An outline of polar spaces: basics and advances Part 3

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- Part 3a: Theory of Projective Embeddings of classical thick dual polar spaces: Homogeneous, Polarized, Universal and Minimal embedding.

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References:

✓ R. J. Blok, I. Cardinali, B. De Bruyn, A. Pasini. Polarized and homogeneous embeddings of dual polar spaces. *J. Alg. Combin.*, 30 (2009), 381–399.

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- Part 3b: Projective embeddings of Orthogonal, Hermitian, Symplectic dual polar spaces.

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Reference:

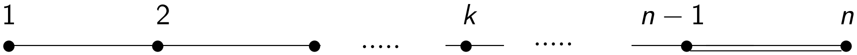
✓ R. J. Blok and B. N. Cooperstein. The generating rank of the unitary and symplectic grassmannians. J. *Combin. Theory Ser. A* 119, no. 1 (2012), 113.

✓ B. N. Cooperstein and E. E. Shult, A note on embedding and generating dual polar spaces, *Adv. Geom.* 1 (2001), 37-48.

✓ De Bruyn and A. Pasini. Generating symplectic and Hermitian dual polar spaces over arbitrary fields nonisomorphic to $\mathbb{F}_2$. *Electron. J. Combin.* 14 (2007) Research Paper 54.

✓ B. De Bruyn and A. Pasini. Minimal scattered sets and polarized embeddings of dual polar spaces, European J. Combin. 28 (2007), no. 7, 1890-1909.

Π: classical thick polar space of rank $n \geq 2$



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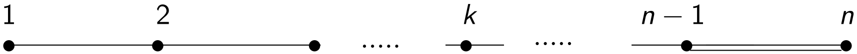

Δ: dual polar space of rank n associated to Π

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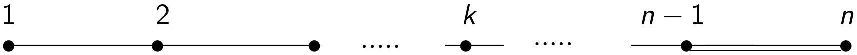

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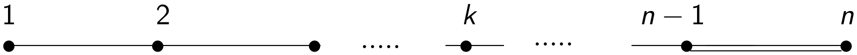

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Definition

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(PE3) $\varepsilon(l) \neq \varepsilon(m)$ *for any distinct lines* $l, m \in \mathcal{L}$.

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ε: full projective embedding $\Rightarrow \varepsilon_\Omega$: full projective embedding

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ε_Ω: polarized embedding $\Rightarrow$ ε: polarized embedding

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$$\varepsilon_1 : \Delta \rightarrow \Sigma_1, \;\; \varepsilon_2 : \Delta \rightarrow \Sigma_2$$

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Definition

Let Δ be a dual polar space. The embedding $\varepsilon_{univ} : \Delta \to \widetilde{\Sigma}$ is (absolutely) universal *if for any full embedding ε of Δ we have $\varepsilon_{univ} \geq \varepsilon$.*

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- ε_{univ}: universal embedding of $\Delta \Rightarrow \varepsilon \cong \varepsilon_{univ}/\Omega$ for any embedding ε of Δ and a suitable subspace Ω of $\widetilde{\Sigma}$.

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- If a universal embedding exists then it is uniquely determined up to isomorphism.
- Universal embedding of $\Delta \leftrightarrow$ hull of all linear embeddings of Δ.

Definition

$$\varepsilon\colon \Delta \to \Sigma,\ \Sigma = \mathrm{PG}(\mathbb{V}),\ \ g \in Aut(\Delta)$$

- *g lifts to Σ through ε* if $\exists\ \varepsilon(g) \in P\Gamma L(\mathbb{V})$ s.t. $\varepsilon(g)\varepsilon = \varepsilon g$.

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  (*) If $\varepsilon(g)$ *exists then it is uniquely determined by* g.

  $\varepsilon(g)$*:*

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- *$G \leq \mathrm{Aut}(\Delta)$. If all elements of G lift to Σ through ε then we say that G lifts to Σ.*

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- *$G \leq \mathrm{Aut}(\Delta)$. If all elements of G lift to Σ through ε then we say that G lifts to Σ.*

- *The embedding $\varepsilon\colon \Delta \to \Sigma$ is G-homogeneous if G lifts to Σ through ε*

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Theorem

If a thick dual polar space Δ of rank $n \geq 2$ admits at least one full projective embedding then Δ admits the absolutely universal embedding.

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Up to isomorphisms, there exists a unique full polarized embedding ε_{min} such that every full polarized embedding ε of Δ has a quotient isomorphic to ε_{min}.

ε_{min}: the minimal full polarized embedding of Δ.

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