An analogue of Bott vanishing result to simply laced Schubert varieties - connection with torus semi-stable points

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Notations

Let G be a simple, simply connected, simply laced algebraic group of rank l over the field $\mathbb{C}$ of complex numbers.
Let T be a maximal torus of G.
Let $N_G(T)$ denote the normaliser of T in G. Let $W = N_G(T)/T$ denote the Weyl group of G with respect to T.
We denote by $\mathfrak{g}$ the Lie algebra of G.
We denote by $\mathfrak{h} \subseteq \mathfrak{g}$ the Lie algebra of T.
Let R denote the roots of G with respect to T.

Notations

Let R^+ denote the set of positive roots. Let B^+ be the Borel sub group of G containing T with respect to R^+.
Let $S = \{\alpha_1, \ldots, \alpha_l\}$ denote the set of simple roots in R^+, where l is the rank of G.
Let B be the Borel subgroup of G containing T with respect to the set of negative roots $R^- = -R^+$.
For $\beta \in R^+$ we also use the notation $\beta > 0$.
The simple reflection in the Weyl group corresponding to α_i is denoted by s_{α_i}.

Notations

We have $X(T)\bigotimes\mathbb{R} = Hom_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$, the dual of the real form of $\mathfrak{h}$.
The positive definite W-invariant form on $Hom_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$ induced by the Killing form of the Lie algebra $\mathfrak{g}$ of G is denoted by $(\ ,\ )$.
We use the notation $\langle\ ,\ \rangle$ to denote $\langle\nu, \alpha\rangle = \frac{2(\nu,\alpha)}{(\alpha,\alpha)}$.

Notations

Let $\leq$ denote the partial order on $X(T)$ given by $\mu \leq \lambda$ if $\lambda - \mu$ is a non negative integral linear combination of simple roots. We also say that $\mu < \lambda$ if $\mu \leq \lambda$, and $\mu \neq \lambda$.

We denote by $X(T)^+$ the set of dominant characters of T with respect to B^+.

Let ρ denote the half sum of all positive roots of G with respect to T and B^+.

Notations

For any simple root α, we denote the fundamental weight corresponding to α by ω_α.
For $w \in W$, let $l(w)$ denote the length of w. We define the dot action by $w \cdot \lambda = w(\lambda + \rho) - \rho$.
Let w_0 denote the longest element of the Weyl group W.

Notations

Let $\leq$ denote the Bruhat order on W. We also say that $w < \tau$ if $w \leq \tau$, and $w \neq \tau$.
Let s_i denote the simple reflection corresponding to the simple root α_i.
For a subset $J \subseteq S$ denote $W^J = \{w \in W | w(\alpha) > 0, \alpha \in J\}$.

Preliminaries

Coxeter element

An element w in W is said to be a Coxeter element if w is of the form $s_{i_1}s_{i_2}\cdots s_{i_l}$, where $i_j \neq i_k$ when ever $j \neq k$

We denote by G/B, the flag variety of all Borel subgroups of G.

Schubert variety

For any $w \in W$, we denote by $X(w) = \overline{BwB/B} \subset G/B$ the Schubert Variety corresponding to w.

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For any $w \in W$, we denote by $X(w) = \overline{BwB/B} \subset G/B$ the Schubert Variety corresponding to w.

We note that $X(w)$ is stable for the left action of T on G/B.

Priliminaries

Let $\lambda \in X(B)$, we have an action of B on $\mathbb{C}$, namely
$b.k = \lambda(b^{-1})k, \ b \in B, \ k \in \mathbb{C}$.
$\mathcal{L}_\lambda := G \times \mathbb{C}/\sim$, where $\sim$ is the equivalence relation defined by $(gb, b.k) \sim (g, k), g \in G, b \in B, k \in \mathbb{C}$. Then $\mathcal{L}_\lambda$ is the total space of a line bundle over G/B associated to λ.

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$X(w)^{ss}_T(\mathcal{L}_\lambda)$ denote the set of all semistable points with respect to the line bundle $\mathcal{L}_\lambda$ and for the action of T.
Here, a point $x \in X(w)$ is said to be semistable if there is a $n \in \mathbb{N}$ and a section $s \in H^0(X(w), \mathcal{L}_\lambda^{\otimes n})^T$ such that $s(x) \neq 0$.

Proposition

Proposition (K , S.K.Pattanayak , 2009)

Let $\lambda = \sum_{\alpha \in S} a_\alpha \varpi_\alpha$ *be a dominant character of* T *which is in the root lattice. Let* $J = Supp(\lambda) = \{\alpha \in S : a_\alpha \neq 0\}$ *and let* $w \in W^{J^c}$*, where* $J^c = S \setminus J$ *. Then* $X(w)^{ss}_T(\mathcal{L}_\lambda) \neq \emptyset$ *if and only if* $w\lambda \leq 0$*.*

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Special case:

If $\lambda = \alpha_0$, the highest root, we have
$X(w)^{ss}_T(\mathcal{L}_{\alpha_0}) \neq \emptyset$ if and only if $w(\alpha_0)$ is a negative root.

Minimal parabolic subgroup and its Levi factor

Unipotent radical of B

We denote by U (resp. U^+) the unipotent radical of B (resp B^+). We denote by P_α the minimal parabolic subgroup of G containing B and s_α. Let L_α denote the Levi subgroup of P_α containing T. We denote by B_α the intersection of L_α and B. Then L_α is the product of T and a homomorphic image G_α of $SL(2)$ via a homomorphism $\psi : SL(2) \longrightarrow L_\alpha$. (cf. [7,II.1.1.4]).

One dimensional reprsentation of B

We refer to [6] for notation and preliminaries on semisimple Lie algebras and their root systems.

For a fixed $w \in W$, the set of all positive roots α which are made negative by w by $R^+(w)$.

For any character λ of B, we denote by $\mathbb{C}_\lambda$ the one dimensional representation of B corresponding to λ.

Cohomology of line bundles on $\mathbb{P}^1$

We make use of following points in computing cohomologies.
Since G is simply connected, the morphism $\psi : SL(2) \longrightarrow G_\alpha$ is an isomorphism, and hence $\psi : SL(2) \longrightarrow L_\alpha$ is injective. We denote this copy of $SL(2)$ in L_α by $SL(2, \alpha)$ We denote by B'_α the intersection of B_α and $SL(2, \alpha)$ in L_α.
We also note that the morphism $SL(2, \alpha)/B'_\alpha \hookrightarrow L_\alpha/B_\alpha$ induced by ψ is an isomorphism.
Since $L_\alpha/B_\alpha \hookrightarrow P_\alpha/B$ is an isomorphism, to compute the cohomology $H^i(P_\alpha/B, V)$ for any B- module V, we treat V as a B_α- module and we compute $H^i(L_\alpha/B_\alpha, V)$

Bott-Samelson- Demazure-Hansen sheme

We recall some basic facts and results about Schubert varieties. A good reference for all this is the book by Jantzen (cf [7, II, Chapter 14]).
Let $w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \ldots s_{\alpha_{i_n}}$ be a reduced expression for $w \in W$. Define

$$Z(w) = \frac{P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \ldots \times P_{\alpha_{i_n}}}{B \times \ldots \times B},$$

where the action of $B \times \ldots \times B$ on $P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \ldots \times P_{\alpha_{i_n}}$ is given by $(p_1, \ldots, p_n)(b_1, \ldots, b_n) = (p_1 \cdot b_1, b_1^{-1} \cdot p_2 \cdot b_2, \ldots, b_{n-1}^{-1} \cdot p_n \cdot b_n)$, $p_j \in P_{\alpha_{i_j}}$, $b_j \in B$. We denote by ϕ_w the birational surjective morphism $\phi_w : Z(w) \longrightarrow X(w)$.
We note that for each reduced expression for w, $Z(w)$ is smooth, however, $Z(w)$ may not be independent of a reduced expression.

$\mathbb{P}^1$-fibration

Let $f_n : Z(w) \longrightarrow Z(ws_{\alpha_n})$ denote the map induced by the projection $P_{\alpha_1} \times P_{\alpha_2} \times \ldots \times P_{\alpha_n} \longrightarrow P_{\alpha_1} \times P_{\alpha_2} \times \ldots \times P_{\alpha_{n-1}}$. Then we observe that f_n is a $P_{\alpha_n}/B \simeq \mathbf{P}^1$-fibration.

Let V be a B-module. Let $\mathcal{L}_w(V)$ denote the pull back to $X(w)$ of the homogeneous vector bundle on G/B associated to V. *By abuse of notation* we denote the pull back of $\mathcal{L}_w(V)$ to $Z(w)$ also by $\mathcal{L}_w(V)$, when there is no cause for confusion.

$\mathbb{P}^1$-fibration

$$R^i f_{n*}\mathcal{L}_w(V) = \mathcal{L}_{ws_{\alpha_n}}(H^i(P_{\alpha_n}/B, \mathcal{L}_w(V)).$$

This together with easy applications of Leray spectral sequences is the constantly used tool in what follows. We term this the *descending 1-step construction*.

$\mathbb{P}^1$-base

We also have the *ascending 1-step construction* which too is used extensively in what follows sometimes in conjunction with the descending construction. We recall this for the convenience of the reader.
Let the notations be as above and write $\tau = s_\gamma w$, with $l(\tau) = l(w) + 1$, for some simple root γ. Then we have an induced morphism

$$g_1 : Z(\tau) \longrightarrow P_\gamma/B \simeq \mathbf{P}^1,$$

with fibres given by $Z(w)$. Again, by an application of the Leray spectral sequences together with the fact that the base is a $\mathbf{P}^1$, we obtain for every B-module V the following exact sequence of P_γ-modules:

Short exact sequence of B-modules

$$(0) \longrightarrow H^1(P_\gamma/B, R^{i-1}g_{1*}\mathcal{L}_w(V)) \longrightarrow H^i(Z(\tau), \mathcal{L}_\tau(V)) \longrightarrow H^0(P_\gamma/B, R^i g_{1*}\mathcal{L}_w(V)) \longrightarrow (0).$$

This short exact sequence of B-modules will be used frequently.
So, we denote this short exact sequence by SES when ever this is being used.

Vanishing of higher direct images

We also recall the following well-known isomorphisms:

- $\phi_{w*}\mathcal{O}_{Z(w)} = \mathcal{O}_{X(w)}$.
- $R^q\phi_{w*}\mathcal{O}_{Z(w)} = 0$ for $q > 0$.

This together with [7, II. 14.6] implies that we may use the Bott-Samelson schemes $Z(w)$ for the computation and study of all the cohomology modules $H^i(X(w), \mathcal{L}_w(V))$. Henceforth in this paper we shall use the Bott-Samelson schemes and their cohomology modules in all the computations.

The Notation $H^i(w, \lambda)$

Simplicity of Notation If V is a B-module and $\mathcal{L}_w(V)$ is the induced vector bundle on $Z(w)$ we denote the cohomology modules $H^i(Z(w), \mathcal{L}_w(V))$ by $H^i(w, V)$.
In particular if λ is a character of B we denote the cohomology modules $H^i(Z(w), \mathcal{L}_\lambda)$ by $H^i(w, \lambda)$.

Some constructions from Demazure's paper

We recall briefly two exact sequences from Demazure's paper for short proof of the Borel-Weil-Bott theorem .
Let α be a simple root and let $\lambda \in X(T)$ be a weight such that $\langle \lambda, \alpha \rangle \geq 0$. For such a λ, we denote by $V_{\lambda,\alpha}$ the module $H^0(P_\alpha/B, \mathcal{L}_\lambda)$.
Let $\mathbb{C}_\lambda$ denote the one dimensional B- module.

Lemma due to Demazure

Lemma (Demazure)

$(0) \longrightarrow K \longrightarrow V_{\lambda,\alpha} \longrightarrow \mathbb{C}_{\lambda} \longrightarrow (0).$
$(0) \longrightarrow \mathbb{C}_{s_{\alpha}(\lambda)} \longrightarrow K \longrightarrow V_{\lambda-\alpha,\alpha} \longrightarrow (0).$

A consequence of the Lemma

Lemma (Demazure)

1. *Let* $\tau = ws_\alpha$, $l(\tau) = l(w) + 1$. *If* $\langle \lambda, \alpha \rangle \geq 0$ *then* $H^j(\tau, \lambda) = H^j(w, V_{\lambda,\alpha})$ *for all* $j \geq 0$.
2. *Let* $\tau = ws_\alpha$, $l(\tau) = l(w) + 1$. *If* $\langle \lambda, \alpha \rangle \geq 0$, *then* $H^i(\tau, \lambda) = H^{i+1}(\tau, s_\alpha \cdot \lambda)$. *Further, if* $\langle \lambda, \alpha \rangle \leq -2$, *then* $H^i(\tau, \lambda) = H^{i-1}(\tau, s_\alpha \cdot \lambda)$.
3. *If* $\langle \lambda, \alpha \rangle = -1$, *then* $H^i(\tau, \lambda)$ *vanishes for all* $i \geq 0$ *(cf. Prop 5.2(b), [7]).*

Another consequence of the lemma

Lemma

Let V be an irreducible L_α- module. Let λ be a character of B_α. Then, we have

1. *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^0(L_\alpha/B_\alpha, V \bigotimes \mathbb{C}_\lambda)$ is isomorphic to the tensor product of V and $H^0(L_\alpha/B_\alpha, \mathbb{C}_\lambda)$, and $H^j(L_\alpha/B_\alpha, V \bigotimes \mathbb{C}_\lambda) = (0)$ for all $j \geq 1$.*
2. *If $\langle \lambda, \alpha \rangle \leq -2$, $H^0(L_\alpha/B_\alpha, V \bigotimes \mathbb{C}_\lambda) = (0)$, and $H^1(L_\alpha/B_\alpha, V \bigotimes \mathbb{C}_\lambda)$, is isomorphic to the tensor product of V and $H^0(L_\alpha/B_\alpha, \mathbb{C}_{s_\alpha \cdot \lambda})$,*
3. *If $\langle \lambda, \alpha \rangle = -1$, then $H^j(L_\alpha/B_\alpha, V \bigotimes \mathbb{C}_\lambda) = (0)$ for all $j \geq 0$.*

Indecomposable B_α-modules

Lemma (Attributed to Verma)

Any indecomposable finite dimensional B_α module V is isomorphic to $V' \bigotimes \mathbb{C}_\lambda$ for some irreducible representation V' of L_α, and $\mathbb{C}_\lambda$ is an one dimensional representation of L_α given by a character λ of B_α.

Bott Vanishing Result

Let $\mathfrak{b}$ be the Lie algebra of B.
Consider the quotient B- module $\mathfrak{g}/\mathfrak{b}$.
We have the following consequence of Bott's Vanishing theorem on Homogeneous Vector bundles:

Theorem (Bott, 1957)

1. $H^i(G/B, \mathcal{L}(\mathfrak{g}/\mathfrak{b})) = 0$ *for* $i \geq 1$.
2. $H^0(G/B, \mathcal{L}(\mathfrak{g}/\mathfrak{b})) = \mathfrak{g}$.

Question

Question:

For which $\tau \in W$, the following statements are true ?

1. $H^i(X(\tau), \mathcal{L}(\mathfrak{g}/\mathfrak{b})) = 0$ for $i \geq 1$.
2. $H^0(X(\tau), \mathcal{L}(\mathfrak{g}/\mathfrak{b})) = \mathfrak{g}$.

Example

Let $\tau \in W$.
Let $h^0(\tau, \lambda)$ denote the character of the T- module $H^0(\tau, \lambda)$.

Example

Take $G = SL(3, \mathbb{C})$, $\tau = s_2 s_1$.

(1). $h^0(s_1, \alpha_1) = e^{\alpha_1} + e^0 + e^{-\alpha_1}$.

(2). $h^0(s_1, \alpha_2) = 0$. (since $\langle \alpha_2, \alpha_1 \rangle = -1$)

(3). $h^0(s_1, \alpha_1 + \alpha_2) = e^{\alpha_1 + \alpha_2} + e^{\alpha_2}$.

(4). $h^0(s_2 s_1, \alpha_1) = e^0 + e^{-\alpha_1} + e^{-\alpha_1 - \alpha_2}$.

(5). $h^0(s_2 s_1, \alpha_2) = 0$.

(6). $h^0(s_2 s_1, \alpha_1 + \alpha_2) = e^{\alpha_1 + \alpha_2} + e^{\alpha_2} + e^{\alpha_1} + e^0 + e^{-\alpha_2}$.

(7). $h^0(s_2 s_1, \alpha_1) + h^0(s_2 s_1, \alpha_2) + h^0(s_2 s_1, \alpha_1 + \alpha_2) =$
$e^{\alpha_1 + \alpha_2} + e^{\alpha_1} + e^{\alpha_2} + 2e^0 + e^{-\alpha_1} + e^{-\alpha_2} + e^{-\alpha_1 - \alpha_2} = char(\mathfrak{g})$.

A Combinatorial Lemma

We state a combinatorial lemma. For completeness, we give a proof here.

Lemma

Let G be a simple simply laced algebraic group. Let $\alpha \in S$, and β be a root different from both α and $-\alpha$. Then, $\langle \beta, \alpha \rangle \in \{-1, 0, 1\}$.

Proof.

Since β and α are not proportional, the product $\langle \beta, \alpha \rangle \, \langle \alpha, \beta \rangle$ is an integer lying in $\{0, 1, 2, 3\}$.
Since G is simply laced, we have $\langle \beta, \alpha \rangle = \langle \alpha, \beta \rangle$.
Since $\langle \beta, \alpha \rangle$ is an integer, $\langle \beta, \alpha \rangle \in \{-1, 0, 1\}$.

□

A Vanishing Result

Let V_1 be a B-sub module of $\mathfrak{g}$ containing $\mathfrak{b}$. Let V_2 be a B-sub module of V_1 containing $\mathfrak{b}$. Let $\tau \in W$.
The natural projection $\Pi : V_1 \longrightarrow V_1/V_2$ of B-modules induces a homomorphism of B-modules $\Pi_\tau : H^0(\tau, V_1) \longrightarrow H^0(\tau, V_1/V_2)$ of B-modules.
We now deduce the follwing lemma as a consequence of the above lemma.

Lemma (K, 2012)

1. $H^i(\tau, V_1/V_2)$ *is zero for all* $i \geq 1$.
2. $\Pi_\tau : H^0(\tau, V_1) \longrightarrow H^0(\tau, V_1/V_2)$ *is surjective.*

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Corollary (K, 2012)

Let $\tau \in W$. *Let* α *be a positive root. Then,* $H^i(\tau, \alpha) = (0)$ *for every* $i \geq 1$.

Answer to the question

Recall the question:

Question:

For which $\tau \in W$, the following statements are true ?

1. $H^i(X(\tau), \mathcal{L}(\mathfrak{g}/\mathfrak{b})) = 0$ for $i \geq 1$.
2. $H^0(X(\tau), \mathcal{L}(\mathfrak{g}/\mathfrak{b})) = \mathfrak{g}$.

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2. $H^0(X(\tau), \mathcal{L}(\mathfrak{g}/\mathfrak{b})) = \mathfrak{g}$.

The following is the answer for the question:
Let $\tau \in W$.

Theorem (K,2012)

$X(\tau^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ *is non-empty if and only if*

1. $H^i(X(\tau), \mathcal{L}(\mathfrak{g}/\mathfrak{b})) = 0$ *for* $i \geq 1$.
2. $H^0(X(\tau), \mathcal{L}(\mathfrak{g}/\mathfrak{b})) = \mathfrak{g}$.

Character of $\mathfrak{g}$

Let $h^0(\tau, \mu)$ denote the character of the T- module $H^0(\tau, \mu)$.

Corollary (K,2012)

$\sum_{\alpha \in R^+} h^0(\tau, \alpha) = Char(\mathfrak{g})$ *if and only if the set of semi stable points* $X(\tau^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ *is non-empty.*

Character of $\mathfrak{g}$

Let $\alpha \in S$. Let Q_α denote the maximal parabolic subgroup of G containing B all s_β, where β running over all simple roots different from α. Let w_α denote the unique minimal representative of the longest element w_0 of W with respect to the maximal parabolic subgroup Q_α.
Recall $R^+(\tau) := \{\beta \in R^+ : \tau(\beta) \in -R^+\}$.
Let $\tau \in W$ be such that $\tau \geq w_\alpha$. The following theorem describe the character of $\mathfrak{g}$ in terms of the sum of characters of $H^0(\tau, \beta)$, β running over all elements of $R^+(\tau)$.

Theorem (K,2012)

For any $\tau \geq w_\alpha$, we have $\sum_{\beta \in R^+(\tau)} h^0(\tau, \beta)) = Char(\mathfrak{g})$

Corollary

Let $c \in W$ be a Coxeter element. Denote C by the cyclic subgroup of W generated by c.

Theorem (Yang-Zelevinsky, 2008)

For any Coxeter element c and for any fundamental weight ω_i, there exist a $\tau \in C$ such that $\tau\omega_i = w_0\omega_i$.

Corollary

Let $c \in W$ be a Coxeter element. Denote C by the cyclic subgroup of W generated by c.

Theorem (Yang-Zelevinsky, 2008)

For any Coxeter element c and for any fundamental weight ω_i, there exist a $\tau \in C$ such that $\tau\omega_i = w_0\omega_i$.

Corollary (K,2012)

Given a Coxeter element c, there is a $\tau \in C$ such that $\sum_{\beta \in R^+(\tau)} h^0(\tau, \beta)) = Char(\mathfrak{g})$

THANK YOU