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No representation of Moore groups and affine groups has any rate of random mixing

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Abstract

A sequence $a_n \downarrow 0$ forms a rate of random mixing for an unitary system $(G, \mu, \pi, \mathcal{H})$ if for any $u, v \in \mathcal{H}$

$$\limsup \frac{| < \pi(g_n^\omega)u, v > |}{a_n} < \infty$$

a.e. ω in the probability space $(G^{\mathbb{N}}, \mu^{\mathbb{N}})$ of the random walk induced by μ. We study the class of locally compact groups none of whose representation has any rate of random mixing and prove that this class contains Moore groups and certain solvable groups which includes the group of affine transformations on a local field.

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1 Introduction and Preliminaries

Let G be a locally compact group and $\mathcal{P}(G)$ be the space of regular Borel probability measures on G with weak topology which is the coarsest topology for which the functions $\mu \mapsto \mu(f)$ are continuous for all continuous bounded function f on G.

For $\mu \in \mathcal{P}(G)$ and $n \geq 1$, let μ^n denote the n-th convolution product of μ with itself.

A probability measure $\mu \in \mathcal{P}(G)$ is called *adapted* if it is not supported on a proper closed subgroup and μ is called *strictly aperiodic* if it is not supported on a coset of a proper closed normal subgroup.

A probability measure $\mu \in \mathcal{P}(G)$ is called *spread-out* if μ^k is not singular with respect to a Haar measure on G for some $k \geq 1$.

Let μ be a probability measure on G. Let $G^{\mathbb{N}}$ be denote the product space $\Pi_{i=1}^{\infty} G$ which is the space of paths $\omega = (\omega_n)$ of the random walk and $\mu^{\mathbb{N}}$ be the product measure

$\Pi_{i=1}^{\infty}\mu$. Then $(G^{\mathbb{N}}, \mu^{\mathbb{N}})$ is called the probability space of the random walk induced by μ. For any sequence $\omega = (\omega_n) \in G^{\mathbb{N}}$, we define the nth random product by

$$g_n^{\omega} = w_n w_2 \cdots w_1$$

for any $n \geq 1$.

We say that $(G, \mu, \pi, \mathcal{H})$ is an unitary system if G is a locally compact group, $\mu \in \mathcal{P}(G)$ is adapted and strictly aperiodic spread-out probability measure and π is an unitary representation of G on a Hilbert space $\mathcal{H}$.

Definition Let $(G, \mu, \pi, \mathcal{H})$ be an unitary system. A decreasing sequence (a_n) with $a_n \to 0$ is said to form a *rate of random mixing* for $(G, \mu, \pi, \mathcal{H})$ if for every $u, v \in \mathcal{H}$, with probability one:

$$\limsup \frac{| < \pi(g_n^{\omega})u, v > |}{a_n} < \infty.$$

The notion of rate of random mixing was introduced in [8] to bound the critical exponential rate of mixing from below. Since every irreducible representation of a locally compact abelian group is one-dimensional and by Theorem 2.13 of [8], we get that no unitary representation of a locally compact abelian group has any rate of random mixing. Shalom raises the question of characterizing groups with this property. It follows from Corollary 5.5 of [1] and Theorem 2.8 of [8] that there are unitary representations of non-amenable groups having a rate of random mixing.

In this note we prove that no unitary representation of Moore groups (that is, groups for which all irreducible unitary representations are finite-dimensional) and certain solvable groups (which includes the group of affine transformations on a local field of characteristic zero) has any rate of random mixing.

We now prove the following useful lemmas first of which is easy to verify and we omit the proof.

Lemma 1.1 *Let $(G, \mu, \pi, \mathcal{H})$ be a unitary system and H be a normal subgroup of G contained in the kernel of π. Then (a_n) is a rate of random mixing for $(G, \mu, \pi, \mathcal{H})$ if and only if (a_n) is a rate of random mixing for $(G/H, \tilde{\mu}, \tilde{\pi}, \mathcal{H})$ where $\tilde{\mu}$ is the projection of μ onto G/H and $\tilde{\pi}$ is the factor representation of π.*

Lemma 1.2 *Let $(G, \mu, \pi, \mathcal{H})$ be a unitary system with π irreducible. Then there exists a compact normal subgroup K of G such that K is contained in the kernel of π and G/K is second countable and (a_n) is a rate of random mixing for $(G, \mu, \pi, \mathcal{H})$ if and only if (a_n) is a rate of random mixing for $(G/K, \tilde{\mu}, \tilde{\pi}, \mathcal{H})$ where $\tilde{\mu}$ is the projection of μ onto G/K and $\tilde{\pi}$ is the factor representation of π.*

Proof Since μ is adapted, G is σ-compact and hence G can be approximated by second countable groups. Let $v \in \mathcal{H}$ be a unit vector. Then there exists a compact normal subgroup K of G such that G/K is second countable and $||\pi(g)v - v|| < 1$ for all $g \in K$. This implies that $||\pi(\omega_K)v - v|| < 1$ where ω_K is the normalized Haar measure on K. Thus, $\pi(\omega_K)v$ is a K-invariant non-zero vector in $\mathcal{H}$. Since K is normal, the space of K-invariant vectors is an invariant subspace. Since π is irreducible, K is contained in the kernel of π. Now the second part follows from Lemma 1.1.

In view of Lemma 1.2, we may assume that the class of groups under consideration consists of second countable groups.

2 Moore groups

Proposition 2.1 *Let $(G, \mu, \pi, \mathcal{H})$ be any unitary system. Suppose the unitary representation π is of finite-dimension. Then for u and v in $\mathcal{H}$, $< \pi(g^\omega_{k_n})u, v > \to 0$ a.e ω for some subsequence (k_n) of positive integers if and only if there exist orthogonal invariant subspaces U and V of $\mathcal{H}$ such that $u \in U$ and $v \in V$.*

Remark 2.1 We would like to remark that Proposition 2.1 is strictly stronger than not admitting any rate of random mixing and it may be seen by showing that the only if part of Proposition 2.1 is not true for the left regular representation of non-compact groups. Since the left regular representation of amenable groups weakly contain the trivial representation, left regular representation of amenable groups have no rate of random mixing. Let G be a non-compact locally compact group and R be the left regular representation of G. Suppose μ is any adapted and strictly aperiodic spread-out probability measure on G and $f \in L^2(G)$ be any non-negative function. Then

$$\int < R(g^\omega_n)f, f > d\mu^{\mathbb{N}}(\omega) = \int < R(g)f, f > d\mu^n(g) =< R(\mu)f, f > \to 0$$

by Theorem 2.8 of [2]. Since $< R(g)f, f > \geq 0$ for all $g \in G$, there exists a subsequence (k_n) such that

$$< R(g^\omega_{k_n})f, f > \to 0$$

a.e. ω (see for example Theorem 3.12 of [7]).

Proof Let $\mathcal{U}$ be the group of unitary operators on $\mathcal{H}$. Since $\mathcal{H}$ is of finite-dimension, $\mathcal{U}$ is compact. Let K be the closure of $\pi(G)$. Then $K \subset \mathcal{U}$ and hence K is compact.

Suppose there exists a subsequence (k_n) of positive integers and vectors u and v in $\mathcal{H}$ such that $< \pi(g^\omega_{k_n})u, v > \to 0$ a.e. ω. For $n \geq 1$, let $f_n: G^{\mathbb{N}} \to \mathbb{R}$ be defined by

$$f_n(\omega) = | < \pi(g^\omega_{k_n})u, v > |$$

for all $\omega \in G^{\mathbb{N}}$. Then f_n is uniformly bounded and $f_n(w) \to 0$ a.e. ω. By Lebesque dominated convergence theorem,

$$\int | < \pi(g)u, v > | d\mu^{k_n}(g) = \int f_n(w) d\mu^{\mathbb{N}}(\omega) \to 0$$

as $n \to \infty$.

Let $f \colon K \to \mathbb{R}$ be defined by

$$f(t) = | < t(u), v > |$$

for all $t \in K$. Then f is a continuous bounded function and

$$\int f(t) d\pi(\mu)^{k_n}(t) = \int f(\pi(g)) d\mu^{k_n}(g) \to 0$$

as $n \to \infty$. Since μ is adapted and strictly aperiodic in G, $\pi(\mu)$ is also adapted and strictly aperiodic in K. Thus, by Kawada-Ito theorem $\pi(\mu)^n \to \omega_K$ in $\mathcal{P}(K)$ where ω_K is the normalized Haar measure on K (see [4]). Since f is a continuous bounded function on K, we get that

$$\int f(t) d\pi(\mu)^n(t) \to \int f(t) d\omega_K(t)$$

as $n \to \infty$. Thus,

$$\int f(t) d\omega_K(t) = 0.$$

Since $f \geq 0$ is a continuous function on K, $f(t) = 0$ for all $t \in K$. This in particular, implies that $< \pi(g)u, v >= 0$ for all $g \in G$. Let U be the subspace of $\mathcal{H}$ spanned by $\{\pi(g)u \mid g \in G\}$. Then U is a closed invariant subspace of $\mathcal{H}$ such that $u \in U$ and $v \in U^{\perp}$. Thus, vectors u and v are in two orthogonal invariant subspaces. Converse is easy to prove.

We now apply Proposition 2.1 to finite-dimensional representations and in particular to Moore groups. A locally compact group G is called a *Moore group* if irreducible unitary representations of G are finite-dimensional.

Theorem 2.1 *Let $(G, \mu, \pi, \mathcal{H})$ be any unitary system. If π is finite-dimensional, then π has no rate of random mixing. In particular, no representation of a Moore group has any rate of random mixing.*

Proof Suppose the sequence (a_n) forms a rate of random mixing for $(G, \mu, \pi, \mathcal{H})$. Since π is finite dimensional, $\mathcal{H}$ contains a non-trivial irreducible subspace and let u and v be from a non-trivial irreducible subspace. Then

$$\limsup \frac{| < \pi(g_n^{\omega})u, v > |}{a_n} < \infty$$

for a.e. ω. This implies that $| < \pi(g_n^\omega)u, v > | \to 0$ a.e. ω. By Proposition 2.1, u and v are from two invariant orthogonal subspaces. This is a contradiction. Hence the unitary system $(G, \mu, \pi, \mathcal{H})$ has no rate of random mixing.

Suppose G is a Moore group. Let $(G, \mu, \pi, \mathcal{H})$ be an unitary system. Using Theorem 2.13 of [8], we may assume that π is an irreducible unitary representation of G. Then $\mathcal{H}$ is of finite-dimension. By the previous case, we get that $(G, \mu, \pi, \mathcal{H})$ has no rate of random mixing.

3 Affine groups

We now show that no representation of affine groups has any rate of random mixing.

Theorem 3.1 *Let $\mathbb{K}$ be a local field of characteristic zero. Let G be the semidirect product of $\mathbb{K}^*$, the multiplicative group of non-zero elements in $\mathbb{K}$ and $\mathbb{K}^n$ where action of $\mathbb{K}^*$ on $\mathbb{K}^n$ is given by*

$$a \cdot (a_1, \cdots, a_n) = (aa_1, \cdots, aa_n)$$

for all $a \in \mathbb{K}^*$ and $(a_1, \cdots, a_n) \in \mathbb{K}^n$. Then no representation of G has any rate of random mixing.

Proof Let π be an irreducible unitary representation of G. Then by Mackey's Theorem there exists a character χ of $\mathbb{K}^n$ such that π is induced from an irreducible representation, say ρ, of the stabilizer G_χ of χ (see for example Theorem 6.38 of [3]). If χ is trivial, then $G_\chi = G$ and hence π is an one-dimensional representation. Thus, π has no rate of random mixing.

If χ is non-trivial, then $G_\chi = \mathbb{K}^n$ and π is induced from the one-dimensional representation χ of $\mathbb{K}^n$. Let dm be a Haar measure on $\mathbb{K}^*$. Then π is the representation defined on $L^2(\mathbb{K}^*)$ by

$$\pi(x)f(b) = \chi(b^{-1} \cdot u)f(a^{-1}b)$$

for all $f \in L^2(\mathbb{K}^*)$, $x = (a, u) \in G$ and $b \in \mathbb{K}^*$.

Let V_0 be the subspace of $\mathbb{K}^n$ of co-dimension one such that $\chi = 1$ on V_0. Now for $f \in L^2(\mathbb{K}^*)$ and $v \in V_0$,

$$\pi(v)f(b) = \chi(b^{-1} \cdot v)f(b) = f(b)$$

for all $b \in \mathbb{K}^*$. This shows that π is trivial on V_0. Replacing $\mathbb{K}^n$ by $\mathbb{K}^n/V_0$, we may assume that $n = 1$. Thus, the group G is the semidirect product of $\mathbb{K}^*$ and $\mathbb{K}$.

We now claim that π weakly contains the trivial representation of G. Since $\mathbb{K}^*$ is amenable, there exists a summing sequence of non-null compact sets (B_n) in $\mathbb{K}^*$ (see 4.15 and 4.16 of [5]). For $n \geq 1$, define

$$f_n = \frac{1}{\sqrt{dm(B_n)}} 1_{B_n}$$

where 1_{B_n} is the indicator function on the set B_n. Then (f_n) is asymptotically translation invariant in $L^2(\mathbb{K}^*)$, that is,

$$||\pi(a)f_n - f_n|| = ||R(a)f_n - f_n|| \to 0$$

as $n \to \infty$ for all $a \in \mathbb{K}^*$ where R is the left regular representation of $\mathbb{K}^*$ on $L^2(\mathbb{K}^*)$. Now for $u \in \mathbb{K}$,

$$\begin{array}{rcl} ||\pi(u)f_n - f_n||^2 & = & \frac{1}{dm(B_n)} \int_{B_n} |\chi(b^{-1}u) - 1|^2 dm(b) \\ & \leq & \frac{1}{dm(B_n)} \int_{B_n \setminus B_k} |\chi(b^{-1}u) - 1|^2 dm(b) + 2\frac{dm(B_k)}{dm(B_n)} \end{array}$$

for any $k \leq n$. Since $\chi(b^{-1}u) \to 1$ as $b \to \infty$, we get that

$$||\pi(u)f_n - f_n||^2 \to 0$$

as $n \to \infty$ for all $u \in \mathbb{K}$. Thus, the group $\{g \in G \mid ||\pi(g)f_n - f_n|| \to 0\}$ contains $\mathbb{K}$ and $\mathbb{K}^*$ and hence $||\pi(g)f_n - f_n|| \to 0$ for all $g \in G$. This implies that the trivial representation is weakly contained in π (see Remark 2.1 of [6]). By Theorem 2.13 of [8], any rate of random mixing of π is also a rate of random mixing of the trivial representation but the trivial representation has no rate of random mixing. Hence π has no rate of random mixing.

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