Permutation modules and p-ranks of Incidence Matrices
Part 2: Spaces with forms

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Groups and Geometries, ISI Bangalore, December 2012

Outline

- G finite classical group (symplectic, orthogonal, unitary).
- P the set of singular points of the standard module V
- k, algebraically closed field of (defining) characteristic p.
- We consider the permutation module k^P.
- Main difficulty is that for orthogonal an unitary groups, P is a proper subset of $\mathbf{P}(V)$.

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Symplectic groups in odd characteristic

Symplectic groups in characteristic 2

Other groups, hyperplane incidences

$\mathrm{Sp}(2m, q)$, $q = p^t$ odd

- We consider the submodule structures of $k[V]$, $A[d]$, and Y_P, under the action of $\mathrm{Sp}(V)$.
- S^λ :=truncated symmetric power (prev. $\overline{S}^\lambda$) with $0 \le \lambda_j \le 2m(p-1)$.
- S^λ remain simple except when $\lambda = m(p-1)$, in which case we have
$$S^{m(p-1)} = S^+ \oplus S^-.$$
- S^+ and S^- are simple $k\,\mathrm{Sp}(V)$-modules,
$$\dim(S^+) = (d_{(p-1)m} + p^m)/2, \quad \dim(S^-) = (d_{(p-1)m} - p^m)/2.$$

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The modules S^+ and S^-.

- (Wong, Lahtohnen (1990))
- Use multi-index notation $X^\alpha Y^\beta$ for monomials in symplectic coords X_i, Y_i, $0 \le i \le m$.
- For any multi-index β, we define $|\beta| = \sum_{i=1}^m b_i$, $\beta! = \prod_{i=1}^m b_i!$, and $\overline{\beta} = (p-1-b_1, \ldots, p-1-b_m)$.
- Denote monomials in the quotient module $S^{m(p-1)}$ using bars. The map

$$\tau : S^{m(p-1)} \to S^{m(p-1)}, \quad \overline{X}^\alpha \overline{Y}^\beta \mapsto (-1)^{|\beta|} \alpha! \beta! \overline{X}^{\overline{\beta}} \overline{Y}^{\overline{\alpha}}$$

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Submodules of k^P for $\mathrm{Sp}(2m, q)$

- (Chandler-Sin Xiang, 2007)
- Construct a special basis of $k[V]$, of *symplectic basis functions*.
- Describe the submodule structure of the kG-submodule of $k[V]$ and k^P generated by an arbitrary symplectic basis function.
- Describe the part of the submodule lattice of $k[V]$ and k^P involving the above submodules.
- This includes images of incidence maps $\eta_r : k^{\mathcal{I}_r} \to k^P$, where $\mathcal{I}_r$ is the set of totally isotropic r-subspaces.
- Symplectic analogue of Hamada's p-rank formula,
- When $m = 2$, get a closed formula for the p-rank of the symplectic GQ.

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Basis of special functions

- S^+ and S^- have bases consisting of images of monomials:

$$x^{\boldsymbol{\alpha}} y^{\overline{\boldsymbol{\alpha}}},$$

and sums and differences of monomials:

$$x^{\boldsymbol{\alpha}} y^{\boldsymbol{\beta}} \pm (-1)^{|\boldsymbol{\beta}|+m} \alpha! \beta! x^{\overline{\boldsymbol{\beta}}} y^{\overline{\boldsymbol{\alpha}}}$$

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- Symplectic basis functions of type $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{t-1})$.

$$f = f_0 f_1^p \cdots f_{t-1}^{p^{t-1}}.$$

  where each f_j, which we will call the j-th digit of f, is either a basis monomial or binomial of $k[V]$ of degree λ_j. If $\lambda_j \neq (p-1)m$, then f_j can be any basis monomial of degree λ_j in which the degree in each variable is at most $p-1$. If $\lambda_j = (p-1)m$, then f_j can be any of the S^+ and S^- basis functions.
- The union of these sets of functions over all λ is our special basis for $k[V]$.
- By restricting the types for the symplectic basis functions we can obtain bases for $A[d]$, and k^P.

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The posets $\mathcal{S}$ and $\mathcal{S}[d]$

- $\mathbf{\Lambda} = \{(\lambda_0, \ldots, \lambda_{t-1} \mid 0 \leq \lambda_j \leq 2m(p-1)\,\forall j\}$, “Types”
- Definition
  For $\lambda \in \Lambda$, let $\mathbf{s}$ be the corresponding $\mathcal{H}$-type in $\mathcal{H}[d]$. Set

  $$J(\mathbf{s}) = \{j \mid 0 \leq j \leq t-1,\ \lambda_j = m(p-1)\}.$$

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  $Z(\mathbf{s}, \mathbf{s}') = \{j \mid s'_j = s_j,\ s'_{j+1} = s_{j+1},\ \lambda_j = m(p-1)\}$. We define

  $$\mathcal{S}[d] = \{(\mathbf{s}, \epsilon) \mid \mathbf{s} \in \mathcal{H}[d],\ \epsilon \subseteq J(\mathbf{s})\}.$$

  In the case $[d] = [0]$, we also define

  $$\mathcal{S} = \{(\mathbf{s}, \epsilon) \mid \mathbf{s} \in \mathcal{H}, \epsilon \subseteq J(\mathbf{s})\}.$$

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Definition

To each symplectic basis function of $k[V]$ we associate a pair $(\mathbf{s}, \epsilon) \in \mathcal{S}[d]$ for some $[d] \in \mathbb{Z}/(q-1)\mathbb{Z}$, as follows. If f is of type $\boldsymbol{\lambda}$, then $\mathbf{s}$ is the corresponding $\mathcal{H}$-type. The set $\epsilon \subseteq J(\mathbf{s})$, called the *signature,* is defined to be the set of $j \in J(\mathbf{s})$ for which the image of the j-th digit f_j of f in $S^{m(p-1)}$ belongs to S^+.

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- In terms of $\mathcal{H}$-types, we see that each $\mathcal{H}$-type gives a $k\,\mathrm{GL}(V)$-composition factor and then the choice of signs determines the simple $k\,\mathrm{Sp}(V)$ composition factor of this simple $k\,\mathrm{GL}(V)$-module.
- Thus $\mathcal{S}$ indexes the $k\,\mathrm{Sp}(V)$-composition factors of Y_P, $\mathcal{S}[d]$, $[d] \neq [0]$ label the $k\,\mathrm{Sp}(V)$-composition factors of $A[d]$.
- However it should be noted that different elements of $\mathcal{S}$ or $\mathcal{S}[d]$ can label isomorphic composition factors, due to the fact that $S^{\lambda} \cong S^{2m(p-1)-\lambda}$ as $k\,\mathrm{Sp}(V)$-modules.
- $L(\mathbf{s}, \epsilon)[d]$ denotes the simple summand of $L(\mathbf{s})[d]$ where we take the $+$ summand for each $j \in \epsilon$ and the $-$ summand for each $j \in J(\mathbf{s}) \setminus \epsilon$. When $\mathbf{s} \in \mathcal{H}$, we may use the simpler notation $L(\mathbf{s}, \epsilon)$.

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Symplectic Analogue of Hamada's formula

Next we give the symplectic analogue of Hamada's formula for the p-rank of the incidence matrix between points and m-flats of $\mathrm{W}(2m-1,q)$ in terms of t, where $q = p^t$, p an odd prime.

Theorem

Let $A^m_{1,m}(p^t)$ be the incidence matrix between points and m-flats of $\mathrm{W}(2m-1,p^t)$. Assume that p is odd. Then

$$\mathrm{rank}_p(A^m_{1,m}(p^t)) = 1 + \sum_{\forall j, 1\le s_j \le m} \prod_{j=0}^{t-1} d_{(s_j,s_{j+1})},$$

where

$$d_{(s_j,s_{j+1})} = \begin{cases} \dim(S^+) = (d_{m(p-1)} + p^m)/2, & \text{if } s_j = s_{j+1} = m, \\ d_{\lambda_j}, \ \text{where } \lambda_j = ps_{j+1} - s_j, & \text{otherwise.} \end{cases}$$

$\mathrm{Sp}(4, q)$ generalized quadrangle q odd.

- We consider the case where $m = 2$ and $r = 2$.
- Symplectic polar space $\mathrm{W}(3, q)$ is a classical generalized quadrangle.
- In the case where $q = p$ is an odd prime, de Caen and Moorhouse determined the p-rank of $A^2_{1,2}(p)$.

- Theorem

  Let p be an odd prime and let $t \geq 1$ be an integer. Then the p-rank of $A^2_{1,2}(p^t)$ is equal to

$$1 + \alpha_1^t + \alpha_2^t,$$

where

$$\alpha_1, \alpha_2 = \frac{p(p+1)^2}{4} \pm \frac{p(p+1)(p-1)}{12}\sqrt{17}.$$

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Outline

- Chandler-Sin-Xiang (2010)
- $q = 2^t$, V a $2m$-dimensional symplectic $\mathbf{F}_q$-vector space.
- The trucated symmetric powers S^λ are exterior powers $\wedge^\lambda(V)$ and are not simple or semisimple, but rather have filtrations by Weyl modules, The Weyl modules themselves are not simple or semisimple.
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Analogue of Hamada's formula

- $B_{r,1} = B_{r,1}(t)$ denote the incidence matrix between $P = \mathcal{I}_1$ and $\mathcal{I}_r$.

Theorem

Let $m \geq 2$ and $1 \leq r \leq 2m-1$. Let A be the $(2m-r) \times (2m-r)$-matrix whose (i,j)-entry is

$$a_{i,j} = \binom{2m}{2j-i} - \binom{2m}{2j+i+2r-4m-2-2(m-r)\delta(r \leq m)}.$$

Then

$$\mathrm{rank}_2(B_{r,1}(t)) = 1 + \mathrm{Trace}(A^t).$$

($\delta(P) = 1$ if a statement P holds, and $\delta(P) = 0$ otherwise.)

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- So far, no analogous results on submodule structure of k^P for orthogonal or unitary groups.
- We need to understand the submodules of the homogeneous coordinate ring of the projective variety of singular points in the algebraically closed case.
- $\bigoplus_{r\geq 0} H^0(r\omega), \omega \in \{\omega_1, \omega_1 + \omega_\ell\}$.
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Orthogonal, unitary groups, hyperplanes, opposites

- $V = V(q)$, quadratic form or $V(q^2)$ with Hermitian form.
- P be the set of singular points, P^* the set of polar hyperplanes
- $\widehat{P}$,$\widehat{P}^*$ sets of all points and hyperplanes.
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Orthogonal case

Theorem

Suppose $\dim V(q) = n \geq 4$. *The following hold.*

(a) *Assume* $p = 2$. *Then*

$$\operatorname{rank}_p A_{11} = \begin{cases} 1 + n^t, & \text{if } n \text{ is even,} \\ 1 + (n-1)^t, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem

(cont'd)

(b) *Assume $p > 2$. Then the p-rank depends on whether there exists a positive integer u such that*

$$u \equiv n \pmod{2} \quad \textit{and} \quad n-3 \leq up \leq p+n-5.$$

If u exists then

$$\operatorname{rank}_p A_{11} = 1 + (\binom{n+p-2}{n-1} - \binom{n+p-4}{n-1} - \binom{up+2}{n-1} + \binom{up}{n-1})^t.$$

Otherwise,

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Remark

When n is even, there are two types of nondegenerate forms, distinguished by the Witt index . However, the p-rank of A_{11} is the same for both types.

Hermitian case

Theorem

Suppose $\dim V(q^2) = n \geq 4$. *The p-rank depends on the existence of a positive integer u satisfying*

$$n - 2 \leq up \leq p + n - 3$$

If u exists then

$$\mathrm{rank}_p A_{11} = 1 + (\binom{n+p-2}{n-1}^2 - \binom{n+p-3}{n-1}^2 - \binom{up+1}{n-1}^2 + \binom{up}{n-1}^2)^t.$$

Otherwise,

$$\mathrm{rank}_p A_{11} = 1 + \left[\binom{n+p-2}{n-1}^2 - \binom{n+p-3}{n-1}^2\right]^t.$$

- When $n = 3$ or 4, the totally isotropic subspaces of dimensions one and two form the points and lines of the Hermitian generalized quadrangle.
- The p-rank of the incidence relation of points and lines of this generalized quadrangle is still unknown in general.
- we can also compute point-hyperplane p-ranks for $DH(4, q^2)$.

Theorem

The p-rank of the point-hyperplane incidence matrix A_{11} for the dual Hermitian generalized quadrangle $DH(4, q^2)$ is as follows.

(a) *If $p > 2$ then*

$$\operatorname{rank}_p A_{11} = 1 + \left(\frac{p(p+1)}{32}\binom{2p+2}{3}^2 - \frac{p(p-1)}{32}\binom{2p}{3}^2 + \frac{p}{2}\binom{p+1}{3}^2\right)^e$$

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