$K(\pi, 1)$ problem for Artin groups

Part III

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Theorem (Tits). Let $w \in W$, and let μ, μ' be two reduced expressions of w.
Then there is a finite sequence of elementary M-transformations joining μ to μ'.

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$$\tau(w) = \sigma_{s_1} \cdots \sigma_{s_\ell} \, .$$

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The definition of $\tau(w)$ does not depend on the choice of the reduced expression.

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Theorem (Bourbaki). (W_X, X) is the Coxeter system of Γ_X.

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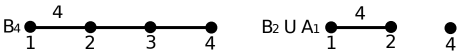

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Definition. Let X, Y be two subsets of S.
We say that an element $w \in W$ is **(X, Y)-minimal** if it is of minimal length in the double-coset $W_X w W_Y$.

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Then w is $(\emptyset, X)$-minimal if and only if $\lg(wu) = \lg(w) + \lg(u)$ for all $u \in W_X$.

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For $\Delta = \{s_0, s_1, \dots, s_p\}$ in A, we set

$$|\Delta| = \left\{ t_0 e_{s_0} + t_1 e_{s_1} + \cdots + t_p e_{s_p} \mid 0 \le t_0, t_1, \dots, t_p \le 1 \text{ and } \sum_{i=0}^{p} t_i = 1 \right\}.$$

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Note that $|\Delta|$ is a (geometric) simplex of dimension p.

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Example. If $(E, \leq)$ is a partially ordered set, then the nonempty finite chains of E form a simplicial complex, called **derived complex** of $(E, \leq)$ and denoted by $E' = (E, \leq)'$.

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Lemma. Let $\preceq$ be the relation on $W \times \mathcal{S}^f$ defined by

$$(u, X) \preceq (v, Y)$$

if

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Then $\preceq$ is a (partial) ordering relation.

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Theorem (Charney, Davis). Take a Vinberg system (W, S) and denote by Γ the Coxeter graph of (W, S).
Then there exists a homotopy equivalence $f : \mathrm{Sal}(\Gamma) \to M(W, S)$ equivariant under the actions of W and that induces a homotopy equivalence $\bar{f} : \mathrm{Sal}(\Gamma)/W \to M(W, S)/W = N(W, S)$.

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The 0-skeleton of $\overline{\mathrm{Sal}}(\Gamma)$ is reduced to a point that we denote by x_0.

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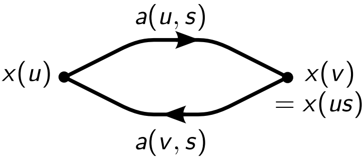

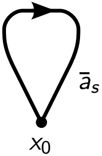

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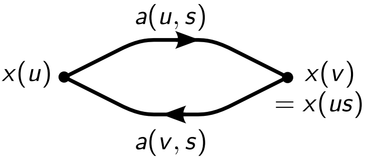

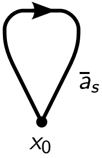

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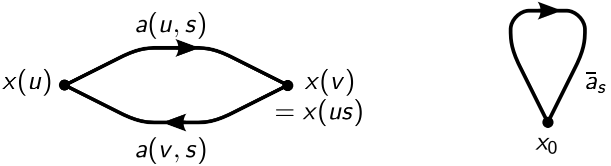

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For each $s \in S$ there is an arrow $\bar{a}_s$ in $\overline{\mathrm{Sal}}(\Gamma)$ from x_0 to x_0.

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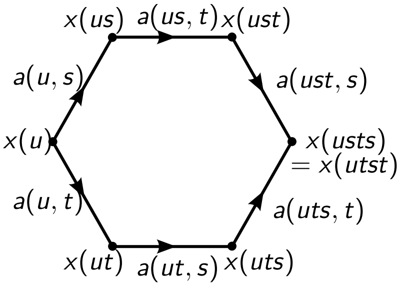
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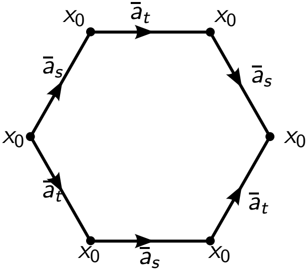
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ā_t
x0
ā_s
ā_s
x0
x0
ā_t
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Definition. The **Artin monoid** of Γ is the monoid A_Γ^+ defined by

$$A_\Gamma^+ = \langle \Sigma \mid \Pi(\sigma_s, \sigma_t : m_{s,t}) = \Pi(\sigma_t, \sigma_s : m_{s,t}) \\ \text{for all } s, t \in S,\ s \neq t,\ m_{s,t} \neq \infty \rangle^+ .$$

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And $\tilde{\theta}$ extends to the universal cover $\widetilde{\mathrm{Sal}}(\Gamma) \to \mathrm{Sal}(\Gamma)$.

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Definition. Γ is of **spherical type** if W_Γ is finite.

Corollary (not obvious but true). $\mathrm{Sal}(\Gamma)$ is an Eilenberg MacLane space if Γ is of spherical type.

Theorem (Godelle, P.). Let Γ be a Coxeter graph, let S be its set of vertices, and let X be a subset of S.

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Then there are "natural" maps $\iota_X : \mathrm{Sal}(\Gamma_X) \to \mathrm{Sal}(\Gamma)$ and $\pi_X : \mathrm{Sal}(\Gamma) \to \mathrm{Sal}(\Gamma_X)$ such that $\pi_X \circ \iota_X = \mathrm{Id}$.

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The inclusion $W_X \times \mathcal{S}_X^f \hookrightarrow W \times \mathcal{S}$ induces an embedding $\iota_X : \mathrm{Sal}_{\Gamma_X} \hookrightarrow \mathrm{Sal}(\Gamma)$.

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Then $\pi_X : \mathrm{Sal}(\Gamma) \to \mathrm{Sal}(\Gamma_X)$ is induced by
$\pi_X : W \times \mathcal{S}^f \to W_X \times \mathcal{S}_X^f$.

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Then φ_X is injective.