JRF Analysis and Probability Courses

Core Courses

- **Measure Theory**
- **Complex Analysis**
- **Functional Analysis**
- **Analysis of Several Variables**
- **Partial Differential Equations**
- **Fourier Analysis**
- **Advanced Functional Analysis**
- **Analysis on Graphs**
- **Operator Algebras**
- **Basics of Quantum Probability and Quantum Information theory**
- **Probability Theory**
- **Introduction to Stochastic Processes**
- **Topics in Discrete Probability**
- **Topics in Gaussian Processes**
- **Martingale Theory**
- **Topics in Large deviations**

Detailed Syllabi

Core Courses

- **Measure Theory**

The concept of sigma-algebra, Borel subsets of real line, Construction of Lebesgue and Lebesgue-Stieltjes measures on the real line following outer measure.

Abstract measure theory: definition and examples of measure space, measurable functions, Lebesgue integration, convergence theorems (Fatou's Lemma, Monotone convergence and dominated convergence theorem).

Caratheodory extension theorem, completion of measure spaces.

Product measures and Fubini's theorem.

L^p-spaces, Riesz-Fischer Theorem, approximation by step functions and continuous functions.

Absolute continuity, Hahn-Jordan decomposition, Radon-Nikodym theorem, Lebesgue decomposition theorem. Functions of bounded variation.

Complex measures.

References

(a) H. L. Royden and Patrick Fitzpatrick: Real Analysis, Pearson, 4th edition.

(b) Robert B. Ash and Catherine A. Doleans-Dade, Probability and measure theory, GTM(211), Academic Press, 2nd edition.
(c) Elias M. Stein, Rami Shakarchi, Real Analysis: Measure Theory, Integration and Hilbert Spaces , Princeton Lectures in Analysis.
(d) Gerald B. Folland, Real Analysis: Modern Techniques and Their Applications, Pure and Applied Mathematics, A Wiley Series.
(e) G. de Barra, Measure Theory and Integration.

- **Complex Analysis**

Review of sequences and series of functions including power series, Complex differentiation and Cauchy-Riemann equation, Cauchy's theorem and Cauchy's integral formula, Power series expansion of holomorphic function, zeroes of holomorphic functions, Maximum Modulus Principle, Liouville's Theorem, Morera's Theorem.
Complex logarithm and winding number, Singularities, Meromorphic functions, Casorati Weierstrass theorem, Riemann sphere, Laurent series, Residue Theorem and applications to evaluation of definite integrals, Open Mapping Theorem, Rouche's Theorem.

Conformal maps, Schwarz lemma, Linear fractional transformations, automorphisms of a disc, Introduction to Gamma function.

Equicontinuity and Arzela-Ascoli Theorem, Normal family, Montel's theorem and Riemann mapping theorem.

References
(a) Complex Analysis- L. Ahlfors.
(b) Elementary Theory of Analytic Functions of One or Several Complex Variables- H. Cartan
(c) Complex Analysis- E. M. Stein, R. Shakarchi.
(d) Complex Analysis- D. Sarason

- **Functional Analysis**

Quick review of sequences and series of functions, equicontinuity, Arzela-Ascoli theorem.

Normed linear spaces and Banach spaces. Bounded linear operators. Dual of a normed linear space. Hahn-Banach theorem, uniform boundedness principle, open mapping theorem, closed graph theorem. Computing the dual of some well-known Banach spaces. Weak and weak-star topologies, Banach-Alaoglu Theorem. The double dual. L^p-spaces and their duality, Weierstrass and Stone-Weierstrass Theorems.

Hilbert spaces, adjoint operators, self-adjoint and normal operators, spectrum, spectral radius, analysis of the spectrum of a compact operator on a Banach space, spectral theorem for compact self-adjoint operators on Hilbert spaces. Basics of complex measures and statement of the Riesz representation theorem for the space C(X) for a compact Hausdorff space X. Sketch of proof of Riesz representation theorem (if time permits).

References

(a) Real and complex analysis, W. Rudin, McGraw-Hill (1987).
(b) Functional analysis, W. Rudin, McGraw-Hill (1991).
(c) A course in functional analysis, J. B. Conway, GTM (96), Springer-Verlag (1990).
(d) Functional analysis, K. Yosida, Grundlehren der MathematischenWissenschaften (123), Springer-Verlag (1980).

- **Analysis of Several Variables**

Metric Topology of R^n. Topology induced by L^p norms (p = 1, 2, infinity) on R^n and their equivalence. Continuous functions on Rn. Separation properties. Compact subsets of R^n. Path-connectivity. Topological properties of subgroups like GLn(R), GLn(C), O(n), Hilbert-Schmidt norm and operator norm on Mn(R). Sequence and series in Mn(R). Exponential of a matrix.

Differentiation and integration of functions on R^n. Partial derivatives of realvalued functions on R^n. Differentiability of maps from R^m to R^n and the derivative as a linear map. Jacobian theorem. Chain Rule. Mean value theorem. Higher derivatives and Schwarz theorem, Taylor expansions in several variables. Inverse function theorem and implicit function theorems. Local maxima and minima, Lagrange multiplier method.

Vector fields on R^n. Integration of vector fields and flows. Picard's Theorem.

Riemann integration of bounded real-valued functions on rectangles (product of intervals). Existence of the Riemann integral for sufficiently well-behaved functions. Iterated integral and Fubini's theorem. Brief treatment of multiple integrals on more general domains. Change of variable and the Jacobian formula.

Differential forms on R^n. Wedge product of forms. Pullback of differential forms. Exterior differentiation of forms. Integration of compactly supported n-forms on R^n. Change of variable formula revisited. Integration of k-forms along singular k-chains in R^n. Stokes' theorem on chains. [Special emphasis on curves and surfaces in R^2 and R^3. Line integrals, Surface integrals. Gradient, Curl and Divergence operations, Green's theorem and Gauss's (Divergence) theorem.

Reference Texts:

(a) M. Spivak: Calculus on manifolds, Benjamin (1965).
(b) T.Apostol: Mathematical Analysis. S. Lang, Algebra, GTM (211), Springer (Indian reprint 2002).
(c) K. Mukherjea: Differential Calculus in Normed Linear Spaces

- **Partial Differential Equations**

Basics of ODE: (local as well as global) existence and uniqueness results, Picard iteration, Gronwall's inequality, solving some first order and second order equations.

Introduction to PDE: order of a PDE, classification of PDEs into linear, semilinear, quasi-linear, and fully nonlinear equations, Examples of equations from Physics, Geometry, etc. The notion of well-posed PDEs.

First order PDEs: Method of characteristics, existence and uniqueness results of the Cauchy problem for quasilinear and fully nonlinear equations. [9 lectures]
Second order linear PDEs in two independent variables: classification into hyperbolic, parabolic and elliptic equations, canonical forms.

Laplace equation: Definition of Harmonic functions. Mean-value property, Strong Maximum principle for harmonic functions, Liouville's theorem, smoothness of harmonic functions, Poisson's formula. Harmonic functions in rectangles, cubes, circles, wedges, annuli.

Heat equation: Fundamental solution of heat equation, Duhamel's principle, weak and strong maximum principles, smoothness of solutions of heat equation, ill-posedness of backward heat equation.

Wave equation: well-posedness of initial and boundary value problem in 1D and d'Alembert formula. method of descent in 2D and 3D. Duhamel's principle, domain of dependence, range of influence, finite speed of propagation.

Boundary problems: Separation of variables, Dirichlet, Neumann and Robin conditions. The method of seperation of variables for Laplace, Heat and Wave equations.

References
(a) Evans, L. C. Partial Differential Equations, AMS, 2010.
(b) Han, Q. A Basic Course in Partial Differential Equations, AMS, 2011.
(c) McOwen, R. Partial Differential Equations: Methods and Applications, Pearson, 2002.
(d) Pinchover, Y. and Rubinstein, J. An Introduction to Partial Differential Equations, Cambridge.
(e) Fritz John Partial Differential Equations, Springer.
(f) Partial Differential Equations: An introduction by Walter Strauss. ge, 2005.

- **Fourier Analysis**

Fourier Series on T:

a) Dirichlet problem for the unit disc and origin of Fourier series, continuity of translation on L^p(T) and elementary convolution inequalities, approximate identity.

b) Fourier series and its elementary properties, completeness of trigonometric polynomials and Riemann-Lebesgue lemma, Uniqueness of Fourier coefficients and the Fourier inversion, Plancherel theorem, Weyl’s equidistribution theorem.

c) Dirichlet kernel and pointwise convergence of Fourier series for Lipschitz continuous functions, Riemann’s localization principle, existence of a continuous function with divergent Fourier series.

d) Cesaro and Abel summability, Poisson integral and solution of Dirichlet problem for appropriate function classes.

e) Marcinkiewicz interpolation theorem, Young’s inequality and Hausdorff-Young inequality, norm convergence of Fourier series for L^p, $1 < p < 1$.

Fourier transform on R^d:

a) Elementary properties of the Fourier transform involving translation, dilation, rotation, decay and smoothness, Riemann Lebesgue lemma, Fourier transform of Gaussian and the Poisson kernel, the Fourier inversion. Schwartz class functions and its image under Fourier transform.

b) Fourier transform of L^2-functions and the Plancherel theorem, Hausdorff-Young inequality, Paley-Wiener theorem, Poisson summation formula.

c) Tempered distribution and its Fourier transform, computation of some distributional Fourier transform.

d) Weak L^p spaces, Method of maximal function, Lebesgue differentiation theorem, almost everywhere convergence of Poisson integrals.

e) Nontangential convergence of Poisson integral and characterization of Poisson integral of L^p functions (If time permits).

References
(a) Real and complex analysis- W. Rudin.
(b) Topics in Functional analysis- S. Kesavan.
(c) Introduction to Fourier Analysis on Euclidean spaces- E. M. Stein, G. Weiss.
(d) Fourier Analysis- E. M. Stein, R. Shakarchi.
(e) Introduction to Harmonic Analysis-Y. Katznelson

<u>Advanced Functional Analysis</u>

Brief introduction to topological vector spaces (TVS) and locally convex TVS.

Linear Operators. Uniform Boundedness Principle. Geometric Hahn-Banach theorem and applications (Markov-Kakutani fixed point theorem, Haar Measure on locally compact abelian groups, Liapounovs theorem). Extreme points and

Krein-Milman theorem.

In addition, one of the following topics:

(a) Geometry of Banach spaces: vector measures, Radon-Nikodym Property and geometric equivalents. Choquet theory. Weak compactness and Eberlein-Smulian Theorem. Schauder Basis.

(b) Banach algebras, spectral radius, maximal ideal space, Gelfand transform.

(c) Unbounded operators, Domains, Graphs, Adjoints, spectral theorem.

References

(a) N. Dunford and J. T. Schwartz, Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space, Interscience Publishers, John Wiley (1963).

(b) Walter Rudin, Functional analysis, Second edition, International Series in Pure and Applied Mathematics. McGraw-Hill (1991).

(c) K. Yosida, Functional analysis, Springer (Indian reprint 2004).

(d) J. Diestel and J. J. Uhl, Jr., Vector measures, Mathematical Surveys (15), AMS (1977).

Operator Algebras

Definition and examples of Banach and C^* algebras, Gelfand theory for abelian Banach and C^* algebras, Spectral theorem for bounded normal operators on a Hilbert spaces.

General noncommutative C^* algebras: basic properties, Positive elements and states, GNS theorem, ideals and quotients of C^* algebras, C^* algebra of compact operators.

Definition of examples of von Neumann algebras, the double commutant theorem and Kaplansky density theorem, abelian von Neumann algebras and the L^1-functional calculus.

Time permitting: discussion on some concrete examples like AF algebras, group C^* algebras for discrete countable groups, Cuntz algebras, noncommutative tori etc.

References

(a) C_ Algebras by Examples, K R Davidson;

(b) Functional A Course in Functional Analysis, J B Conway

(c) C_ Algebra and Operator Theory, G. J. Murphy

(d) Fundamentals of the theory of operator algebras, volume I and II, R.V. Kadison and J. R. Ringrose

Analysis on Graphs

Review of Matrices and Graphs. Incidence matrix, Adjacency matrix and Laplace matrix of a graph. The Laplace operator on graphs. Random walks on graphs. Dirichlet Problem. Spectral properties. Eigenvalues and mixing time. Cheeger's inequality. Eigenvalues on infinite graphs. Estimates of the heat kernel. Carne-Varopoulous bound.

References

(a) R. B. Bapat: Graphs and Matrices.

(b) A. Grigor'yan.: Introduction to Analysis on Graphs.

(c) M. T. Barlow: Random Walks and Heat Kernel on Graphs.

(d) R. Lyons and Y. Peres: Probability on trees and networks.

Basics of Quantum Probability and Quantum Information theory.

Finite dimensional quantum probability spaces: Events, observables and states. Pure and mixed states. Tensor products, notions of separability and entanglement for states. Distance measures for density matrices. Bell's inequality, Dynamics in finite dimensional quantum probability spaces. Basic notions of spectral integration. Self-adjoint operators viewed as observables.

Completely positive maps, Stinespring's theorem, Choi-Kraus representations. State preserving maps and quantum channels. Majorization. Generalized measurements (POVMs) and Naimark's theorem.

Books suggested:

1) An Introduction to Quantum Stochastic Calculus, K. R. Parthasarathy.
2) The Theory of Quantum Information, J. Watrous.
3) Quantum Computation and Quantum Information, M. A. Nielsen, I. L. Chuang

Probability Theory (Basic Core Course)

- Revision of Measure theory: probability spaces, distributions, random variables, standard random variable examples, expected value, inequalities (Holder, Cauchy-Schwarz, Jensen, Markov, Chebyshev) , convergence notions(convergence in probability and almost sure, Lp), application of DCT, MCT, Fatou with examples, Revision of Fubini's theorem.
- Independence,sum of random variables, constructing independent random variables, weak law of large numbers, Borel-Cantelli lemmas, First and Second Moment methods, Chernoff bounds and some applications.
- Strong law of large number, Kolmogorov $0-1$ law. Convergence of random series. Kolmogorov's three series theorem.
- Weak convergence, tightness, characteristic functions with examples, Central limit theorem (iid sequence and triangular array).
- Random walks Finite Length: Stopping times and games, The ruin problem, Reflection Principle, Arc Sine law for last visit to origin
- Random walks infinite Length: Higher Dimension, Recurrence versus transience.

References

(a) Rick Durrett: Probability (Theory and Examples).

(b) Patrick Billingsley: Probability and measure.
(c) Robert Ash: Basic probability theory.
(d) Leo Breiman: Probability.

(e) David Williams: Probability with Martingales.

(f) Jeffery Rosenthal: A First Look at Rigorous Probability Theory

B3-Probability III and M2 are slightly different to this but close enough. So M2 (when offered) or B3-Probability III (when offered) instructor can modify things a little.

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The below are a list of courses in Probability that a student can take as additional core courses in Probability.

<u>Introduction to Stochastic Processes</u>

- DISCRETE-TIME MARTINGALES: Optional Stopping theorem, Martingale convergence theorem, Doob's inequality and convergence.
- BRANCHING PROCESSES: Model definition. Connection with martingales. Probability of survival. Mean and variance of number of individuals.

- DISCRETE-TIME MARKOV CHAINS: Classification of states, Stationary distribution, reversibility and convergence. Random walks and electrical networks. Collision and recurrence.
- BASIC PROBABILISTIC INEQUALITIES AND APPLICATIONS: First and Second Moment methods. Applications to Longest increasing subsequences, Random k-Sat problem and connectivity threshold for Erdos-Renyi graphs. Chernoff bounds and Johnson-Lindenstrauss lemma.

References

- (a) N. Lanchier: *Stochastic Modelling.*
- (b) W. Feller: *Introduction to Probability: Theory and Applications - Vol. I and II.*
- (c) L. Levine, Y. Peres and E. Wilmer: *Markov chains and mixing times.*
- (d) Sheldon Ross: *Probability Models.*
- (e) Santosh S. Venkatesh: *Theory of Probability - Explorations and Applications.*
- (f) R. Meester: *A Natural Introduction to Probability Theory.*
- (g) S. R. Athreya and V. S. Sunder: *Measure and Probability.*
- (h) Sebastien Roch: *Modern Discrete Probability: A toolkit. (Notes).*

This is the same as (B.Math elective) and the student can take it along with other students with instructor offering further exercises/independent exploration.

Topics in Discrete Probability

- Review of discrete probability, First and Second Moment methods, Chernoff bounds and some applications.
- Percolation on lattices: Phase-transition phenomena, subcritical and supercritical phases, Uniqueness.
- Random graphs :Phase transition, Influences,Russo's formula and Sharp thresholds. Noise Sensitivity and Stability.
- Introduction to Markov chains and Martingales. Branching processes. Random walks and electrical networks, Uniform spanning trees.

References

- (a) C. Garban and J. Steif: Noise Sensitivity of Boolean Functions and Percolation.
- (b) Sebastien Roch: Modern Discrete Probability: A toolkit. (Notes).

- (c) R. Lyons and Y. Peres: Probability on trees and networks.
- (d) M. Barlow: Random walks and heat kernel on Graphs.
- (e) N. Lanchier: Stochastic Modelling.

This is the same as (M.Math elective) and the student can take it along with other students when offered

Topics in Gaussian Processes (M.Math Elective)

Review of Gaussian random variables. Slepian and Sudakov-Fernique inequalities. Variance bounds, Poincare inequality, Isoperimetric inequality, Log-sobolev inequality, Concentration and transport inequalities. Maxima of Gaussian pro- cesses. Majorizing measures and Generic chaining. Excursion probabilities. Hypercontractivity.

Additional Topics (depending on time and audience interest): Geometry of Gaussian random fields. Stein's method. Introduction to Malliavin calculus.

References

(a) Ramon van Handel: Probability in high dimensions: Notes.

(b) Manjunath Krishnapur's course on Gaussian processes: Notes

(c) R. J. Adler: Introduction to continuity, extrema and related topics for general Gaussian processes.

(d) R.J. Adler and J. E. Taylor: Random fields and Geometry.

(e) M. Talagrand: Upper and lower bounds for stochastic processes.

This is the same as (M.Math elective) and the student can take it along with other students when offered

Martingale Theory

- Absolute continuity and singularity of measures. Hahn-Jordon decomposition, Radon-Nikodym Theorem, Lebesgue decomposition. Conditional expectation - Definition and Properties. Regular conditional probability, proper RCP. Regular conditional distribution.
- Discrete parameter martingales, sub-and super-martingales. Doob's Maximal Inequality, Upcrossing inequality, martingale convergence theorem, Lp inequality, uniformly integrable martingales, reverse martingales, Levy's upward and downward theorems. Stopping times, Doob's optional sampling theorem. Discrete martingale transform, Doob's Decomposition Theorem.

- Applications of martingale theory: Azuma-Hoeffding Inequality and some ap- plications. SLLN for i.i.d. random variables. Infinite products of probability spaces, Hewitt-Savage 0-1 Law. Finite and infinite exchangeable sequence of random variables, de Finetti's Theorem. SLLN for U-Statistics for exchange- able data.
- Introduction to continuous parameter martingales: definition, examples and basic properties.
- (If time permits) Martingale Central Limit Theorem and applications.

References :

(a) Y. S. Chow and H. Teicher: Probability Theory
(b) Leo Breiman: Probability Theory
(c) Jacques Nevue: Discrete Parameter Martingales
(d) P. Hall , C. C. Heyde: Martingale Limit Theory and its Application (e) R. Durret: Probability Theory and Examples

(f) P. Billingsley: Probability and Measures

<u>This is the same as (M.Math elective) and the student can take it along with other students when offered</u>

<u>Theory of Large Deviations</u>

- Introduction to large deviations.
- Sanov's theorem and Cramer's theorem for finitely supported random variables.
- General notion of large deviation principle on Polish spaces: Laplace principle, Varadhan's lemma, weak large deviation principle, exponential tightness, goodness of rate function, contraction principle, Bryc's lemma.
- Cramer's theorem for general random variables and vectors.
- Exponential tightness of (a) sample averages of i.i.d. Banach space valued random variables and (b) empirical measures of i.i.d. Polish space valued random variables.
- Cramer's theorem on locally convex separable Hausdorff topological vector spaces.
- Large deviations of Brownian paths: Schilder's theorem.
- Sanov's theorem on Polish spaces:Donsker-Varadhan variational formula.Gartner and Ellis theorem.

References

1. (a) Large Deviations Techniques and Application by A. Dembo and O. Zeitouni
2. (b) Large Deviations by Deuschel and Stroock
3. (c) Large Deviations by Hollander
4. (d) Large Deviations and Applications by S. R. S. Varadhan

5. (e) A Weak Convergence Approach to the Theory of Large Deviations by P. Dupuis and T. Elliss

This is the same as (M.Math elective) and the student can take it along with other students when offered