isib/ms/2005/8
February 18th, 2005
http://www.isibang.ac.in/~statmath/eprints

Powers of a matrix and combinatorial identities

J. Mc Laughlin and B. Sury

Indian Statistical Institute, Bangalore Centre
8th Mile Mysore Road–560 059, India

POWERS OF A MATRIX AND COMBINATORIAL IDENTITIES

J. MC LAUGHLIN AND B. SURY

ABSTRACT. In this article we obtain a general polynomial identity in k variables, where $k \geq 2$ is an arbitrary positive integer.

We use this identity to give a closed-form expression for the entries of the powers of a $k \times k$ matrix.

Finally, we use these results to derive various combinatorial identities.

1. Introduction

In [4], the second author had observed that the following 'curious' polynomial identity holds:

$$\sum (-1)^i \binom{n-i}{i} (x+y)^{n-2i}(xy)^i = x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n.$$

The proof was simply observing that both sides satisfied the same recursion. He had also observed (but not published the result) that this recursion defines in a closed form the entries of the powers of a 2×2 matrix in terms of its trace and determinant and the entries of the original matrix. The first author had independently discovered this fact and derived several combinatorial identities as consequences [2].

In this article, for a general k, we obtain a polynomial identity and show how it gives a closed-form expression for the entries of the powers of a $k \times k$ matrix. From these, we derive some combinatorial identities as consequences.

2. Main Results

Throughout the paper, let K be any fixed field of characteristic zero. We also fix a positive integer k. The main results are the following two theorems:

Theorem 1. *Let $x_1, \cdots, x_k$ be independent variables and let $s_1, \cdots, s_k$ denote the various symmetric polynomials in the x_i's of degrees $1, 2 \cdots, k$ respectively. Then, in the polynomial ring $K[x_1, \cdots, x_k]$, for each positive integer n, one has the identity*

$$\sum_{r_1+\cdots+r_k=n} x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k} =$$

$$\sum_{2i_2+3i_3+\cdots+ki_k \leq n} c(i_2, \cdots, i_k, n) s_1^{n-2i_2-3i_3-\cdots-ki_k} (-s_2)^{i_2} s_3^{i_3} \cdots ((-1)^{k-1} s_k)^{i_k},$$

where

$$c(i_2, \cdots, i_k, n) = \frac{(n - i_2 - 2i_3 - \cdots - (k-1)i_k)!}{i_2! \cdots i_k!(n - 2i_2 - 3i_3 - \cdots - (ki_k)!}.$$

Date: November 30th, 2004.

Key words and phrases. Polynomial and Combinatorial Identities.

Theorem 2. *Suppose $A \in M_k(K)$ and let*

$$T^k - s_1 T^{k-1} + s_2 T^{k-2} + \cdots + (-1)^k s_k\, I$$

denote its characteristic polynomial. Then, for all $n \geq k$, one has

$$A^n = b_{k-1} A^{k-1} + b_{k-2} A^{k-2} + \cdots + b_0\, I,$$

where

$$\begin{aligned}
b_{k-1} &= a(n-k+1),\\
b_{k-2} &= a(n-k+2) - s_1 a(n-k+1),\\
&\vdots\\
b_1 &= a(n-1) - s_1 a(n-2) + \cdots + (-1)^{k-2} s_{k-2} a(n-k+1),\\
b_0 &= a(n) - s_1 a(n-1) + \cdots + (-1)^{k-1} s_{k-1} a(n-k+1)\\
&= (-1)^{k-1} s_k a(n-k).
\end{aligned}$$

and

$$a(n) = c(i_2, \cdots, i_k, n) s_1^{n-i_2-2i_3-\cdots-(k-1)i_k} (-s_2)^{i_2} s_3^{i_3} \cdots ((-1)^{k-1} s_k)^{i_k},$$

with

$$c(i_2, \cdots, i_k, n) = \frac{(n - i_2 - 2i_3 - \cdots - (k-1)i_k)!}{i_2! \cdots i_k! (n - 2i_2 - 3i_3 - \cdots - (ki_k)!}.$$

as in Theorem 1.

Proof of Theorems 1 and 2. In Theorem 1, if $a(n)$ denotes either side, it is straightforward to verify that

$$a(n) = s_1 a(n-1) - s_2 a(n-2) + \cdots + (-1)^{k-1} s_k a(n-k).$$

Theorem 2 is a consequence of Theorem 1 on using induction on n.

□

The special cases $k = 2$ and $k = 3$ are worth noting for it is easier to derive various combinatorial identities from them.

Corollary 1. *(i) Let $A \in M_3(K)$ and let $X^3 = tX^2 - sX + d$ denote the characteristic polynomial of A. Then, for all $n \geq 3$,*

$$A^n = a_{n-1} A + a_{n-2} Adj(A) + (a_n - t a_{n-1})\, I, \tag{2.1}$$

where

$$a_n = \sum_{2i+3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} t^{n-2i-3j} s^i d^j$$

for $n > 0$ and $a_0 = 1$.

(ii) Let $B \in M_2(K)$ and let $X^2 = t\,X - d$ denote the characteristic polynomial of B. Then, for all $n \geq 2$,

$$B^n = b_n I + b_{n-1} Adj(B)$$

for all $n \geq 2$, *where*

$$b_n = \sum \binom{n-i}{i} (-1)^i t^{n-2i} d^i.$$

Corollary 2. *Let* $\theta \in K$, $B \in M_2(K)$ *and* t *denote the trace and* d *the determinant of* B. *We have the following identity in* $M_2(K)$ *:*

$$(a_{n-1} - \theta a_{n-2})B + (a_n - (\theta + t)a_{n-1} + \theta a_{n-2} t)I$$
$$= y_{n-1} B + (y_n - t\, y_{n-1})I,$$

where

$$a_n = \sum_{2i+3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} (\theta + t)^{n-2i-3j} (\theta t + d)^i (\theta d)^j$$

and

$$y_n = \sum \binom{n-i}{i} (-1)^i t^{n-2i} d^i.$$

In particular, for any $\theta \in K$, *one has*

$$b_n - (\theta + 1) b_{n-1} + \theta b_{n-2} = 1,$$

where

$$b_n = \sum_{2i+3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} (\theta + 2)^{n-2i-3j} (1 + 2\theta)^i \theta^j.$$

Proof. This is obtained by applying corollary 1 to the 3×3 matrix

$$A = \begin{pmatrix} B & 0 \\ 0 & \theta \end{pmatrix}.$$

□

Corollary 3. *The numbers* $c_n = \sum_{2i+3j=n} (-1)^i \binom{i+j}{j} 2^i 3^j$ *satisfy*

$$c_n + c_{n-1} - 2c_{n-2} = 1.$$

Proof. This is the special case of Corollary 2 where we take $\theta = -2$. Note that the sum defining c_n is over only those i, j for which $2i + 3j = n$. □

Note than when $k = 3$, Theorem 1 can be rewritten as follows:

Theorem 3. *Let* n *be a positive integer and* x, y, z *be indeterminates. Then*

$$\sum_{2i+3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} (x+y+z)^{n-2i-3j} (xy+yz+zx)^i (xyz)^j \tag{2.2}$$
$$= \frac{x\,y\,\left(x^{n+1} - y^{n+1}\right) - x\,z\,\left(x^{n+1} - z^{n+1}\right) + y\,z\,\left(y^{n+1} - z^{n+1}\right)}{(x-y)\,(x-z)\,(y-z)}.$$

Proof. In Corollary 1, let

$$A = \begin{pmatrix} x+y+z & 1 & 0 \\ -x\,y - x\,z - y\,z & 0 & 1 \\ x\,y\,z & 0 & 0 \end{pmatrix}.$$

Then $t = x+y+z$, $s = xy+xz+yz$ and $d = xyz$. It is easy to show (by first diagonalizing A) that the $(1,2)$ entry of A^n equals the right side of (2.2), with $n+1$ replaced by n, and the $(1,2)$ entry on the right side of (2.1) is a_{n-1}. □

Corollary 4. *(i) Let x and z be indeterminates and $n \geq 3$ a positive integer. Then*

$$\sum_{2i+3j\leq n} (-1)^i \binom{i+j}{j}\binom{n-i-2j}{i+j}(2x+z)^{n-2i-3j}(x^2+2xz)^i(x^2z)^j$$
$$= \frac{x^{2+n} + n\,x^{1+n}\,(x-z) - 2\,x^{1+n}\,z + z^{2+n}}{(x-z)^2}.$$

(ii) For each positive integer $n \geq 3$

$$\sum_{2i+3j\leq n} (-1)^i \binom{i+j}{j}\binom{n-i-2j}{i+j}3^{n-i-3j} = \frac{(n+1)(n+2)}{2}.$$

Proof. (i) Let $y \to x$ in Theorem 3. For (ii), let $z \to x$ in (i). □

Some interesting identities can be derived by specialising the variables in Theorem 1. For instance, in [5], it was noted that Binet's formula for the Fibonacci numbers is a consequence of Theorem 1 for $k=2$. This can be generalized as follows :

Corollary 5. *(Generalization of Binet's formula)*
Let the numbers $F_k(n)$ be defined by the recursion

$$F_k(0) = 1, F_k(r) = 0 \forall r < 0,$$
$$F_k(n) = F_k(n-1) + F_k(n-2) + \cdots + F_k(n-k).$$

Then, we have

$$F_k(n) = \sum_{2i_2+\cdots+ki_k\leq n} \frac{(n-i_2-2i_3-\cdots-(k-1)i_k)!}{i_1!i_2!\cdots i_k!(n-2i_2-3i_3-\cdots-ki_k)!}.$$

Further, this equals $\sum_{r_1+\cdots+r_k=n} \lambda_1^{r_1}\cdots\lambda_k^{r_k}$ where $\lambda_i, 1 \leq i \leq k$ are the roots of the equation $T^k - T^{k-1} - T^{k-2} - \cdots - 1 = 0$.

Proof. The recursion defining $F_k(n)$'s corresponds to the case $s_1 = -s_2 = \cdots = (-1)^{k-1}s_k = 1$ of the theorem. □

Corollary 6.

$$\sum c(i_2,\cdots,i_k,n)k^n \prod_{j=2}^{k}\left((-1)^{j-1}k^{-j}\binom{k}{j}\right)^{i_j} = \binom{n+k-1}{k}.$$

where

$$c(i_2,\cdots,i_k,n) = \frac{(n-i_2-2i_3-\cdots-(k-1)i_k)!}{i_2!\cdots i_k!(n-2i_2-3i_3-\cdots-ki_k)!}.$$

Proof. Take $x_i = 1$ for all i in Theorem 1. The left side of Theorem 1 is simply the sum $\sum_{r_1+\cdots+r_k=n} 1$. □

From Theorem 3 we have the following binomial identities as special cases.

Proposition 1. *(i)*

$$\sum_{2i+3j\le n} (-1)^j \binom{i+j}{j} \binom{n-i-2j}{i+j} = [n/2] + 1.$$

(ii)

$$\sum \binom{n-2j}{j} (-4)^j 3^{n-3j} = \frac{(3n+4)2^{n+1} + (-1)^n}{9}.$$

(iii)

$$\sum \binom{n-2j}{j} 3^{n-3j}(-2)^j$$
$$= \frac{(1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1}}{2\sqrt{3}} + \frac{(1+\sqrt{3})^{n+1} + (1-\sqrt{3})^{n+1}}{6} - \frac{1}{3}.$$

(iv)

$$1 + \sum_{3j\le n+2} (-1)^j \binom{n+2-2j}{j} 2^{n+2-3j} = F_n.$$

(v) For $w \neq 1$,

$$\sum_{2i+3j\le n} (-1)^j \binom{i+j}{j} \binom{n-i-2j}{i+j} w^{i+j} = \frac{w^{\lceil (n+1)/2 \rceil} - 1}{w-1}.$$

(vi)

$$\sum_{2i+3j\le n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} 2^{n-i-3j} = \begin{cases} 0, & n \equiv 4, 5 \mod 6, \\ 1, & n \equiv 0, 3 \mod 6, \\ 2, & n \equiv 1, 2 \mod 6, \end{cases}.$$

Proof. For (i), we take $x = y = 1, z = -1$ in theorem 3 and simplify the right hand side. Note that the right hand side has to be interpreted in the obvious manner when two of the variables are specialised to the same value. For (ii), take $x = y = 1, z = -\frac{1}{2}$ and for (iii), take $x = 1, y = 1-\sqrt{3}, z = 1+\sqrt{3}$. To prove (iv), set $x = 1, y = (1-\sqrt{5})/2$, $z = (1+\sqrt{5})/2$ and simplify. For (v), set $x = 1, y = \sqrt{w}$ and $z = -\sqrt{w}$. Part (vi) follows upon setting $x = 1, y = (1 - \imath\sqrt{3})/2$, $z = (1 + \imath\sqrt{3})/2$. □

3. Commuting Matrices

In this section we derive various combinatorial identities by writing a general 3×3 Matrix A as a product of commuting matrices.

Proposition 2. *Let A be an arbitrary 3×3 matrix with characteristic equation $x^3-tx^2+s\,x-d=0$, $d\neq 0$. Suppose p is arbitrary, with $p^3+p^2t+ps+d\neq 0$, $p\neq 0, -t$. If n is a positive integer, then*

$$A^n=\left(\frac{p\,d}{p^3+p^2t+sp+d}\right)^n\sum_{r=0}^{3n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k} \times\left(\frac{-p(p+t)^2}{d}\right)^j\left(\frac{-(p+t)}{p}\right)^k\left(\frac{-A}{p+t}\right)^r. \tag{3.1}$$

Proof. This follows from the identity

$$A=\frac{-1}{p^3+p^2t+sp+d}\left(pA^2-Ap(p+t)-d\,I\right)(A+p\,I),$$

after raising both sides to the n-th power and collecting powers of A. Note that the two matrices $pA^2-Ap(p+t)-d\,I$ and $A+p\,I$ commute. □

Corollary 7. *Let p, x, y and z be indeterminates and let n be a positive integer. Then*

$$\begin{aligned}\sum_{r=0}^{3\,n}\sum_{j=0}^{n}\sum_{k=0}^{n}&\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{j+k+r}\left(\frac{p(p+x+y+z)^2}{xyz}\right)^j\\
&\times\left(\frac{p+x+y+z}{p}\right)^k\frac{x\,y\,(x^r-y^r)-x\,z\,(x^r-z^r)+y\,z\,(y^r-z^r)}{(p+x+y+z)^r}\\
&=(x\,y\,(x^n-y^n)-x\,z\,(x^n-z^n)+y\,z\,(y^n-z^n))\\
&\qquad\times\left(\frac{p^3+p^2\,(x+y+z)+p\,(x\,y+x\,z+y\,z)+x\,y\,z}{p\,x\,y\,z}\right)^n.\end{aligned}$$

Proof. Let A be the matrix from Theorem 3 and compare $(1,1)$ entries on both sides of (3.1). □

Corollary 8. *Let p, x and z be indeterminates and let n be a positive integer. Then*

$$\begin{aligned}\sum_{r=0}^{3\,n}\sum_{j=0}^{n}\sum_{k=0}^{n}&\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{j+k+r}\left(\frac{p(p+2x+z)^2}{x^2z}\right)^j\\
&\qquad\times\left(\frac{p+2x+z}{p}\right)^k\frac{r\,x^{1+r}-x^r\,z-r\,x^r\,z+z^{1+r}}{(p+2x+z)^r}\\
&=\left(n\,x^{1+n}-x^n\,z-n\,x^n\,z+z^{1+n}\right)\\
&\qquad\times\left(\frac{p^3+p^2\,(2x+z)+p\,\left(x^2+2x\,z\right)+x^2\,z}{p\,x^2\,z}\right)^n.\end{aligned}$$

Proof. Divide both sides in the corollary above by $x-y$ and let $y\to x$. □

Corollary 9. *Let p and x be indeterminates and let n be a positive integer. Then*

$$\sum_{r=0}^{3n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{j+k+r}\left(\frac{p(p+3x)^2}{x^3}\right)^j \times\left(\frac{p+3x}{p}\right)^k\frac{r\,(1+r)\,x^{-1+r}}{2(p+3x)^r} = \frac{n\,(1+n)\,x^{-1+n}}{2}\left(\frac{(p+x)^3}{p\,x^3}\right)^n.$$

Proof. Divide both sides in the corollary above by $(x-z)^2$ and let $z \to x$. □

Corollary 10. *Let p be an indeterminate and let n be a positive integer. Then*

$$\sum_{r=0}^{3n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{j+k+r}p^{j-k}(p+3)^{2j+k-r}\frac{r\,(1+r)}{2} = \frac{n\,(1+n)}{2}\frac{(p+1)^{3n}}{p^n}.$$

Proof. Replace p by $p\,x$ in the corollary above and simplify. □

Various combinatorial identities can be derived from Theorem 3 by considering matrices A such that particular entries in A^n have a simple closed form. We give four examples.

Corollary 11. *Let n be a positive integer.*
(i) If $p \neq 0, -1$, then

$$\sum_{r=0}^{3n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{j+k+r}p^{j-k}(p+3)^{2j+k-r}r = n\frac{(1+p)^{3n}}{p^n}.$$

(ii) Let F_n denote the n-th Fibonacci number. If $p \neq 0, -1, \phi$ or $1/\phi$ (where ϕ is the golden ratio, then

$$\sum_{r=0}^{3n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{k+r}p^{j-k}(p+2)^{2j+k-r}F_r = F_n\frac{(1+p)^n\left(-1+p+p^2\right)^n}{(-p)^n}.$$

(iii) If $p \neq 0, -1$ or -2, then

$$\sum_{r=0}^{3n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{j+k+r}p^{j-k}(p+4)^{2j+k-r}2^{-j}(2^r-1) = (2^n-1)\left(\frac{(1+p)^2(p+2)}{2p}\right)^n.$$

(iv) If $p \neq 0, -1, -g$ or $-h$ and $gh \neq 0$, then

$$\sum_{r=0}^{3n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{j+k+r}p^{j-k}(p+1+g+h)^{2j+k-r} \times \frac{g^r+h^r}{(g\,h)^j} = (g^n+h^n)\left(\frac{(1+p)(g+p)(h+p)}{ghp}\right)^n.$$

Proof. The results follow from considering the $(1,2)$ entries on both sides in Theorem 3 for the matrices

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&1&0\\1&0&0\\0&0&1\end{pmatrix},\begin{pmatrix}3&1&0\\-2&0&0\\0&0&1\end{pmatrix},\begin{pmatrix}\frac{g+h}{2}&\frac{(g-h)^2}{4}&0\\1&\frac{g+h}{2}&0\\0&0&1\end{pmatrix},$$

respectively. □

4. A Result of Bernstein

In [1] Bernstein showed that the only zeros of the integer function

$$f(n) := \sum_{j\geq 0}(-1)^j\binom{n-2j}{j}$$

are at $n=3$ and $n=12$. We use Corollary 1 to relate the zeros of this function to solutions of a certain cubic Thue equation and hence to derive Bernstein's result.

Let

$$A = \begin{pmatrix}1&1&0\\0&0&1\\-1&0&0\end{pmatrix}.$$

With the notation of Corollary 1, $t=1$, $s=0$, $d=-1$, so that

$$a_n = \sum_{3j\leq n}(-1)^j\binom{n-2j}{j} = f(n),$$

and, for $n\geq 4$,

$$\begin{aligned}A^n &= f(n-2)A^2 + (f(n)-f(n-2))A + (f(n)-f(n-1))\,I\\ &= \begin{pmatrix}f(n)&f(n-1)&f(n-2)\\-f(n-2)&-f(n-3)&-f(n-4)\\-f(n-1)&-f(n-2)&-f(n-3)\end{pmatrix}.\end{aligned}$$

The last equality follows from the fact that $f(k+1)=f(k)-f(k-2)$, for $k\geq 2$.

Now suppose $f(n-2) = 0$. Since the recurrence relation above gives that $f(n-4) = -f(n-1)$ and $f(n) = f(n-1) - f(n-3)$, it follows that

$$(-1)^n = \det(A^n) = \begin{vmatrix} f(n-1) - f(n-3) & f(n-1) & 0 \\ 0 & -f(n-3) & f(n-1) \\ -f(n-1) & 0 & -f(n-3) \end{vmatrix}$$

$$= -f(n-1)^3 - f(n-3)^3 + f(n-1)f(n-3)^2.$$

Thus $(x, y) = \pm(f(n-1), f(n-3))$ is a solution of the Thue equation

$$x^3 + y^3 - x\,y^2 = 1.$$

One could solve this equation in the usual manner of finding bounds on powers of fundamental units in the cubic number field defined by the equation $x^3 - x + 1 = 0$. Alternatively, the Thue equation solver in PARI/GP [3] gives unconditionally (in less than a second) that the only solutions to this equation are

$$(x, y) \in \{(4, -3), (-1, 1), (1, 0), (0, 1), (1, 1)\}\ ,$$

leading to Bernstein's result once again.

References

[1] Bernstein, Leon *Zeros of the functions* $f(n) = \sum_{i=0}(-1)^i\binom{n-2i}{i}$. J. Number Theory **6** (1974), 264–270.
[2] James Mc Laughlin, *Combinatorial Identities Deriving from the n-th Power of a* 2×2 *Matrix.* Integers **4** (2004), A19, 14 pp.
[3] PARI/GP: http://pari.math.u-bordeaux.fr/
[4] B. Sury, *A curious polynomial identity.* Nieuw Arch. Wisk. (4) **11** (1993), no. 2, 93–96.
[5] ——, *A parent of Binet's forumla?* Math. Magazine **77** (2004), no. 4, 308-310.

Mathematics Department, Trinity College, 300 Summit Street, Hartford, CT 06106-3100
E-mail address: james.mclaughlin@trincoll.edu

Stat-Math Unit, Indian Statistical Institute, 8th Mile Mysore Road, Bangalore 560 059, India.,
E-mail address: sury@isibang.ac.in