if there is a smooth embedding $i\colon M \to N \times (0,) \subset N \times [0,1]$ so that $i(M)$ separates the two boundary components $N \times \{0\}$ and $N \times \{1\}$, with the *appropriate orientation.* Observe that if M strongly dominates N, the the composition $\pi \circ i$ of the embedding i with the projection $\pi\colon N \times [0,1] \to N$ has degree ± 1. We say that the embedding has the *appropriate orientation* if the degree of this map is one.

This definition is motivated by the relation to ((3+1)-dimensional) *topological field theories*, in particular the invariants of Ozsvath and Szabo. Recall that a degree-one map $f\colon M \to N$ induces a surjection

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f_* on the fundamental group. Hence if $\pi_1(N)$ is non-trivial so is $\pi_1(M)$. Further $f^*\colon H^*(N) \to H^*(M)$ is an injection, which shows that if $H^k(N) \neq 0$ then $H^k(M) \neq 0$.

We see that an analogous result holds for any topological field theory, with dominance replaced by strong dominance. Recall that a $(3+1)$-dimensional topological field theory associates to each closed, oriented 3-manifold M a vector space $V(M)$ and to each cobordism W from M to another closed, oriented 3-manifold N a linear transformation $T(W)\colon V(M) \to V(N)$. Further this satisfies functorial properties, namely a product cobordism induces the identity map and if W_1 is a cobordism from M_1 to M_2 and W_2 is a cobordism from M_2 to M_3 then for the cobordism $W_1 \coprod_{M_2} W_2$ from M_1 to M_3, $T(W_1 \coprod_{M_2} W_2) = T(W_2) \circ T(W_1)$.

Suppose M strongly dominates N, then splitting $N \times [0,1]$ along the given embedding of M gives two cobordisms, W_1 from N to M and W_2 from M to N. The composition of these is the product cobordism $N \times [0,1]$, which induced the identity map on $T(N)$. It follows that the identity map on $V(N)$ factors through $V(M)$, and in particular $V(N) \neq 0$ implies that $V(M) \neq 0$. This is the analogue of the corresponding results for π_1 and H^* with respect to degree-one maps. Thus *strong dominance* plays the same role in the *bordism category* as dominance in the homotopy category.

We shall see that relation of strong dominance is stronger than dominance. In particular we shall see in Theorem 4.1 that for lens spaces $L(p,q)$ and $L(p,q')$, if $L(p,q)$ strongly dominates $L(p,q')$ then $L(p,q)$ and $L(p,q')$ are homeomorphic. This holds even if the embedding is only assumed to be topologically locally flat (rather than smooth). However, there are degree-one maps, in fact homotopy equivalences, between lens spaces which are not homeomorphic (for example $L(7,1)$ and $L(7,2)$). We get more subtle examples using Gauge theory, namely the Poincaré homology sphere does not (smoothly) strongly dominate S^3. However there is a topologically locally flat embedding of the Poincaré homology sphere in $S^3 \times [0,1]$.

We show, however, that dominance is equivalent to a relation obtained using 4-manifolds similar to the above one except that we allow 'positive and negative blow-ups'.

Theorem 1.2. *For closed orientable 3-manifolds M and N, there is a degree-one map from M to N if and only if there is a smooth embedding of M in $int(W)$, $W = (N \times I)\#_n\overline{\mathbb{C}P^2}\#_m\mathbb{C}P^2$ for some $m > 0$, $n > 0$ which separates the boundary components of W, with the embedding having the appropriate orientation.*

There is a relation in between dominance and strong dominance which is of interest. Namely, we say that M *negatively dominates* N if there is an embedding of M into $W = (N \times I)\#_n\overline{\mathbb{C}P^2}$, for some $n \geq 0$, which separates the two boundary components. This is of interest because the Ozsvath-Szabo invariants (as also the Seiberg-Witten invariants) behave well under blowing up.

We shall see (in Theorem 4.2) that the Poincaré homology sphere does not even negatively dominate S^3. We shall study negative dominance elsewhere.

2. Degree-one maps and Surgery

In this section, we give a proof of Theorem 1.1. We first recall the (well known) easy part of the result (see, for example, [1]). Namely, let M be obtained from N by surgery about a link $L \subset N$ each of whose components K_i is homotopically trivial. Then we have a corresponding link $L' \subset M$ with components K_i', and an identification $M - int(Nbd(L')) = N - int(Nbd(L))$. We construct a degree-one map $f\colon M \to N$ as follows. Define f on $M - int(Nbd(L')) = N - int(Nbd(L))$ to be the identity map. We extend this to each solid torus $X_i = Nbd(K_i') = D^2 \times S^1$.

Fix an identification $X_i = D^2 \times S^1$, $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$ and let $X_i' \subset X_i$ be the solid torus $\{z \in \mathbb{C} : |z| \leq 1/2\} \times S^1$. Let $T_i = \partial X_i'$. We identify $A_i = T_i \times [1/2, 1] = X_i - int(X_i')$ with $S^1 \times S^1 \times [1/2, 1]$ so that the curves of the form $S^1 \times \{p\} \times \{1\}$ are mapped by f to meridians of $Nbd(K_i)$. Then f extends to a map on A_i, with image contained in $Nbd(K_i)$, so that curves of the form $S^1 \times \{p\} \times \{1/2\}$ are mapped to points on K_i.

Now identify X_i' with $D^2 \times S^1$, which gives a new identification of T_i with $S^1 \times S^1$. Under this identification, we can assume that curves of the form $S^1 \times \{p\}$ are mapped to $K_i = S^1$ by a smooth map h, which we identify with a map $h\colon S^1 \to K$. As K is homotopic to the trivial curve, this extends to a smooth map $h\colon D^2 \to K$. Define f on $X_i' = D^2 \times S^1$ by $f(x, y) = h(x)$. Thus the map f is defined on all of M. We see that this has degree one as any point in N disjoint from $Nbd(L) \cup image(h)$ is a regular value for f and has inverse image a single point P, with f the identity on a neighbourhood of P.

The converse is based on the following theorem of Haken [4] and Waldhausen [14] (see also [12]).

Theorem (Haken-Waldhausen)**.** *Let $f\colon M \to N$ be a degree-one map and let $N = H_1 \cup H_2$ be a Heegaard decomposition of N with H_1 and H_2 handlebodies. Then f is homotopic to a map g such that $g|_{g^{-1}(H_1)}\colon g^{-1}(H_1) \to H_1$ is a homeomorphism.*

Such a map is called a 1-pinch. Thus if M dominates N, there is a 1-pinch $g\colon M \to N$.

Proof of Theorem 1.1. Assume M dominates N and let g, H_1 and H_2 be as above. We shall first show that M can be obtained from N by surgery about a link each of whose components is homotopically trivial.

Consider a collection of properly embedded discs D_i, $1 \leq i \leq n$ in H_2 such that on splitting H_2 along D_i, we get a 3-ball B. We can assume that g is transversal to D_i for all i, $1 \leq i \leq n$. Let $F_i = g^{-1}(D_i)$ and let $K = g^{-1}(H_2)$. Note that $g|_{\partial H_2}$ is a homeomorphism and hence F_i consists of a compact surface with a single boundary component and a (possibly empty) collection of closed surfaces. First, note that by performing a homotopy of g we can assume that each F_i is connected. This follows (as the induced map on π_1 is a surjection) by using standard techniques using binding ties as in Stalling's proof of the Knesser conjecture (see for example the proof of Knesser's conjecture in [6]). Hence F_i, $1 \leq i \leq n$, is a compact surface with a single boundary component.

We first consider the special case when each F_i is a disc. After a homotopy of g, we can assume that F_i maps homeomorphically onto D_i for $1 \leq i \leq n$. On splitting K along the properly embedded discs F_i, $1 \leq i \leq n$, we get a manifold K' with boundary a 2-sphere. By the theorem of Lickorish and Wallace, this can be obtained from B by surgery about a link L in B. Note that each component of L is contained in the 3-ball B and hence is homotopically trivial. Indeed the Lickorish-Wallace theorem says that we can take each component to be an unknot.

We now consider the general case. For each i, $1 \leq i \leq n$, consider a collection L_i of disjoint, embedded simple closed curves on F_i which do not separate F_i and are maximal with respect to this property. Then $L' = \cup_i L_i$ is a link in M. We consider a corresponding framed link (also denoted L'), with the framing of a component of L_i given by the normal to F_i. Let M' be the manifold obtained from M by surgery about the framed link L'.

We shall see that the map g induces a degree-one map g' from M' to N with $g'^{-1}(D_i)$ obtained from F_i by compressing along the components of L_i. By the choice of L_i it follows that $g'^{-1}(D_i)$ is a disc for all i, $1 \leq i \leq n$.

Let the components of L_i be C^i_j and let T^i_j denote a regular neighbourhood of C^i_j. On the complement of $\bigcup_{i,j} int(T^i_j)$, we let $g' = g$. After surgery, each T^i_j is replaced by a solid torus $X = X^i_j$ with the same boundary as T^i_j. Now, $F_i \cap \partial X$ consists of two meridinal curves (on ∂X) which bound properly embedded discs E_1 and E_2 in X. As $g(\partial E_i) \subset D_i$, the map g' extends to E_i with $g'(E_i) \subset D_i$. Using transversality of g to D_i, we see that we can extend g' to a regular neighbourhood $E_i \times [-1, 1]$ of E_i with $g'(E_i \times [-1, 1] - E_i) \cap D_i = \phi$. Finally, $X - (E_1 \times (-1, 1)) - (E_2 \times (-1, 1))$ is the union of two balls, each of whose boundaries is mapped by g' into B. Thus g' extends to a map on X with $(g'|_T)^{-1}(D_i) = E_1 \cup E_2$. Making this construction for each X^i_j, we get a map g' as claimed.

Now g' is a pinch which is as in the special case, i.e. $g'^{-1}(D_i)$ is a disc for all i, $1 \leq i \leq n$. Thus M' can be obtained from N by surgery about a link $L_0 \subset N$, each of whose components is homotopically trivial in N. We now perform surgeries about knots $\gamma^i_j \subset M'$ so that the surgery about γ^i_j cancels the surgery about $C^i_j \subset L'$. Thus on performing such surgeries we obtain M. The knots γ^i_j can be regarded as knots in $H_2 \subset N$ and their union, with framing corresponding to the cancelling surgeries, is a framed link $L_1 \subset N$.

Thus M is obtained from N be surgery about the framed link $L = L_0 \cup L_1$, with each component of L_0 homotopically trivial. We next show that the knots γ^i_j, regarded as curves in N, are trivial. Recall that, as γ^i_j is the knot corresponding to the surgery cancelling the surgery about C^i_j, it is given by an unknot bounding a disc that intersects C^i_j transversally in a single point and is contained in a ball around a point in C^i_j. Hence we can assume that γ^i_j intersects F_i transversally in two points, with opposite signs of intersection, and γ^i_j is disjoint from F_k for $k \neq i$.

Note that $\pi_1(H_2)$ is a free group with generators α_i, $1 \leq i \leq n$, corresponding to the discs D_i. Further, if γ is a curve transversal to the discs D_i, $1 \leq i \leq n$, then (up to conjugacy) the word represented by γ is determined by the intersection points with the discs D_i. Namely, if the points of $\gamma \cap (\cup_i D_i)$, in cyclic order around γ, are contained in $D_{i_1}, \dots D_{i_k}$ with signs of intersection $\epsilon_j = \pm 1$, then $\gamma = \alpha^{\epsilon_1}_{i_1} \dots \alpha^{\epsilon_k}_{i_1}$ up to conjugacy.

As γ^i_j intersects F_i (hence D_i) transversally in two points, with opposite signs of intersection, and γ^i_j is disjoint from F_k (hence D_k) for $k \neq i$, it follows that γ^i_j represents the trivial word in $\pi_1(H_2)$, and hence is homotopically trivial in N.

Now as each compoenent of L is homotopically trivial, there is a sequence of crossing changes so that on performing these crossing changes we obtain a link all of whose components are unknots. Observe that each crossing change of a knot κ is locally of a standard form. Namely, there is a ball $B \subset M$ which intersects κ in a pair of arcs c_1 and c_2, and the crossing change corresponds to a crossing of these arcs to give new arcs c'_1 and c'_2 with the same endpoints as c_1 and c_2.

Further, if K_i is an unknot in B unlinked from the arcs c_i with framing ± 1, then on performing the Kirby moves of sliding c_1 and c_2 over K_i, with opposite orientations, we get the knot obtained by crossing c_1 and c_2. In this manner we can obtain both positive and negative crossing changes.

Note that replacing L by its union with unknots and performing the Kirby moves as above does not change the resulting manifold. Thus, we can replace L by a framed link in N, each of whose components is an unknot, so that the result of surgery about the link is M.

□

3. Surgery and 4-manifolds

We now characterise dominance in terms of 4-manifolds.

Proof of Theorem 1.2. Suppose M embeds in W as in the hypothesis. Then $W - M$ has two components with closures K_1 and K_2 so that $\partial[K_1] = [M] - [N \times \{0\}]$. Hence $[M]$ is homologous to $[N \times \{0\}]$. Now by identifying all the points in each $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$ in $W = (M \times I) \#_n \overline{\mathbb{C}P^2} \#_m \mathbb{C}P^2$ to a single point, we get a blow-down map $\pi\colon W \to N \times [0, 1]$. By composing with the projection, we get a map

$p\colon W \to N$ with $p\colon N \times \{0\} \to N$ being the identity map. This restricts to a map $p\colon M \to N$. As $[M]$ is homologous to $[N \times \{0\}]$, $p_*([M]) = [N]$, i.e., M has degree one.

Conversely, assume M and N are as in the hypothesis with $\pi_1(M) = 1$. By Theorem 1.1, M can be obtained from N by surgery about a framed link L, all of whose components are unknots in N. Hence L can be obtained from an unlink $L_0 \subset N$ by a sequence of (say p) crossings.

Let $K_1, \dots K_n$ a collection of unknots in M, with $n \geq p$ to be specified later, so that $L_0 \cup \{K_1, \dots K_n\}$ forms an unlink. Let W be obtained by attaching a 2-handle with framing ± 1 (with signs to be chosen later) to $M \times [0, 1/2]$ along each of K_0, $K_1, \dots K_n$. Note that $W = (M \times [0,1]) \#_k \overline{\mathbb{C}P^2} \#_l \mathbb{C}P^2$ for some k and l.

We shall construct a different Kirby diagram for W. Corresponding to the p crossings of L_0 required to make it isotopic to L we can find disjoint balls B_i, $1 \leq i \leq p$, in which the crossing is made. By an isotopy, we can assume that for $1 \leq i \leq p$, K_i is contained in B_i. Performing the Kirby moves corresponding to the corossing changes in each of these B_i, we get a Kirby diagram for W with a sublink isotopic to L. Furthermore, by performing the Kirby move of sliding over the unknots $K_{p+1}, \dots$, K_n (with framing ± 1) we can ensure this sublink is isotopic to L as a framed link (as such a Kirby move changes the framing by ± 1 without changing the link). Consider the corresponding Morse function for W with the 2-handles corresponding to components of L attached first. The level set on attaching L is the result of surgery about L. But this is M, and hence we get an embedding of M separating the boundary components of W. □

4. Examples

We now consider the relation of strong dominance where we allow no blowing-up. We see that we have restrictions in this case even for topologically locally flat embeddings.

Theorem 4.1. *If the lens space $L(p, q)$ (topologically) strongly dominates the lens space $L(p, q')$ for a prime p then $L(p, q)$ and $L(p, q')$ are homeomorphic.*

Proof. By hypothesis we have an embedding of $L(p, q)$ in $L(p, q') \times [0, 1]$ separating the two boundary components. Splitting along $L(p, q)$, we get a cobordism from $L(p, q')$ to $L(p, q)$. By the Seifert-Van Kampen theorem and a Mayer-Vietoris argument we see that this is an h-cobordism. By a result of Milnor [7], it follows that $L(p, q)$ and $L(p, q')$ are homeomorphic. □

More subtle obstructions can be obtained using Gauge theory. Any 3-manifold dominates the 3-sphere. Furthermore, by a theorem of Freedman [3], we have a topological locally flat embedding of any homology sphere into $S^3 \times [0, 1]$. However in the smooth case, we see that S^3 is not a minimal element with respect to strong dominance or even negative dominance.

Theorem 4.2. *For $n \geq 0$, there is no embedding of the Poincare homology sphere in $(S^3 \times I) \#_n \overline{\mathbb{C}P^2}$ which separates the boundary components.*

Proof. Note that the Poincaré homology sphere can be obtained from S^3 by surgery on the E_8 link, and hence, with one of its orientations, bounds a 4-manifold W with positive definite intersection pairing. Denote the Poincaré homology sphere with this orientation as M.

Supppose, for some $n \geq 0$, there is an embedding of the Poincare homology sphere in $(S^3 \times I) \#_n \overline{\mathbb{C}P^2}$ which separates the boundary components. Then by capping off the boundary components $S^3 \times \{0\}$ and $S^3 \times \{1\}$, we get an embedding of the Poincare homology sphere in $\#_n \overline{\mathbb{C}P^2}$. Splitting along the embedding and using the Mayer-Vietoris sequence, we get 4-manifolds W_1 and W_2 bounding M and $-M$ with positive definite intersection forms. If $\partial W_1 = -M$, then $Y = W \coprod_M W_1$ is a smooth 4-manifold with $H_1(Y, \mathbb{Z}) = 0$ and the intersection form on $H_2(Y, \mathbb{Z})$ being positive definite but not diagonal, contradicting Donaldson's theorem [2].

$\square$

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