Permutation modules and p-ranks of Incidence Matrices I

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Groups and Geometries, ISI Bangalore, December 2012

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Outline

Incidence matrices, permutation modules

$\mathrm{GL}(V)$ acting on points and vectors

Nonzero intersection

Affine group action

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Incidence matrices

- X, Y sets $I \subset X \times Y$ incidence relation.
- A incidence matrix over a field k.
- $\eta : k^X \to k^Y$, $x \mapsto \sum_{(x,y)\in I} y$
- If a group G acts on X and Y, preserving I then η is a kG-module homomorphism.
- $\mathrm{Im}\,\eta$ is a kG-submodule of k^Y of dimension $\mathrm{rank}\,A$.
- Study submodule structure of k^Y to study incidence, and vice versa.

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- $q = p^t$, $V = V(q)$ an $(n+1)$-dimensional $\mathbf{F}_q$-vector space.
- $G = \mathrm{GL}(V) \cong \mathrm{GL}(n+1, q)$.
- k algebraically closed field of characteristic p.
- $P = \{$1-diml. subspaces of V$\}$, the points.
- $k^P = k1 \oplus Y_P$ as kG-modules,

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The set $\mathcal{H}$

- Let $\mathcal{H}$ denote the set of t-tuples $(s_0, \ldots, s_{t-1})$ of integers satisfying (for $j = 0, \ldots, t-1$)
  1. $1 \leq s_j \leq n$;
  2. $0 \leq ps_{j+1} - s_j \leq (p-1)(n+1)$. (Subcripts mod t.)
- Let $\mathcal{H}$ be partially ordered in the natural way: $(s'_0, \ldots, s'_{t-1}) \leq (s_0, \ldots, s_{t-1})$ if and only if $s'_j \leq s_j$ for all j.

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Theorem

(a) *The module* k^P *is multiplicity free and has composition factors* $L(s_0, \ldots, s_{t-1})$ *parametrized by the set* $\mathcal{H} \cup \{(0, \ldots, 0)\}$.

(b) *For* $(s_0, \ldots, s_{t-1}) \in \mathcal{H}$*, let* $\lambda_j = ps_{j+1} - s_j$*. Then the simple* kG*-module* $L(s_0, \ldots, s_{t-1})$ *is isomorphic to the twisted tensor product*

$$\bigotimes_{j=0}^{t-1} (\overline{S}^{\lambda_j})^{(p^j)},$$

where $\overline{S}^{\lambda}$ *denotes the component of degree* λ *in the truncated polynomial ring* $\overline{S} = k[X_0, \ldots, X_n]/(X_i^p)_{i=0}^n$ *and the superscripts* (p^j) *indicate twisting by powers of the Frobenius map.*

Cont'd

Theorem

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(c) *For each submodule M of Y_P, let $\mathcal{H}_M \subseteq \mathcal{H}$ be the set of its composition factors. Then $\mathcal{H}_M$ is an ideal of the partially ordered set $(\mathcal{H}, \leq)$, i.e if $(s_0, \ldots, s_{t-1}) \in \mathcal{H}_M$ and $(s'_0, \ldots, s'_{t-1}) \leq (s_0, \ldots, s_{t-1})$, then $(s'_0, \ldots, s'_{t-1}) \in \mathcal{H}_M$.*

(d) *The mapping $M \mapsto \mathcal{H}_M$ defines a lattice isomorphism between the submodule lattice of Y_P and the lattice of ideals, ordered by inclusion, of the partially ordered set $(\mathcal{H}, \leq)$*

Stabilization of module structure

- Condition (2) in the definition of $\mathcal{H}$ is automatically satisfied when $t = 1$, (i.e. $q = p$) or when $p \geq n$.
- Thus, in both of these cases, the submodule lattice of Y_P is isomorphic to the lattice of ideals in the t-fold product of the integer interval $[1, n]$.
- In particular, it does not depend on p.
- When $t = 1$ the submodules of k^P are well in coding theory, as generalizations of the Reed-Muller codes.

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Submodule generated by an element

- To apply the theorem, we need to be able to read off the submodule generated by a given element.
- k^P has a monomial basis, each monomial defines an element of $\mathcal{H} \cup \{(0, \dots, 0)\}$.
- For $f \in k^P$, let $\mathcal{H}_f \subseteq \mathcal{H} \cup \{(0, \dots, 0)\}$ denote the set of tuples of the basis monomials appearing with nonzero coefficients in the the expression for f.

Theorem

The kG-submodule of k^P generated by f is the smallest submodule having all the $L(s_0, \dots, s_{t-1})$ for $(s_0, \dots, s_{t-1}) \in \mathcal{H}_f$ as composition factors.

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Hamada's Formula

- $\mathcal{C}_r \subseteq k^P$, subspace spanned by the r-dimensional subspaces of P.
- $\mathcal{C}_r$ is equal to $kG\chi_L$, where L is defined by the equations $X_i = 0$, $i = r+1\ldots,n$. Its characteristic function can be written as

$$\chi_L = \prod_{i=r+1}^{n}(1 - x_i^{q-1}) = \sum_{I\subseteq\{r+1,\ldots,n\}}(-1)^{|I|}x_I^{q-1}.$$

For $I \neq \emptyset$ the monomial x_I^{q-1} has $\mathcal{H}$-tuple $(|I|,\ldots,|I|)$, which lies below the $\mathcal{H}$-tuple $(n-r,\ldots,n-r)$ of $\prod_{i=r+1}^{n} x_i^{q-1}$.

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Hamada's Formula

$$\dim \mathcal{C}_r = 1 + \sum_{(s_0,\dots,s_{t-1})} \prod_{j=0}^{t-1} \sum_{i=0}^{\lfloor \frac{ps_{j+1}-s_j}{p} \rfloor} (-1)^i \binom{n+1}{i} \binom{n + ps_{j+1} - s_j - ip}{n},$$

summed over $(s_0, \dots, s_{t-1}) \in \mathcal{H}$ with $1 \le s_j \le n - r$.

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Delsarte's Theorem

Corollary

1. *$f \in k^P$ belongs to $\mathcal{C}_r{}^{\perp}$ iff every monomial that occurs in f belongs to $\mathcal{C}_r{}^{\perp}$.*
2. *$\mathcal{C}_r{}^{\perp}$ has a basis of monomials of type $(s_0, \dots, s_t)$ such that $s_j < r$ for some r.*

Delsarte (1970). Glynn-Hirschfield call this the "main theorem on geometric codes"

Action on vectors

- Action of $Z(G)$ on $k[V(q)]$ yields

$$k[V(q)] = \bigoplus_{[d] \in \mathbb{Z}/(q-1)\mathbb{Z}} A[d],$$

  where $A[d]$ is the span of the images of monomials of degree congruent to d mod $q-1$.
- $A[0] \cong k \oplus k^P$.
- Similar methods give structure of $A[d]$ for $[d] \neq [0]$.
- Write $d = d_0 + d_1 p + \cdots + d_{t-1} p^{t-1}, \qquad (0 \leq d_j \leq p-1)$.

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Theorem

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(b) *For* $(r_0, \ldots, r_{t-1}) \in \mathcal{H}[d]$, *let* $\lambda_j = d_j + pr_{j+1} - r_j$. *Then the simple kG-module* $L[d](r_0, \ldots, r_{t-1})$ *is isomorphic to the twisted tensor product*

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Structure of symmetric powers

- $S^d \subseteq k[X_0, \ldots, X_n]$, the space of homogeneous polynomials of degree d.
- View as module for the algebraic group $\mathrm{GL}(n+1, k)$.
- When $d < q - 1$ the map $S^d \to A[d]$ is an embedding of kG-modules with image $\mathcal{F}_0[d]$. Thus, S^d corresponds to the ideal

$$\mathcal{H}[d]_{S^d} = \{(r_0, \ldots, r_{t-1}) \in \mathcal{H}[d] \mid r_0 = 0\}.$$

  This gives the submodule structure of S^d as a module for $G = \mathrm{GL}(n+1, p^t)$.
- Fix d and replace p^t by a higher power p^N. Let $A[d](N)$, $\mathcal{H}[d](N)$, etc. denote the corresponding objects for $G(N) = \mathrm{GL}(n+1, p^N)$. Then

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- The p-adic expression for d is unchanged, so that we have $d_j = 0$ for $t \leq j \leq N-1$. Let $(r_0, \ldots, r_{N-1}) \in \mathcal{H}[d](N)_{S^d}$. Then from the definitions we have $0 \leq d_{N-1} + pr_0 - r_{N-1} = -r_{N-1}$, which forces $r_{N-1} = 0$. Repeating this, we obtain $r_j = 0$ for $t \leq j \leq N-1$. Moreover, the conditions on the entries r_j for $0 \leq j \leq t-1$ are exactly the conditions for the t-tuple $(r_0, \ldots, r_{t-1})$ to belong to $\mathcal{H}[d]_{S^d}$.

Theorem

The submodule lattice of S^d is the same for all of the groups $\mathrm{GL}(n+1, p^t)$ for $p^t - 1 > d$. Consequently, it is also the same for the algebraic group $\mathrm{GL}(n+1, k)$. This lattice is isomorphic to the lattice of ideals in the partially ordered set $\mathcal{H}[d]_{S^d}$

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Nonzero intersection

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- For $1 \leq d, e \leq n+1$, let $A(d,e)$ be the incidence matrix for d-subspaces versus e-subspaces, with incidence being nonzero intersection.

Theorem

The p-rank of $A(d,e)$ is given by the formula

$$\mathrm{rank}_p A(d,e) = 1 + \sum_{\substack{\mathbf{s}\in\mathcal{H} \\ (\underline{e})\leq\mathbf{s}\leq(\underline{n-d+1})}} \prod_{j=0}^{t-1} m(n+1, ps_{j+1}-s_j, p-1)$$

- When $d = 1$ this is Hamada's formula.
- If $\mathbf{s} = (s_0, \ldots, s_{t-1})$ satisfies $e \leq s_j \leq n-d+1$ for all j but does not belong to $\mathcal{H}$ then there is some j' for which $m(n+1, ps_{j'+1}-s_{j'}, p-1) = 0$. So we can sum over all tuples $\mathbf{s}$ with $e \leq s_j \leq n-d+1$ instead of just those belonging to $\mathcal{H}$.

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Generating function formulation

- Eric Moorhouse gave a generating function formulation.
- Let $D = D(n, p, d, e)$ be the matrix with rows and columns indexed by $\{e, e+1, \ldots, n-d+1\}$ given by $D_{s,s'} = m(n+1, ps' - s, p-1)$. Then rank formula can be rewritten as

$$\begin{aligned} \operatorname{rank}_p A(d,e) &= 1 + \operatorname{trace} D^t \\ &= 1 + (\text{coefficient of } x^t \text{ in } \operatorname{trace}[(I - xD)^{-1}]) \end{aligned}$$

- Study of $A(d, e)$ (partially) motivated by partial m-systems (Shult-Thas).

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