$K(\pi, 1)$ problem for Artin groups

Part II

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Definition. For a nonempty open convex cone I in a real vector space V, and a hyperplane arrangement $\mathcal{A}$ in I, we set

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$$N(W, S) = M(W, S)/W .$$

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Then $\pi_1(N(W, S)) = A_\Gamma$, $\pi_1(M(W, S)) = CA_\Gamma$, and the short exact sequence associated with the regular covering $M(W, S) \to N(W, S)$ is

$$1 \longrightarrow CA_\Gamma \longrightarrow A_\Gamma \xrightarrow{\theta} W \longrightarrow 1 \ .$$

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The set $\mathcal{R}$ of reflections coincides with the set of transpositions, and $\mathcal{A} = \{H_{i,j} \mid 1 \leq i < j \leq n+1\}$.

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$$M(\mathfrak{S}_{n+1}, S) = \mathbb{C}^{n+1} \setminus \left(\bigcup_{i<j} \mathbb{C} \otimes H_{i,j} \right)$$

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$$N(\mathfrak{S}_{n+1}, S) = M(\mathfrak{S}_{n+1}, S)/\mathfrak{S}_{n+1}$$

is the **space of (non-ordered) configurations** of $n+1$ points in $\mathbb{C}$.

Theorem (Artin) $\pi_1(N(\mathfrak{S}_{n+1}, S)) = \mathcal{B}_{n+1}$, the braid group on $n+1$ strands.

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The **Sylvester matrix** of f and g is

$$\mathrm{Sylv}(f,g) = \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \ddots & \vdots & b_1 & b_0 & \ddots & \vdots \\ \vdots & a_1 & \ddots & 0 & \vdots & b_1 & \ddots & 0 \\ a_m & \vdots & \ddots & a_0 & b_n & \vdots & \ddots & b_0 \\ 0 & a_m & & a_1 & 0 & b_n & & b_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_m & 0 & \cdots & 0 & b_n \end{pmatrix}$$

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Corollary. Let $f \in \mathbb{C}[x]$ be a polynomial of degree $d \geq 2$. Then f has a multiple root if and only if $\mathrm{Res}(f, f') = 0$.

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Example. If $f = ax^2 + bx + c$, then $\mathrm{Disc}(f) = b^2 - 4ac$.

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Thus

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$\mathcal{D}$ is the ***n-th discriminant***.

Proposition. $N(\mathfrak{S}_n, S) = \mathbb{C}_n[x] \setminus \mathcal{D}$.

Proof. Let $\Phi : M(\mathfrak{S}_n) \to \mathbb{C}_n[x] \setminus \mathcal{D}$ be

$$\Phi(z_1, \ldots, z_n) = (x - z_1) \cdots (x - z_n)\,.$$

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Thus $\mathbb{C}_n[x] \setminus \mathcal{D} \simeq M(\mathfrak{S}_n)/\mathfrak{S}_n = N(\mathfrak{S}_n)$. □

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(3) Any graph is an Eilenberg MacLane space.

By (1), in order to prove that $N(\mathfrak{S}_n)$ is an Eilenberg MacLane space, it suffices to prove that $M(\mathfrak{S}_n)$ is an Eilenberg MacLane space.

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Suppose $n = 2$.
Then

$$M(\mathfrak{S}_2) = \mathbb{C}^2 \setminus \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = z_2\} \simeq \mathbb{C} \times \mathbb{C}^* .$$

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$\mathbb{C}$ is an Eilenberg MacLane space because it is contractible.

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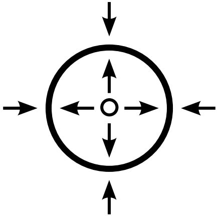

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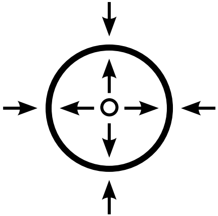

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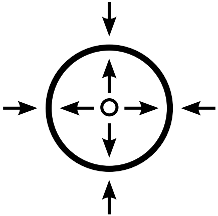

thus $\mathbb{C}^*$ has the same homotopy type as the circle, therefore $\mathbb{C}^*$ is an Eilenberg MacLane space by (3).
By (2) we conclude that $M(\mathfrak{S}_2)$ is an Eilenberg MacLane space.

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$$\begin{array}{ccc} M(\mathfrak{S}_{n+1}) & \to & M(\mathfrak{S}_n) \\ (z_1, \dots, z_n, z_{n+1}) & \mapsto & (z_1, \dots, z_n) \end{array}$$

is a locally trivial fibration.
The fiber above $(1, \dots, n)$ is

$$\{(1, \dots, n, z_{n+1}) \mid z_{n+1} \notin \{1, \dots, n\}\} \simeq \mathbb{C} \setminus \{1, \dots, n\}\,.$$

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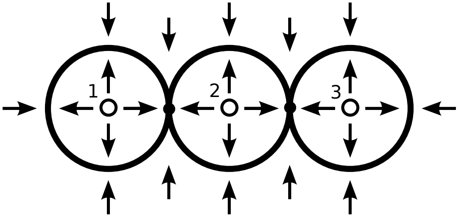

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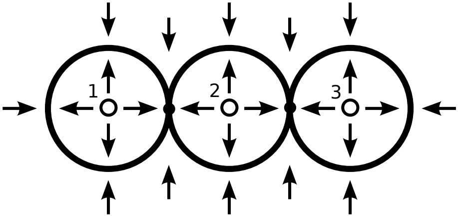

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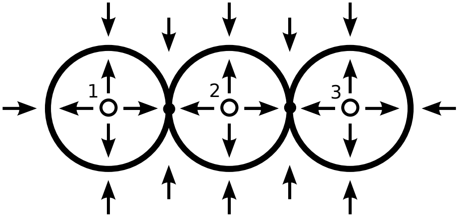

Thus $\mathbb{C} \setminus \{1, \ldots, n\}$ is an Eilenberg MacLane space by (3).
We conclude by (2) that $M(\mathfrak{S}_{n+1}, S)$ is an Eilenberg MacLane space. □

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Define $B : V \times V \to \mathbb{R}$ by

$$B(e_s, e_t) = \begin{cases} -\cos(\frac{\pi}{m_{s,t}}) & \text{if } m_{s,t} \neq \infty \\ -1 & \text{if } m_{s,t} = \infty \end{cases}$$

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For $s \in S$ define $\rho_s \in \mathrm{GL}(V)$ by

$$\rho_s(x) = x - 2\,B(x, e_s)e_s\,, \quad x \in V\,.$$

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ρ_s is a linear reflection for all $s \in S$.
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$\rho : W \to \mathrm{GL}(V)$.

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$$\bar{C}_0 = \{\alpha \in V^* \mid \langle \alpha, e_s \rangle \geq 0 \text{ for all } s \in S\}.$$

Theorem (Tits, Bourbaki).

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In particular, $(\rho^*(W), \rho^*(S))$ is a Vinberg system whose associated Coxeter graph is Γ.

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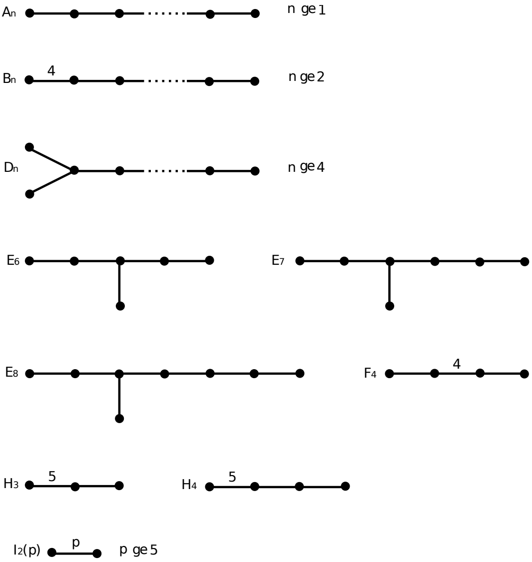
(2) The spherical type connected Coxeter graphs are precisely those listed in the following figure.



An
n ge 1
Bn
4
n ge 2
Dn
n ge 4
E6
E7
E8
F4
4
H3
5
H4
5
I2(p)
p
p ge 5

Theorem (Deligne). Let (W, S) be a Vinberg system. If W is finite, then $N(W)$ is an Eilenberg MacLane space.