Exceptional groups of Lie type: subgroup structure and unipotent elements

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Outline:

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Maximal subgroups of exceptional algebraic groups

Let G be a simple algebraic group of exceptional type, defined over an algebraically closed field k of characteristic $p \geq 0$.

Let $M \subset G$ be a maximal positive-dimensional closed subgroup. By the Borel-Tits theorem, if M° is not reductive, then M is a maximal parabolic subgroup of G.

Now if M° is reductive, it is possible that M° contains a maximal torus of G. It is straightforward to describe subgroups containing maximal tori of G via the Borel-de Siebenthal algorithm. Then M is the full normalizer of such a group, and finding such M which are maximal is again straightfoward.

So finally, one is left to consider the subgroups M such that

- M° is reductive, and
- M does not contain a maximal torus of G.

A classification of the positive-dimensional maximal closed subgroups of G was completed in 2004 by Liebeck and Seitz. (Earlier work of Seitz had reduced this problem to those cases where $\mathrm{char}(k)$ is 'small'.)

We may assume the exceptional group G to be an adjoint type group, as maximal subgroups of an arbitrary simple algebraic group $\tilde{G}$ must contain $Z(\tilde{G})$ and hence will have an image which is a maximal subgroup of the adjoint group.

We adopt the convention $p = \infty$ when $\mathrm{char}(k) = 0$.

The first result classifies the subgroups which are maximal among proper closed connected subgroups.

Maximal closed connected subgroups of the exceptional algebraic groups

Theorem (Seitz, 1991, Liebeck-Seitz, 2004)

Let G be an exceptional algebraic group defined over an algebraically closed field of characteristic p. Let $X < G$ be a closed subgroup. Then X is maximal among proper closed connected subgroups of G if and only if one of the following:

(1) *X is a maximal parabolic subgroup.*

(2) *X is a maximal rank subsystem subgroup as in Table 1 below.*

(3) *X and G are as in Tables 2 and 3 below.*

Table 1: Maximal maximal-rank connected subgroups

G	X
G_2	$A_1\tilde{A}_1$, A_2, $\tilde{A}_2$ $(p=3)$
F_4 $(p \neq 2)$	B_4, A_1C_3, $A_2\tilde{A}_2$
F_4 $(p = 2)$	B_4, C_4, $A_2\tilde{A}_2$
E_6	A_1A_5, $A_2A_2A_2$
E_7	A_1D_6, A_7, A_2A_5
E_8	D_8, A_1E_7, A_8, A_2E_6, A_4A_4

Here $\tilde{\Phi}$ signifies a subsystem containing short roots.

Table 2: Simple maximal connected subgroups, of rank $< \mathrm{rank} G$

G	X simple
G_2	A_1 $(p \geq 7)$
F_4	A_1 $(p \geq 13)$, G_2 $(p = 7)$
E_6	A_2 $(p \neq 2, 3)$, G_2 $(p \neq 7)$, C_4 $(p \neq 2)$, F_4
E_7	A_1 (2 classes, $p \geq 17, 19$ respectively), A_2 $(p \geq 5)$
E_8	A_1 (3 classes, $p \geq 23, 29, 31$ respectively), B_2 $(p \geq 5)$

Table 3: Non-simple maximal connected subgroups, of rank $< \mathrm{rank} G$

G	X semisimple, non-simple
F_4	$A_1 G_2$ $(p \neq 2)$
E_6	$A_2 G_2$
E_7	$A_1 A_1$ $(p \neq 2, 3)$, $A_1 G_2$ $(p \neq 2)$, $A_1 F_4$, $G_2 C_3$
E_8	$A_1 A_2$ $(p \neq 2, 3)$, $G_2 F_4$

Maximal closed positive-dimensional subgroups

Theorem (Liebeck-Seitz, 2004)

Let G be an exceptional algebraic group of adjoint type, defined over an algebraically closed field of characteristic p. Let $M < G$ be a closed subgroup. Then M is maximal among positive-dimensional closed subgroups of G if and only if one of the following holds:

(1) *M is a maximal parabolic subgroup.*

(2) *$G = E_7$, $p \neq 2$ and $M = (Z_2^2 \times D_4).S_3$.*

(3) *$G = E_8$, $p \neq 2, 3, 5$ and $M = A_1 \times S_5$.*

(4) *$M = N_G(X)$ for X as in (3) of the previous theorem.*

(5) *$G = E_8$, $p \neq 2$, $M = N_G(X)$, where $X = A_1G_2G_2$, with A_1G_2 maximal closed connected in F_4, and $M/X = Z_2$.*

(6) *$M = N_G(X)$ where X is connected reductive of maximal rank and the pair $(X, M/X)$ is as in the table below, where T_i indicates an i-dimensional subtorus of G:*

G	X	M/X
G_2	$A_1\tilde{A}_1$, A_2, $\tilde{A}_2$ $(p=3)$	1, Z_2, Z_2
F_4 $(p \neq 2)$	B_4, D_4, A_1C_3, $A_2\tilde{A}_2$	1, S_3, 1, Z_2
F_4 $(p = 2)$	B_4, C_4, D_4, $\tilde{D}_4$, $A_2\tilde{A}_2$	1, 1, S_3, S_3, Z_2
E_6	A_1A_5, $(A_2)^3$, D_4T_2, T_6	1, S_3, S_3, $W(E_6)$
E_7	A_1D_6, A_7, A_2A_5, $(A_1)^3D_4$,	1, Z_2, Z_2, S_3,
	$(A_1)^7$, E_6T_1, T_7	$\mathrm{PSL}_3(2)$, Z_2, $W(E_7)$
E_8	D_8, A_1E_7, A_8, A_2E_6,	1, 1, Z_2, Z_2,
	$(A_4)^2$, $(D_4)^2$,	Z_4, $S_3 \times Z_2$,
	$(A_2)^4$, $(A_1)^8$, T_8	$\mathrm{GL}_2(3)$, $\mathrm{AGL}_3(2)$, $W(E_8)$

Remark

When M is the normalizer of one of the non-maximal rank maximal connected subgroups, the index $|M : X|$ is 1 or 2. In all cases where X has a factor of type A_2, M induces a non-trivial graph automorphism of this factor.

Questions

- How effectively can one apply the above results to the study of the subgroup lattice of G.
- For example, if one wants to classify all A_1- or B_2-type subgroups in G (up to conjugacy) how does one use the above theorem ?

- In principle, for $Y \subset G$ connected reductive, if Y lies in a reductive maximal subgroup M, then we can apply induction and use results on the subgroup structure of the classical algebraic groups, which are quite complete in small rank, and determine Y up to conjugacy.
- However, if Y lies in a parabolic subgroup P of G, then one is faced with the difficult question: does Y lie in a Levi subgroup of P? If so, then again, one can proceed by induction; if not, this question can be quite complicated.

David Stewart will address this issue in his talks.

Maximal subgroups of the finite exceptional groups

Reduction theorem:

Theorem (Borovik, 1989 Liebeck-Seitz, 1990)

Let G be an exceptional simple algebraic group, $F : G \to G$ an endomorphism with finite fixed-point subgroup G^F, and $H < G^F$ maximal. Then one of the following holds:

(1) $H = M^F$ *for* M *an* F*-stable maximal positive-dimensional closed subgroup,*

(2) H *is an exotic local subgroup as described below,*

(3) $G = E_8$, $p > 5$, $H = (Alt_5 \times S_6).2$, *or*

(4) H *is almost simple.*

(We will call such an F a *Steinberg endomorphism*.)

Jordan subgroups

Definition

Let G be a simple algebraic group, defined over an algebraically closed field k. An elementary abelian r-subgroup R of G, with $r \neq \mathrm{char}(k)$, is called a *Jordan subgroup* of G if it satisfies the following conditions:

1. $C_G(R)$ (and hence $N_G(R)$) is finite,
2. R is a minimal normal subgroup of $N_G(R)$,
3. $N_G(R)$ is maximal subject to conditions 1. and 2., and
4. there is no non-trivial connected $N_G(R)$-invariant proper subgroup of G.

Exotic locals

Theorem (Borovik, Cohen–Liebeck–Saxl–Seitz (1992))

Let G be a simple exceptional algebraic group of adjoint type with Steinberg endomorphism $F : G \to G$. Then the Jordan subgroups R in G^F and their normalizers $H = N_{G^F}(R)$ are given as follows:

(1) $G = G_2$, $R = 2^3$ *and* $H = 2^3.\mathrm{SL}_3(2)$,

(2) $G = F_4$, $R = 3^3$, $H = 3^3.\mathrm{SL}_3(3)$,

(3) $G = E_6$, $R = 3^3$, $H = 3^{3+3}.\mathrm{SL}_3(3)$, *or*

(4) $G = E_8$, $R = 2^5$, $H = 2^{5+10}.\mathrm{SL}_5(2)$, *or* $R = 5^3$, $H = 5^3.\mathrm{SL}_3(5)$.

In each case, these are unique up to G^F-conjugation.

In the statement, the notation r^a, with r a prime and $a \geq 1$, denotes an elementary abelian r-group of this order, while r^{a+b} stands for an (unspecified) extension of an elementary abelian r-group of order r^a by one of order r^b.

Definition

The normalizers $N_{G^F}(R)$ of these Jordan subgroups are called *exotic local subgroups* of exceptional groups of Lie type.

The Liebeck-Seitz-Borovik result reduces the problem of determining the maximal subgroups of the finite exceptional groups to that of determining the almost simple maximal subgroups $H \subset G^F$.

This problem naturally separates into two cases

- H is of Lie type in characteristic p,
- H is not of Lie type in characteristic p.

The latter case has been studied by Liebeck and Seitz. If we include the consideration of finite subgroups of the exceptional groups defined over $\mathbb{C}$, there is a much longer story, written by various authors: Cohen, Wales, Griess, Ryba, Serre.

Note that this is much more complicated in the finite classical groups, and essentially comes down to difficult questions in modular representation theory.

In the case which interests us here, that is of the finite exceptional groups, the Liebeck-Seitz result is

Theorem

Let S be a finite simple group, some cover of which is contained in an exceptional algebraic group G in characteristic $p > 0$. Assume that S is not a group of Lie type in characteristic p. Then the possibilities for S and G are given below.

$S \subset G$, G of type G_2, F_4, E_6 in characteristic p and $S \notin \mathrm{Lie}(p)$

G	S
G_2	$\mathrm{Alt}_5, \mathrm{Alt}_6, L_2(7), L_2(8), L_2(13), U_3(3)$ $\mathrm{Alt}_7\ (p=5), J_1\ (p=11), J_2\ (p=2)$
F_4	$\mathrm{above, plus:}\ \mathrm{Alt}_r, r=7,8,9,10, L_2(17), L_2(25), L_2(27)$ $L_3(3), U_4(2), \mathrm{Sp}_6(2), \Omega_8^+(2), {}^3D_4(2), J_2$ $\mathrm{Alt}_{11}\ (p=11), L_3(4)\ (p=3), L_4(3)\ (p=2)$ ${}^2B_2(8)\ (p=5), M_{11}\ (p=11)$
E_6	$\mathrm{above, plus:}\ \mathrm{Alt}_{11}, L_2(11), L_2(19), L_3(4), U_4(3), {}^2F_4(2)', M_{11},$ $\mathrm{Alt}_{12}\ (p=2,3), G_2(3)\ (p=2), \Omega_7(3)\ (p=2), M_{22}\ (p=2,7),$ $J_3\ (p=2), Fi_{22}\ (p=2), M_{12}\ (p=2,3,5)$

$S \subset G$, G of type E_7, E_8 in characteristic p and $S \not\in \mathrm{Lie}(p)$

G	S
E_7	$\text{above, plus}: \mathrm{Alt}_{12}, \mathrm{Alt}_{13}, L_2(29), L_2(37), U_3(8)$ $M_{12}, \mathrm{Alt}_{14}\ (p=7), M_{22}\ (p=5), Ru\ (p=5), HS\ (p=5)$
E_8	$\text{above, plus}: \mathrm{Alt}_r, 14 \le r \le 17, L_2(q), q = 16, 31, 32, 41, 49, 61$ $L_3(5), \mathrm{PSp}_4(5), G_2(3), {}^2B_2(8),$ $\mathrm{Alt}_{18}\ (p=3), L_4(5)\ (p=2), Th\ (p=3), {}^2B_2(32)\ (p=5)$

Questions

In each case, they establish the existence of such a specified covering group of S in the exceptional algebraic group.

- Determine the conjugacy classes of such subgroups.

The above result reduces the case where H is not of Lie type in the defining characteristic to a very finite list of configurations to study.

So we see that the remaining cases are those where H is of Lie type in characteristic p.

Lifting results

The question we consider here is the following:

Given very precise information about the subgroup structure of a simple algebraic group G, defined over an algebraically closed field of positive characteristic, how much can one deduce about the subgroup structure of finite groups which occur as fixed point subgroups G^σ of some rational endomorphism $\sigma : G \to G$?

Best case scenario: Steinberg

Theorem (Steinberg, 1963)

Let G be a simple algebraic group defined over $\bar{\mathbb{F}}_p$. Let $\sigma : G \to G$ be a rational endomorphism with finite fixed-point subgroup G^σ and let $\rho : G^\sigma \to \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ be an irreducible representation. Then there exists an irreducible rational representation $\bar{\rho} : G \to \mathrm{GL}_n(\bar{\mathbb{F}}_p)$, such that $\rho = \bar{\rho}|_{G^\sigma}$.

One can in fact show more, namely,

if $\rho(G^\sigma)$ fixes a nondegenerate bilinear or quadratic form on the associated $\bar{\mathbb{F}}_p G^\sigma$-module, then $\bar{\rho}(G)$ fixes the same form. (Seitz, 1988)

Hence, the embedding of $\rho(G^\sigma)$ in the corresponding finite classical group 'lifts' to an embedding of the algebraic group $\bar{\rho}(G)$ in the corresponding simple classical type algebraic group.

No completely general lifting result

Of course one cannot hope for a completely general lifting result, as there are indecomposable representations of finite groups of Lie type which do not arise as restrictions of representations of the corresponding algebraic group.

Example

Take $G^\sigma = \mathrm{SL}_2(3)$, and let $N \subset G^\sigma$ be a (normal) 2-Sylow subgroup, with quotient $G^\sigma/N = \langle cN \rangle$. Define a representation of G^σ/N by

$cN \mapsto \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. This defines an indecomposable representation of G^σ.

Since an element of the Weyl group of $\mathrm{SL}_2(\bar{\mathbb{F}}_3)$ is represented by an element in N, we see that if the representation extends to a representation of $\mathrm{SL}_2(\bar{\mathbb{F}}_3)$, the only weight occurring in the representation (for a fixed maximal torus) is the weight 0 and so a composition series for the module has trivial composition factors, contradicting the simplicity of the algebraic group $\mathrm{SL}_2(\bar{\mathbb{F}}_3)$.

Lifting results for exceptional type groups

Let H be a simple algebraic group defined over $k = \bar{\mathbb{F}}_p$ with endomorphism $\sigma : H \to H$ having finite fixed point subgroup $X = H^\sigma$.

Let G be an exceptional type simple algebraic group defined over k, with endomorphism $F : G \to G$ having finite fixed-point subgroup, and let $\varphi : X \to G$ be a homomorphism whose image lies in G^F.

When does there exist a closed F-invariant subgroup $\bar{X}$ of G such that $\varphi(X) = \bar{X}^F$? One could ask for more, that is, does the homomorphism φ extend to a rational morphism of algebraic groups $\bar{\varphi} : H \to G$, whose image is F-invariant and such that $\varphi(X) = \bar{\varphi}(H)^F$.

Some answers

There are various complementary results which address this question. Let me list them in chronological order.

Notation

- Let $\tilde{X}$ be a semisimple and simply connected algebraic group over $\bar{\mathbb{F}}_p$ and σ an endomorphism of $\tilde{X}$ normalizing each simple factor of $\tilde{X}$ and having finite fixed point group.
- Let $Z < Z(\tilde{X}^\sigma)$ and $\bar{X} = \tilde{X}/Z$. Then σ induces an endomorphism on $\bar{X}$, also denoted by σ.
- Here, a *finite group of Lie type in characteristic p* is a group of the form $X = O^{p'}(\bar{X}^\sigma)$.
- The group $\bar{X}$ is called the *ambient algebraic group corresponding to* X.
- Let $\bar{G}$ be a semisimple algebraic group over $\bar{\mathbb{F}}_p$ and F an endomorphism of $\bar{G}$ such that $G = O^{p'}(\bar{G}^F)$ is finite.

Theorem (Seitz-Testerman (1990))

Let X be a perfect finite group of Lie type in characteristic p, with ambient algebraic group $\bar{X}$. Let $\varphi : X \to G$ be a homomorphism such that $\varphi(X)$ **is contained in no proper parabolic subgroup of** G. *There is an integer N depending on the dimension of the largest simple factor of $\bar{G}$, such that if $p > N$, then φ can be extended to a homomorphism of algebraic groups $\bar{\varphi} : \bar{X} \to \bar{G}$. If each simple factor of $\bar{G}$ is of classical type, then no restriction on p is required.*

The integer N for the exceptional type groups $\bar{G}$ is defined as follows:

Definition

Let $\bar{Y}$ be a simple factor of $\bar{X}$ of minimal dimension. Then $N = 7$ suffices unless $\bar{Y}$ is as in Table 1. In the remaining cases, we take N to be the value in the table corresponding to the pair $(\bar{Y}, \bar{G})$.

Table: $N(\bar{Y}, \bar{G})$

	$\bar{G} = E_8$	E_7	E_6	F_4	G_2
$\bar{Y}$ of type A_4, B_3, C_3	13				
G_2	13	13			
A_3	13	13	13		
B_2	23	19	13	13	
A_2	43	31	19	13	
A_1	113	67	43	43	19

The smaller the rank of the group $\bar{Y}$, the less control one has over the embedding of X in G.

In the particular case where $\bar{X}$ is a simple algebraic group of type A_1, there is an improvement to the above result, if we make some assumption about the $\bar{G}$-class of the unipotent elements in X.

Definition

Let H be a simple algebraic group and let $u \in H$ be a unipotent element. We say that u is *semiregular* if $C_G(u)$ contains non noncentral semisimple elements.

Theorem (Seitz-Testerman (1997))

Let $\mathrm{PSL}_2(p) \cong X \subset \bar{G}$*, where* $\bar{G}$ *is of exceptional type. Assume* X *contains a semiregular unipotent element of* $\bar{G}$*. If* $p \geq 5$ *and* $\mathrm{PGL}_2(p) \subset N_{\bar{G}}(X)$*, then* $N_{\bar{G}}(X) \cong \mathrm{PGL}_2(p)$ *and* $N_{\bar{G}}(X)$ *is contained in a connected subgroup of type* A_1*.*

Notice that considering $\mathrm{PSL}_2(p)$-subgroups rather than $\mathrm{SL}_2(p)$-subgroups is not a restriction, since by the assumption on the unipotent elements in X, we see that the center of X lies in $Z(\bar{G})$ and so we may pass to the adjoint type group, where we have a $\mathrm{PSL}_2(p)$-subgroup.

A second complementary result shows that if we assume in addition that $q > p$, we always find an appropriate positive-dimensional closed subgroup of $\bar{G}$.

Theorem (Seitz-Testerman (1997))

Let $\mathrm{PSL}_2(q) \cong X \subset \bar{G}$, where q is a power of p with $q > p$. Assume X contains a semiregular unipotent element of $\bar{G}$. Then there exists a positive-dimensional connected subgroup $\bar{X} \subset \bar{G}$, with $X \subset \bar{X} \cong \mathrm{PSL}_2(\bar{\mathbb{F}}_p)$. Moreover, except for the case $p = 2$ and $\bar{G} = B_2$, we have $N_{\bar{G}}(X) = N_{\bar{X}}(X) = \mathrm{PGL}_2(q)$.

Remark

The first theorem does not hold for classical groups, while the second theorem does.

The most definitive result to date for the question of lifting homomorphisms of finite groups of Lie type to appropriate morphisms of algebraic groups is

Theorem (Liebeck-Seitz (1998))

Let X be a quasisimple finite group of Lie type in characteristic p, over a finite field $\mathbb{F}_q$, and suppose that $X \subset \bar{G}^F$, where $\bar{G}$ is a simple adjoint algebraic group of exceptional type, defined over $\bar{\mathbb{F}}_p$, with endomorphism $F : \bar{G} \to \bar{G}$ having finite fixed-point subgroup. Moreover assume that $q > t(\bar{G})(2, p-1)$ as defined in Table 2, if $X = A_1(q), {}^2B_2(q)$ or ${}^2G_2(q)$, or $q > 9$ and $X \neq A_2^{\epsilon}(16)$, otherwise.

1. *Then there exists a closed connected F-stable subgroup $\bar{Y}$ of $\bar{G}$, normalized by $N_{\bar{G}}(X)$ with $X \subset \bar{Y}$, such that $\bar{Y}$ stabilizes every X-invariant subspace of $\mathrm{Lie}\bar{G}$. Moreover, if X is not of the same type as $\bar{G}$, then $\bar{Y}$ may be chosen to be a proper subgroup of $\bar{G}$.*
2. *Assume in addition that $p > N'(\bar{X}, \bar{G})$ as defined in Table 3. Then X lies in a closed connected semisimple F-stable subgroup $\bar{Y}$ of $\bar{G}$ where each simple factor of $\bar{Y}$ has the same type as $\bar{X}$.*

Table 2: $t(\bar{G})$

$\bar{G}$	$t(\bar{G})$
G_2	12
F_4	68
E_6	124
E_7	388
E_8	1312

Table 3: $N'(\bar{X}, \bar{G})$

$N'(\bar{X}, \bar{G})$	$\bar{G} = E_8$	E_7	E_6	F_4	G_2
$\bar{X}$ is of type A_1	7	7	5	3	3
A_2	5	5	5	3	
B_2	5	3	3	2	
G_2	7	7	3	2	
B_3	2	2	2	2	
C_3	3	2	2	2	
A_3, B_4, C_4, D_4	2	2	2		

Idea of proofs

Recall the setup: $\varphi : X \to G$ is a homomorphism such that $\varphi(X)$ lies in no proper parabolic subgroup of G. We aim to show that φ extends to a morphism of the associated algebraic groups.

Let us consider the special case where $\bar{G}$ is a simply connected, simple exceptional type algebraic group. We will take $N = 3\dim\bar{G}$, just in order to illustrate how one obtains a bound.

We may assume as well that $\bar{X}$ is simply connected. We proceed by induction, taking $\bar{G}$ as a counterexample of minimal dimension.

Step 1:

We first claim that there is no F-stable, closed connected proper subgroup D of $\bar{G}$ with $\varphi(X) \subset D$.

Indeed, suppose $\varphi(X) \subset D \subset \bar{G}$, $D \neq \bar{G}$, as above. If D is not reductive, then $1 \neq R_u(D)$ is an F-stable unipotent subgroup. Hence, $\varphi(X) \subset D \subset N_{\bar{G}}(R_u(D))$ lies in a proper F-stable parabolic subgroup, contradicting the assumption on $\varphi(X)$.

Hence D is reductive and as X is perfect, $\varphi(X) \subset [D, D]$, a semisimple F-invariant subgroup of dimension less than dim $\bar{G}$. So by minimality of $\bar{G}$, we have the desired extension of φ.

Step 2:

We now construct a 1-dimensional subtorus of the group $\mathrm{GL}(g)$, where $g = \mathrm{Lie}(\bar{G})$, which will play a key role in what follows.

Let $\mathrm{Ad_G} : \bar{\mathrm{G}} \to \mathrm{GL(g)}$ denote the adjoint representation of $\bar{G}$. Let $J \leq X$ with $J \cong \mathrm{SL}_2(p)$.

Now let S be the subgroup of J corresponding to the group of diagonal matrices in $\mathrm{SL}_2(p)$. So $S \subset J$ is isomorphic to $\mathbb{F}_p^\times$; let $\mathbb{F}_p^\times \to S$, $c \mapsto S(c)$, denote such an isomorphism. As S is cyclic, S lies in an F-stable maximal torus T of $\bar{G}$.

Let Φ be the set of roots of $\bar{G}$ with respect to T. Fix a basis $\mathcal{C}_T$ of $\mathrm{Lie}(T)$ and for each $\alpha \in \Phi$, choose $v_\alpha \in g_\alpha \setminus \{0\}$, so that $\mathcal{C} = \mathcal{C}_T \cup \{v_\alpha \mid \alpha \in \Phi\}$ is a basis of g.

Using this basis, identify $\mathrm{GL}(g)$ with the group of invertible $n \times n$ matrices (where $n = \dim g$); so we have $\mathrm{Ad_G(T)} \leq \bar{\mathrm{D}}_\mathrm{n}$, the group of diagonal matrices.

By Steinberg's theorem, the composition factors of J on g are realized as restrictions of restricted irreducible representations of SL_2.

As $p > 3 \dim \bar{G} = 3 \dim g$, the J-composition factors on g are of dimension strictly less than $\frac{p}{3}$.

The action of S on any J-composition factor is diagonalizable with weights $l \in \mathbb{Z}$ satisfying $-\frac{p-1}{3} < l < \frac{p-1}{3}$, where the weights are defined by $S(c) \mapsto c^l$, for $c \in \mathbb{F}_p^\times$.

So in the present situation we have, for $c \in \mathbb{F}_p^\times$, $\mathrm{Ad_G(S(c))v = v}$ for all $v \in \mathcal{C}_T$ and $\mathrm{Ad_G(S(c))v_\alpha = c^{l_\alpha} v_\alpha}$ for some integers l_α satisfying $-\frac{p-1}{3} < l_\alpha < \frac{p-1}{3}$.

We now define a co-character $\gamma : \bar{\mathbb{F}}_p \to \bar{D}_n$: for $a \in \bar{\mathbb{F}}_p$, set

i) $\gamma(a)v = v$ for all $v \in \mathcal{C}_T$.

ii) $\gamma(a)v_\alpha = a^{l_\alpha} v_\alpha$, for $\alpha \in \Phi$.

Set $\overline{S} := \mathrm{Im}(\gamma)$, a 1-dimensional subtorus of $\mathrm{GL}(g)$; so $\gamma(c) = \mathrm{Ad_G(S(c))}$, for all $c \in \mathbb{F}_p^\times$.

Step 3:

$$\overline{S} \subset \mathrm{Ad_G}(\bar{\mathrm{G}})$$

Indeed, we will show that $\overline{S}$ acts as a group of Lie algebra automorphisms of g and so $\overline{S} = \overline{S}^{\circ} \leq \mathrm{Aut}(g)^{\circ} = \mathrm{Ad_G}(\bar{\mathrm{G}})$.
Let $\alpha, \beta \in \Phi$ with $[v_\alpha, v_\beta] \neq 0$. Then considering the action of the torus T, we see that $[v_\alpha, v_\beta]$ is a scalar multiple of $v_{\alpha+\beta}$.

Thus for all $c \in \mathbb{F}_p^{\times}$,

$$c^{l_\alpha} c^{l_\beta}[v_\alpha, v_\beta] = [\gamma(c)v_\alpha, \gamma(c)v_\beta] = \gamma(c)[v_\alpha, v_\beta] = c^{l_{\alpha+\beta}}[v_\alpha, v_\beta]$$

so $c^{l_\alpha + l_\beta} = c^{l_{\alpha+\beta}}$ for all $c \in \mathbb{F}_p^{\times}$.

Using the fact that l_α, l_β lie in the interval $[-\frac{p-1}{3}, \frac{p-1}{3}]$ we see that

$$l_\alpha + l_\beta = l_{\alpha+\beta}$$

and so

$$a^{l_\alpha + l_\beta} = a^{l_{\alpha+\beta}}$$

for all $a \in \bar{\mathbb{F}}_p^\times$. Hence,

$$\gamma(a)[v_\alpha, v_\beta] = [\gamma(a)v_\alpha, \gamma(a)v_\beta].$$

One easily checks that the action on the remaining commutators $[v, v']$, for $v, v' \in \mathcal{C}$ is also preserved by $\gamma(a)$ and so $\overline{S}$ acts as a group of automorphisms of the Lie algebra g, as claimed.

Steps 4 and 5:

Step 4: Using Step 3, one can show that the closed subgroup $R = (\mathrm{Ad}_{\mathrm{G}}^{-1}(\overline{\mathrm{S}}))^{\circ}$ is F-stable.

Step 5: We can now show that X acts irreducibly on g.

For suppose that V is a proper X-invariant subspace of g, so V is spanned by weight vectors for S. We claim that weight vectors for S are in fact weight vectors for $\overline{S}$.

For suppose two $\overline{S}$-weights on g have equal restrictions to S. Then $c^{l_\alpha} = c^{l_\beta}$ for some $\alpha, \beta \in \Phi$ and for some generator $c \in \mathbb{F}_p^{\times}$; that is, $c^{l_\alpha - l_\beta} = 1$, so $(p-1)|(l_\alpha - l_\beta)$. But $-\frac{p-1}{3} \leq l_\alpha, l_\beta \leq \frac{p-1}{3}$, so $l_\alpha = l_\beta$ as claimed.

Now set $D = \langle xRx^{-1} \mid x \in X \rangle$, a closed connected subgroup of G, which is F-stable as R is.

Moreover D stabilizes the subspace V and so D is proper in $\bar{G}$.

But the group $[X, S]$ is normal in X and contains the subgroup J, so we have

$$X = [X, S] \leq [X, RZ(G)] = [X, R] \leq D,$$

which contradicts our standing assumption.

Hence X acts irreducibly as claimed.

Step 6

A lengthy argument relying upon detailed considerations of the representation theory of X and $\bar{G}$, shows that the groups $\bar{X}$ and $\bar{G}$ have isomorphic root systems and the absolutely irreducible representation $\mathrm{Ad_G} \circ \varphi : \mathrm{X} \to \mathrm{GL(g)}$ is the restriction of a twist of the adjoint representation of $\bar{X}$,

that is, there exists a standard Frobenius endomorphism F' of $\bar{X}$ such that $\mathrm{Ad_G} \circ \varphi = (\mathrm{Ad_{\bar{X}}} \circ \mathrm{F'})|_\mathrm{X}$.

We now have that X stabilizes two Lie brackets on g: $[\ ,\ ]_G$ coming from g and $[\ ,\ ]_{\bar{X}}$ coming from $\mathrm{Lie}(\bar{X})$.

Now the existence of a Lie bracket on g shows that $\mathrm{Hom}_{k\bar{G}}(g \wedge g, g)$ is nontrivial. Moreover, using the theory of highest weights and the assumption on p, one can check that for each exceptional group $\bar{G}$, g occurs with multiplicity 1 as a composition factor of $g \wedge g$ and so

$$\mathrm{Hom}_{k\bar{G}}(g \wedge g, g) \text{ is a 1-space.}$$

Conclusion

Hence the two Lie brackets are scalar multiples of each other and $\mathrm{Ad}_{\bar{\mathrm{X}}}(\mathrm{F}'(\bar{\mathrm{X}})) = \mathrm{Ad}_{\mathrm{G}}(\bar{\mathrm{G}})$.

Recall that $\bar{X}$ is simply connected.

Moreover the isogeny $\mathrm{Ad}_{\mathrm{G}} : \bar{\mathrm{G}} \to \mathrm{Ad}_{\mathrm{G}}(\bar{\mathrm{G}})$ satisfies $\ker d\mathrm{Ad} = \ker \mathrm{ad} = 0$, by the restriction on p.

Hence we obtain a morphism $\psi : \bar{X} \to \bar{G}$ such that $\mathrm{Ad}_{\bar{\mathrm{X}}} = \mathrm{Ad}_{\mathrm{G}} \circ \psi$.

But then $\mathrm{Ad}_{\mathrm{G}}(\varphi(\mathrm{x})) = \mathrm{Ad}_{\bar{\mathrm{X}}}(\mathrm{F}'(\mathrm{x})) = \mathrm{Ad}_{\mathrm{G}}(\psi(\mathrm{F}'(\mathrm{x})))$ for all $x \in X$.

Since X is generated by its unipotent elements and Ad_{G} is bijective on unipotent elements, we have

$$\varphi = \psi \circ F'|_X, \text{ whence } \psi \circ F'$$

is the desired extension.

Remarks

- The proof of the Liebeck-Seitz lifitng result is based upon a much more detailed study of the representation theory of the group X and the set of possible composition factors occurring in its action on $\mathrm{Lie}(\bar{G})$. The integer $t(\bar{G})$, calculated by Lawther, is defined as follows:

Definition

Let Φ be an irreducible root system; we call an element of the root lattice $\mathbb{Z}\Phi$ a *root difference* if it is of the form $\alpha - \beta$ for some $\alpha, \beta \in \Phi$. Given a sublattice L of $\mathbb{Z}\Phi$, we write $t(L)$ for the exponent of the torsion subgroup of the quotient $\mathbb{Z}\Phi/L$; we set

$$t(\Phi) = \max\{t(L) \mid L \text{ a sublattice of } \mathbb{Z}\Phi \text{ generated by root differences}\}.$$

- The other ingredient of the Liebeck-Seitz result, which requires the bound $N'(\bar{X}, \bar{G})$, is a result on $\bar{G}$-complete reducibility.

  One needs to know when a closed connected reductive subgroup of a parabolic subgroup of $\bar{G}$ must necessarily lie in a Levi factor of the parabolic subgroup.

Avenues for investigation

- Look carefully at the embeddings of $(\mathrm{P})\mathrm{SL}_2(q)$ subgroups of exceptional groups, according to the class of the unipotent elements.

  In particular, completely settle the lifting problem for $(\mathrm{P})\mathrm{SL}_2(q)$ subgroups containing a *regular* unipotent (i.e. centralizer dimension equals the rank of the group).

  We know that if $q > p$, we can always lift the embedding to an embedding of algebraic groups.

- In case $q = p$, the maximal subgroups of the finite groups G_2 are know (Cooperstein, Kleidman, et al), and using Magaard's thesis and Aschbacher's work on the subgroups of E_6, one should be able to deduce the extension result in these cases. (Containing a regular unipotent element is quite restrictive.)

  For the groups E_7 and E_8, the regular unipotent elements have order p if and only if $p > 17, 29$ respectively.
  Now we apply the semi-regular lifting result and see that it remains to consider the primes $17 < p < 67$ for $\bar{G} = E_7$ and the primes $29 < p < 113$ for $\bar{G} = E_8$, when $N_{\bar{G}}(X)$ does not contain a $\mathrm{PGL}_2(p)$.
  This becomes then a very finite problem.

- One could use the results of Stewart improving the Liebeck-Seitz result on $\bar{G}$-complete reducibility. This should lead to an improvement of the bounds in in the Liebeck-Seitz lifting result.
- Establish a result which depends on the finite group G, that is, for particular choices of F, that is for particular choices of field $\mathbb{F}_q$, over which the finite group $G = \bar{G}^F$ is defined, use the representation theory of X to restrict to a list of possible actions of X on $\mathrm{Lie}(\bar{G})$.

  For specific groups X, enough will be known about the extensions of simple modules and the actions of unipotent elements on these modules, to rule out the configuration.

Overgroups of unipotent elements

There has been much work on studying properties of subgroups which contain representatives of certain classes, for example, subgroups containing long root elements, or subgroups which are centralizers of semisimple elements, either in G or in $\mathrm{Aut}(G)$.

Here we consider subgroups of simple algebraic groups defined by the property of containing representatives of certain classes of unipotent elements (other than root elements).

A_1-type subgroups

We start with a result about overgroups of elements of order p.

Theorem (Testerman, 1995, Proud-Saxl-Testerman, 2000)

Let G be a simple algebraic group defined over an algebraically closed field k of characteristic $p \geq 0$. Let $u \in G$ be unipotent. If $\mathrm{char}(k) > 0$ assume u has order p. Then with one exception, u lies in an A_1-type subgroup of G, that is, there exists a closed connected subgroup $X \subset G$, $X \cong \mathrm{(P)SL}_2(k)$ with $u \in X$. The exception is for the group $G = G_2$, when $\mathrm{char}(k) = 3$ and u lies in the unipotent class $A_1^{(3)}$; here u lies in no closed connected A_1-type subgroup of G.

Analogue for finite groups

Theorem

Let G be as above and assume $\mathrm{char}(k) = p > 0$. Let σ be an endomorphism of G, with finite fixed-point subgroup G^σ. Suppose that $u \in G^\sigma$ has order p. Then u lies in a closed connected σ-invariant A_1-type subgroup of G, except in the following cases:

(i) *$G = G_2$, $p = 3$ and u lies in the $A_1^{(3)}$ class.*

(ii) *$G = G_2$, $p = 3$ and σ is a morphism involving the graph automorphism of G.*

(iii) *$G = B_2$ or $G = F_4$, $p = 2$ and σ is a morphism involving the graph automorphism of G.*

Indeed one has:

Proposition

There are no σ-invariant A_1-type subgroups of G if σ involves a nontrivial graph automorphism of G.

One can still ask whether the finite groups contain finite A_1-subgroups containing u.

1. if $G^\sigma = {}^2B_2(2^{2e+1})$, then G^σ has order $r^2(r^2+1)(r-1)$, for $r = 2^{2e+1}$. Since $3 \not| \, |G^\sigma|$, it follows that G^σ does not contain a subgroup isomorphic to $\mathrm{SL}_2(2)$.

2. In case $G^\sigma = {}^2G_2(3^{2e+1})$, since G^σ has Sylow 2-subgroups which are elementary abelian of order 8, G^σ contains no subgroup isomorphic to $\mathrm{SL}_2(3^i)$, nor $\mathrm{PGL}_2(3^i)$.

On the other hand, we have

Proposition

Let G be a simple algebraic group of type G_2 defined over an algebraically closed field of characteristic 3. Let σ be an endomorphism of G, involving a graph automorphism and such that σ^2 is a q-power Frobenius endomorphism of G. If $u \in G^\sigma$ lies in the class of subregular elements, i.e. in the class $G_2(a_1)$, then u lies in a subgroup of G^σ isomorphic to $\mathrm{PSL}_2(3^{2e+1})$. *Moreover, such a subgroup contains representatives of the two G^σ-classes in $G_2(a_1) \cap G^\sigma$.*

(Note that $u \in G_2(a_1)$ is not semiregular, so the semiregular result does not apply.)

The $A_1^{(3)}$ class in G_2

Proposition

Let G be a simple algebraic group of type G_2, defined over an algebraically closed field of characteristic 3. Let $u \in G$ lie in the $A_1^{(3)}$ class. Then u does not lie in any subgroup of G isomorphic to $\mathrm{PSL}_2(3)$.

Corollary

Let $\tau = q$ or $\tau = gq$ with g a nontrivial graph automorphism of $G = G_2$. So $G^\tau = G_2(q)$, or $G^\tau = {}^2G_2(3q^2)$.

(i) *If $u \in A_1^{(3)}$, then u does not lie in any closed connected A_1-type subgroup of G.*

(ii) *If $u \in A_1^{(3)} \cap G^\tau$, then u does not lie in any finite A_1-type subgroup of G^τ.*

$G = F_4$, $p = 2$, and σ involves a graph automorphism of G

There are two classes of involutions in the finite group ${}^2F_4(2^{2e+1})$, u_1 and u_2 distinguished by the fact that u_1 lies in the center of a Sylow subgroup and u_2 does not. Then we have

Proposition

With the above notation, u_2 lies in a subgroup of G^σ isomorphic to $\mathrm{SL}_2(2^{2e+1})$, while u_1 does not lie in any subgroup of G^σ isomorphic to $\mathrm{SL}_2(2)$.

Questions

- What group should replace an A_1 when u has order greater than p?
- What properties, other than existence, do the A_1 subgroups have ?

Over fields of positive characteristic, there often exist non-conjugate A_1 subgroups containing a fixed element of order p. If we restrict our attention to A_1-subgroups satisfying some particularly nice properties, we do get a conjugacy result.

Good A_1's

Definition

Let G be a simple algebraic group defined over an algebraically closed field of positive characteristic p. Let A be a closed connected A_1-type subgroup of G, wih maximal torus T_A. We will say that A is a *good* A_1 if all weights of T_A on $\mathrm{Lie}(G)$ are at most $2p - 2$.

Theorem (Seitz, 2000)

Let G, p be as above and assume p is a good prime for G. Let $u \in G$ be a unipotent element of order p.

i. *There exists a good A_1 containing u.*
ii. *Any two good A_1's containing u are conjugate by an element of $R_u(C_G(u))$.*
iii. *Let A be a good A_1 containing u, and let $U \subset A$ be a 1-dimensional subgroup containing u. The $C_G(u) = C_G(U) = C_G(\mathrm{Lie}(U))$. Moreover, $C_G(u) = C_G(A)R_u(C_G(u))$ and $C_G(A)$ is reductive.*

Consequences

Proposition

Let G, p and u be as above. There is a unique 1-dimensional unipotent group U containing u such that U is contained in a good A_1 subgroup of G.

In general, if one does not restrict one's attention to good A_1-subgroups, still under certain restrictions on p, one can classify up to conjugacy the A_1-subgroups of the exceptional algebraic groups.

Theorem (Lawther-Testerman, 1999)

Let G be an exceptional algebraic group defined over an algebraically closed field of characteristic $p > 3, 3, 5, 7, 7$, if G is of type G_2, F_4, E_6, E_7, E_8 respectively. All conjugacy classes of A_1-subgroups of G, together with their connected centralizers and their composition factors on $\mathrm{Lie}(G)$ are classified.

Progess on removing the prime restrictions

- David Stewart has classified all reductive subgroups of G_2, in particular the A_1-type subgroups.

- Bonnie Amende in her Oregon PhD thesis (2005) classifies up to conjugacy all G-irreducible A_1-subgroups of $G = F_4$, that is, those subgroups which do not lie in a proper parabolic subgroup of G. (Indeed she considered the groups E_6 and E_7 as well, and determined all possible such G-irreducible A_1-subgroups).

- If X lies in a Levi factor of a parabolic subgroup of G, one can proceed by induction to determine the embedding of X in G (given that we are in a bounded rank setting and the classical groups occurring as Levi factors have small natural representations). This is currently under investigation by Litterick, a PhD student of Martin Liebeck.

- The non G-completely reducible A_1-subgroups of F_4, i.e. those which lie in a proper parabolic subgroup without lying in a Levi factor of the parabolic are classified in the recent Memoir of David Stewart as well.

Remaining questions

Complete Amende's work on G-irreducible A_1-subgroups and carry out Stewart's analysis in exceptional groups of type E_n.

Regular unipotent elements

An element in a semisimple algebraic group is said to be *regular* if its centralizer is of minimal dimension, which is necessarily the rank of the group.

The set of regular unipotent elements in G is a single G-conjugacy class which forms a dense subset of the variety of unipotent elements in G.

In the group $G = \mathrm{SL}_n$, the regular unipotent elements are those having a single Jordan block (in the natural representation), as is also the case for the symplectic groups and the odd-dimensional orthogonal groups. For the even-dimensional orthogonal groups, the regular unipotent elements have two blocks: of sizes $2n-1$ and 1 if p is odd, and of sizes $2n-2$ and 2 if $p = 2$.

If p is large enough, (greater than the height of the highest root of the root system of the group), then u has order p and by the above results we know precisely when u lies in an A_1-type subgroup. One can ask what other reductive (possibly disconnected) subgroups contain a regular unipotent element.

Maximal overgroups of regular unipotent elements

Theorem (Saxl-Seitz, 1997)

Let X be a maximal closed positive-dimensional subgroup of a simple algebraic group G of exceptional type. Assume that X° is reductive. Then X contains a regular unipotent element of G if and only if $X \subset G$ is one of the following:

a) $A_1 \subset G$, *with* $p = 0$ *or* $p \geq h$, *where* h *is the Coxeter number of* G;

b) $F_4 \subset E_6$;

c) $A_2.2 \subset G_2$, *and* $p = 2$;

d) $D_4.S_3 \subset F_4$ *and* $p = 3$;

e) $(D_4 T_2).S_3 \subset E_6$, *and* $p = 3$;

f) $(E_6 T_1).2 \subset E_7$ *and* $p = 2$;

g) $A_1^7.L_3(2) \subset E_7$ *and* $p = 7$.

(They also have a statement for the classical groups.)

Descending through the subgroup lattice

As we have already seen, having a result for maximal subgroups is not sufficient for completing a classification, precisely because of the existence of non G-completely reducible subgroups.

In the particular case of subgroups containing regular unipotent elements, this is relevant since every parabolic subgroup contains a regular unipotent element (in fact a representative of every unipotent class).

So in order to deduce from the above result a statement about *all* reductive subgroups containing regular unipotent elements, we must determine whether there exist non G-completely reducible subgroups of this type.

Connected reductive overgroups of regular unipotent elements are G-irreducible

Theorem (Testerman-Zalesski, 2012)

Let H be a connected reductive subgroup of a connected reductive algebraic group G. Suppose that H contains a regular unipotent element of G. Then H lies in no proper parabolic subgroup of G, that is, H is G-irreducible.

In particular, these groups are G-completely reducible.

From this we deduce the following:

Theorem

Let H be a closed semisimple subgroup of the simple algebraic group G, containing a regular unipotent element of G. Then either the pair G, H is as given in the following table, or H is a $\mathrm{(P)SL}_2$-subgroup and $p = 0$ or $p \geq h$, where h is the Coxeter number for G. Moreover, for each pair of root systems (Φ_G, Φ_H) as in the table, respectively, for (Φ_G, A_1, p), with $p = 0$ or $p \geq h$, there exists a closed simple subgroup $X \subset G$ of type Φ_H, respectively A_1, containing a regular unipotent element of G.

Table: Semisimple subgroups $H \subset G$ containing a regular unipotent element

G	H
A_6	$G_2, p \neq 2$
A_5	$G_2, p = 2$
C_3	$G_2, p = 2$
B_3	$G_2, p \neq 2$
D_4	$G_2, p \neq 2$
	B_3
E_6	F_4
$A_{n-1}, n > 1$	$C_{n/2}, n$ even
	$B_{(n-1)/2}, n$ odd, $p \neq 2$
$D_n, n > 4$	B_{n-1}

The proof of the second theorem is straightforward, using the classification of maximal closed connected subgroups of the exceptional groups, the result of Saxl-Seitz, the theorem on G-irreducibility of reductive subgroups containing regular elements, and representation theory.

Sketch of proof of G-irreducibility of reductive overgroups of regular unipotent elements.

- There is a first reduction to G simple (easy) and then to H simple (This is slightly less obvious, but in fact, no non simple semisimple subgroup of G can contain a regular unipotent element.)

- Assume that G and H are both simple and that H contains u, a regular unipotent element of G.

  One shows by a density argument that u is a regular unipotent element of H.

Suppose H lies in a proper parabolic subgroup of G;

choose a parabolic subgroup P of G minimal with respect to containing H.

Note that the projection of u in a Levi factor of P must be a regular unipotent element of L.

Moreover, H does not lie in a conjugate of L. That is H is non G-completely reducible.

Case I: G classical.

Here H stabilizes a totally singular subspace in its action on the natural module for G.

We have the Jordan block structure of regular unipotent elements on this representation space.

Two ingredients:

- The projection of H in L lies in no proper parabolic of L.
- Work of Suprunenko which determines the irreducible representations $\rho : X \to \mathrm{GL}(V)$ of a simple algebraic group X whose image contains a unipotent element of $\mathrm{GL}(V)$ with precisely one Jordan block.

One must consider the various configurations for the action of H on the natural module of G, together with knowledge of the possible nontrivial extensions among the irreducible modules identified by Suprunenko.

Case II: G exceptional.

Here we rely upon the Seitz-Liebeck theorem which identifies the possible types of simple non G-completely reducible subgroups of G.

One could probably shorten this part of the proof with the recent work of Stewart.

Comparing the order of the regular unipotent elements in the subgroup H and the order of the regular unipotent elements in G, we reduce down to one potential configuration,

that is H of type G_2 in $G = E_7$ when $p = 5$.

Now according to a preprint of Stewart, there is no non G-completely reducible G_2 in E_7 when $p = 5$.

We gave an argument that no such subgroup could contain a regular unipotent element of E_7.

Double centralizers of unipotent elements

As we have seen above, if $u \in G$ has order p, in all but one case u lies in an A_1-type subgroup of G.

In particular, u lies in a closed connected 1-dimensional subgroup of G. Even in the one exceptional case, u still lies in a 1-dimensional closed connected subgroup of G. Moreover, if p is a good prime for G, then there exists a 1-dimensional subgroup U containing u, which has particularly nice properties, for example:

$$C_G(u) = C_G(U) = C_G(\mathrm{Lie}(U))$$

Question

What replaces U, either in bad characteristic, or when u no longer has order p.

The conditions $u \in U$ and $C_G(U) = C_G(u)$ mean that the subgroup U lies $Z(C_G(u)) = C_G(C_G(u))$.

Thinking about the structure of abelian algebraic groups, and in particular abelian connected unipotent groups, one sees that what one should aim for a t-dimensional group if $o(u) = p^t$.

A first result

Theorem (Proud, 2001)

Let G be a simple algebraic group over k and assume $\mathrm{char}(k) = p$ is a good prime for G. Let $u \in G$ be unipotent of order p^t, $t > 1$. Then there exists a closed connected abelian t-dimensional unipotent subgroup $W \leq G$ with $u \in W$

Further questions

Proud's existence result does not point to any particularly canonical properties of the overgroup.

Candidate: $Z(C_G(u)) = C_G(C_G(u))$ is a canonically defined abelian overgroup of u.

But is it unipotent?

And what about the connected component $C_G(C_G(u))^\circ$, which is also a canonically defined subgroup of $C_G(u)$, does it even contain u?

Proud went on to study more general properties of the group $Z(C_G(u))$, work which was later continued by Seitz. They showed (independently)

Proposition

$Z(C_G(u)) = Z(G) \times Z(C_G(u))_u$. Moreover, if p is good for G, then $Z(C_G(u))_u = Z(C_G(u))^\circ$. In particular, if p is good, the group $Z(C_G(u))^\circ$ is a canonically defined connected abelian unipotent overgroup of u.

Remark

If p is a bad (i.e. not good) prime for G, there exist unipotent elements $u \in G$ such that $u \notin C_G(u)^\circ$ and so one cannot hope to find a connected abelian overgroup of u.

Describing $Z(C_G(u))$, good characteristic, joint work with Lawther

We consider as well the case where the field is of characteristic 0, (and obtain new results even in this setting).

Here we have a certain number of very powerful tools available:

- Springer maps: Given G and u, we fix a Springer map, a G-equivariant homeomorphism $\varphi : \mathcal{U} \to \mathcal{N}$, between the variety of unipotent elements in G and the variety of nilpotent elements in $\mathrm{Lie}G$. Such a bijection exists as long as p is good for G (and in fact is an isomorphism of varieties as long as p is very good for G). So we have $C_G(u) = C_G(\varphi(u))$. We will henceforth study the centralizers of nilpotent elements in $\mathrm{Lie}G$.

- Smoothness of centralizers: We also use the result of Slodowy: if $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p$ is a very good prime for G, then $\mathrm{Lie}(C_G(u)) = C_{\mathrm{Lie}G}(u)$.

- For the classical groups, we use a result of Yakimova which gives a basis for $Z(C_{\mathrm{Lie}G}(e))$, for $e \in \mathrm{Lie}G$, nilpotent.

- Bala-Carter-Pommerening classification of unipotent classes/nilpotent orbits. This classification requires the following notion.

Definition

Let H be a connected reductive algebraic group. We say that a nilpotent element $e \in \mathrm{Lie}(H)$ is a *distinguished* nilpotent element in $\mathrm{Lie}(H)$ if $C_H(e)^\circ$ contains no noncentral semisimple elements or, equivalently, each torus of $C_H(e)$ lies in $Z(H)$.

Note: Taking S to be a maximal torus of $C_G(e)$, we have that e is distinguished in the reductive subgroup $C_G(S)$ (a Levi subgroup of G).

- Now let $e \in \mathrm{Lie} G$ be nilpotent. There exists an associated cocharacter for e (Pommerening, Premet), that is,

Definition

A morphism $\tau : k^* \to G$ is said to be an *associated cocharacter for* e if

i. $\tau(c)e = c^2 e$ for all $c \in k^*$, and
ii. $\mathrm{im}(\tau) \subset L'$, the derived subgroup of a Levi factor of G, such that e is a distinguished nilpotent element in $\mathrm{Lie}(L)$.

(Any two cocharacters associated to e are conjugate by an element of $C_G(e)^\circ$.)

We also define a weighted Dynkin diagram associated to the element e (or the G-orbit of e):

Definition

Let $e \in \mathrm{Lie} G$ be nilpotent and τ an associated cocharacter. Embedding $\mathrm{im}(\tau)$ in a maximal torus T of G, there exists a base Δ of the root system $\Phi(G)$ (with respect to T), such that τ has weights $0, 1$ or 2 on the elements of the base; that is, for all $\alpha \in \Delta$, there exists $i_\alpha \in \{0, 1, 2\}$, such that $\alpha(\tau(c)) = c^{i_\alpha}$.

We associate to the G-orbit of e a so-called *weighted Dynkin diagram*, where the node corresponding to α is labelled with the integer i_α. (This is analogous to the usual Kostant-Dynkin theory in characteristic 0.)

Finally, we will need the following notion.

Definition

1. We write $n_2(e)$ for the number of weights equal to 2 on the weighted Dynkin diagram of e.
2. We say that e is *even* if all weights of τ are even, so the weighted Dynkin diagram has all labels either 0 or 2.

For example, distinguished nilpotent elements are even.

A dimension formula

Theorem (Lawther-Testerman, 2011)

Let $e \in \mathrm{Lie} G$ be an even nilpotent element. Then

$$\dim Z(C_G(e)) = n_2(e) = \dim Z(C_G(\mathrm{im}(\tau))).$$

(We establish a more technical dimension formula for non even elements as well, which I'll not state here.)

Examples

- If e is a regular nilpotent element, and so the corresponding class of unipotent elements consists of regular elements, then the weighted Dynkin diagram consists of all weights equal to 2.

  Also $C_G(e)$ is abelian, so indeed $\dim Z(C_G(e)) = \mathrm{rank}(G) = n_2(e)$ and the centralizer of the torus $\mathrm{im}(\tau)$ is a maximal torus of G and so $\dim Z(C_G(\mathrm{im}(\tau)))$ is also equal to $\mathrm{rank}(G)$.

- If $e = 0$, then the labelled diagram for e has only weights 0 and so $n_2(e) = 0$, while $C_G(e) = G$. So indeed $\dim Z(C_G(e)) = 0$ and $\tau : k^* \to G$ is the trivial cocharacter and so $C_G(\mathrm{im}(\tau))$ is also G.

More examples

- Assume $\mathrm{rank}(G) > 2$ and let $e \in \mathrm{Lie}G$ be a regular element in an A_2 Levi factor of $\mathrm{Lie}G$, generated by root vectors corresponding to long roots in $\Phi(G)$. Then e has weighted Dynkin diagram as follows:

A_ℓ $\quad 2\ 0 \cdots\ 0\ 2$, so $\dim Z(C_G(e)) = 2$

$B_\ell, C_\ell, D_\ell \quad (\ell \geq 4) \quad 0\ 2\ 0\ \cdots\ 0$, so $\dim Z(C_G(e)) = 1$

F_4 $\quad 2\ 0\ 0\ 0$, so $\dim Z(C_G(e)) = 1$

E_6 $\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ & & 2 & & \end{matrix}$ so $\dim Z(C_G(e)) = 1$

E_7 $\begin{matrix} 2 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & \end{matrix}$ so $\dim Z(C_G(e)) = 1$

E_8 $\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ & & 0 & & & & \end{matrix}$ so $\dim Z(C_G(e)) = 1$

Note that in the above examples, when $\dim Z(C_G(e)) = 1$, we have a 1-dimensional connected abelian unipotent overgroup U of u, satisfying the properties $C_G(u) = C_G(U)$, as with the good A_1's. Indeed, this must be the same group ($p > 2$, u has order p).

A *mysterious* connection with the degrees of the Weyl group invariants

Note that $\mathrm{im}(\tau)$ normalizes $C_G(e)$ and so acts on the subspace $\mathrm{Lie}(Z(C_G(e)))$, with a certain set of (integral) weights.

Theorem

Let $e \in \mathrm{Lie}G$ be a distinguished nilpotent element, with associated cocharacter τ. Let $d_1, \cdots, d_\ell$ be the degrees of the invariant polynomials of the Weyl group of G, ordered such that d_ℓ is ℓ if G is of type D_ℓ, and otherwise d_ℓ is $\max\{d_i\}$, and $d_i < d_j$ if $i < j < \ell$. Then the weights of $\mathrm{im}(\tau)$ on $\mathrm{Lie}(Z(C_G(e)))$ are the $n_2(\Delta)$ integers $2d_i - 2$ for $i \in S_\Delta$, where

$$S_\Delta = \begin{cases} \{1, \dots, n_2(\Delta) - 1, \ell\} & \text{if } G \text{ is of type } D_\ell \text{ and } \Delta = \cdots {}^{2}_{2}; \\ \{1, \dots, n_2(\Delta) - 1, n_2(\Delta)\} & \text{otherwise.} \end{cases}$$

Examples

- Let e be the regular nilpotent element in $\mathrm{Lie}G$ of type E_6, and so $\dim(Z(C_G(e)) = 6$. Moreover, the degrees of the Weyl group invariants are 2, 5, 6, 8, 9, 12.

  The $\mathrm{im}(\tau)$ weights on $\mathrm{Lie}(Z(C_G(e)))$ are 2, 8, 10, 14, 16, 22.

- Let e_1 be the subregular nilpotent in $\mathrm{Lie}G$ of type E_6, whose weighted Dynkin diagram is $\begin{smallmatrix} 2 & 2 & 0 & 2 & 2 \\ & & 2 & & \end{smallmatrix}$.

  The $\mathrm{im}(\tau)$ weights on $\mathrm{Lie}(Z(C_G(e))$ are 2, 8, 10, 14, 16.

- Let e be a regular nilpotent element in $\mathrm{Lie}G$ of type D_5, where the degrees of the Weyl group invariants are 2, 4, 5, 6, 8, (ordered 2, 4, 6, 8, 5). Then the weights of $\mathrm{im}(\tau)$ on $C_{\mathrm{Lie}G}(e)$ are 2, 6, 10, 14, 8.

- Let e be subregular in $\mathrm{Lie}G$ of type D_5, so the weighted Dynkin diagram of e is 220^2_2.

  The $\mathrm{im}(\tau)$ weights on $\mathrm{Lie}(Z(C_G(e))$ are 2, 6, 10, 8.

Additional information

Under our standing hypotheses on $\mathrm{char}(k)$, $C_G(e) = CR$, a semi-direct product of $R = R_u(C_G(e))$ and a reductive (not necessarily connected) group C.

In fact, $C = C_G(e) \cap C_G(\mathrm{im}(\tau))$. In the case where G is exceptional, we consider the action of C° on $\mathrm{Lie}(R)$, and give for each nilpotent orbit a decomposition of $\mathrm{Lie}(R)$ as a direct sum of indecomposable tilting modules for C°.

About the proof

Let $e \in \mathrm{Lie}G$ be a nilpotent element with associated cocharacter τ.

For an $\mathrm{im}(\tau)$-invariant subspace M of $\mathrm{Lie}G$, we will write M_+ for the sum of the $\mathrm{im}(\tau)$ weight spaces M_j, corresponding to strictly positive weights $j > 0$.

Recall $C_G(e) = CR$, R the unipotent radical and C a reductive complement.

Proposition (Main Tool)

Let e, τ, and C be as above. Assume $\mathrm{char}(k) = 0$ or p a very good prime for G. Then $\mathrm{Lie}(Z(C_G(e))) = (Z(C_{\mathrm{Lie}G}(e)_+))^C$, that is the fixed points of C acting on $Z(C_{\mathrm{Lie}G}(e)_+)$.

So we have 'linearized' the problem;

- find a basis for $Z(C_{\mathrm{Lie}G}(e)_+)$ (if G is classical, this can be deduced from the basis for $Z(C_{\mathrm{Lie}G}(e))$ given by Yakimova),
- determine the fixed point space of the connected reductive group C° acting there, and
- find representatives for the component group C/C° and let them act as well.

In the exceptional groups: lengthy case-by-case considerations are required for most results.

Inductive result

We also have a further inductive result, which is probably related to the notion of induced nilpotent orbits, studied by Lusztig and Spaltenstein.

Definition

Given a weighted Dynkin diagram Δ for the group G, we define the *2-free core* of Δ to be the sub-weighted diagram Δ_0 obtained by removing from Δ all weights equal to 2, together with the corresponding nodes.

Then we let G_0 be a semisimple algebraic group (of any isogeny type) whose root system has the type of the underlying Dynkin diagram of Δ_0.

Theorem

Assume $\mathrm{char}(k)$ *is either 0 or a good prime for* G*. Let* $e \in \mathrm{Lie} G$ *be a nilpotent element with associated weighted Dynkin diagram* Δ*. Let* Δ_0 *and* G_0 *be as above. Then there exists a nilpotent* G_0*-orbit in* $\mathrm{Lie}(G_0)$ *having weighted Dynkin digram* Δ_0*. Moreover if* e_0 *is a representative of this orbit then we have*

$$\dim C_G(e) - \dim C_{G_0}(e_0) = n_2(e) = \dim Z(C_G(e)) - \dim Z(C_{G_0}(e_0)).$$

Example

Let e be a regular nilpotent element in an A_3 Levi factor of E_6. Then the weighted Dynkin diagram for the G-orbit of e is

$$\Delta \qquad \begin{matrix} 1 & 0 & 0 & 0 & 1 \\ & & 2 & & \end{matrix}$$

Now the 2-free core is 1 0 0 0 1, and the corresponding A_5-orbit of nilpotent elements is represented by e_0, whose Jordan normal form has blocks of sizes 2,1,1,1,1 (a root element in A_5).

Then $\dim(Z(C_{G_0}(e_0))) = 1$. So according to the above result, $\dim(Z(C_G(e))) = 2$.

Double centralizers in exceptional groups in bad characteristics

The difficulties:

- No Springer isomorphism; the number of nilpotent classes and unipotent classes is not always the same.

- Centralizers are not smooth, that is, for $u \in G$ unipotent, we do not always have $\mathrm{Lie}(C_G(u)) = C_{\mathrm{Lie}G}(u)$. So now studying $C_G(u)$ and $Z(C_G(u))$ cannot be 'linearized' in the same way as above.

- For u unipotent, we do not necessarily have u in $C_G(u)^\circ$ and so $Z(C_G(u))^\circ$ will not work as a canonically defined connected abelian overgroup of u.

  Springer showed (1966) that for $u \in G$ regular and $\mathrm{char}(k)$ a bad prime for G, then $u \notin C_G(u)^\circ$. Liebeck-Seitz determine all classes u for which $u \notin C_G(u)^\circ$. (AMS monograph, 2012)

Questions

- Does the dimension formula given by Lawther-Testerman hold in bad characteristic?
- Does u lie in $Z(C_G(u))^\circ$, at least when $u \in C_G(u)^\circ$, as in good characteristic?
- What is a characteristic independent description of $Z(C_G(u))^\circ$ which allows us to compute this subgroup?

Answers to the above questions are given in the 2013 PhD thesis of Iulian Simion.

His first result gives a description of $Z(C_G(u))^\circ$, which provides an algorithm for calculating this group.

Choose $T \subset G$ a maximal torus and $B \subset G$ a Borel subgroup containing T. The unipotent radical of B will be denoted by U. The root system is chosen with respect to T and the positive roots are with respect to B.

Theorem

Let $u \in G$ be a unipotent element and suppose that B contains a Borel subgroup of $C_G(u)$. Then

$$Z(C_G(u))^\circ = C_{Z(C_U(u)^\circ)^\circ}(T_u, \tilde{A})^\circ$$

where T_u is a maximal torus of $C_B(u)$ and $\tilde{A}$ is a set of coset representatives for $C_G(u)^\circ$ in $C_G(u)$.

In order to apply this result, he first needs to find a Borel subgroup which contains a Borel subgroup of the centralizer.

Once he has this, he can work out (after long computations) $Z(C_U(u)^\circ)^\circ$. Here there is a partial 'linearization' of the problem, possible because he is working with a connected unipotent group.

His main result is

Theorem

Let $u \in G$ be a unipotent element. Then $\dim Z(C_G(u))$ is explicitly determined. We indicate as well when $u \in Z(C_G(u))^\circ$ and when $u \in Z(C_G(u)^\circ)^\circ$.

In particular, he determines precisely when u does not lie in $Z(C_G(u))^\circ$, which can be the case even when u does lie in $C_G(u)^\circ$.

Result for $G = E_6$

For a fixed unipotent class representative $\tilde{u}$, we denote by C its centralizer $C_G(\tilde{u})$ and by $U_{\tilde{u}}$ the unipotent radical of a Borel subgroup of C. In the second column we give the dimensions of $U_{\tilde{u}}$.

In the fourth and fifth columns we give the dimension of the center of the connected component C° and that of the center of C respectively. Note that this column includes the good characteristic result for comparison.

In the sixth column we mark with $*$ those cases where $\tilde{u}$ is not in $Z(C_G(\tilde{u})^\circ)^\circ$ (in particular $\tilde{u} \notin Z(C_G(\tilde{u}))^\circ$),

with $**$ those cases where $\tilde{u} \in Z(C_G(u)^\circ)^\circ \setminus Z(C_G(u))^\circ$,

and with $\star$ those cases where $u \notin C_G(u)^\circ$.

Class	dim $U_{\tilde{u}}$	A	dim $Z(C^\circ)$	dim $Z(C)$	$u \notin Z(C)^\circ$ 2	3
E_6	6	$Z_{(6,p)}$	6	6	$\star$	$\star$
$E_6(a_1)$	8	1	5	5	$*$	$*$
D_5	9	$Z_{(2,p)}$	4	4	$\star$	$*$
$E_6(a_3)$	12	Z_2	4	3	$*$	$*$
$D_5(a_1)$	13	1	3	3	$*$	$*$
A_5	12	1	3	3	$*$	$*$
A_4A_1	15	1	2	2	$*$	$*$
D_4	13	$Z_{(2,p)}$	2	2	$\star$	$*$
A_4	15	1	3	3	$*$	
$D_4(a_1)$	18	S_3	3	1	$*$	$**$
A_3A_1	19	1	2	2	$*$	
$A_2^2A_1$	22	1	1	1	$*$	

Table: Center of centralizer E_6

Class	dim $U_{\tilde{u}}$	A	dim $Z(C^\circ)$	dim $Z(C)$	$u \notin Z(C)^\circ$ 2	3
A_3	19	1	2	2	$*$	
$A_2A_1^2$	25	1	1	1	$*$	
A_2^2	22	1	2	2		
A_2A_1	26	1	2	2		
A_2	26	Z_2	2	1	$**$	
A_1^3	31	1	1	1		
A_1^2	33	1	1	1		
A_1	36	1	1	1		

Table: Center of centralizer E_6

Remarks

Note that we have

$$Z(C_G(u))^\circ \subset Z(C_G(u)^\circ)^\circ \subset Z(C_G(u)^\circ).$$

Clearly, when $u \notin C_G(u)^\circ$, we have $u \notin Z(C_G(u))^\circ$.

But in fact, there exist u with $u \in C_G(u)^\circ$, but $u \notin Z(C_G(u))^\circ$. There are examples which show that each of the above inclusions is proper.

Questions

- Find a proof in characteristic 0 that for even elements $\dim Z(C_G(e)) \leq n_2(e)$.
- Study the relatonship between the Lusztig-Spaltenstein induced unipotent classes and our inductive formula for the double centralizer dimension.
- Find a case-free, characteristic independent proof of the inequality $\dim Z(C_G(u)) \leq \mathrm{rank}(G)$.

- When is $C_G(u)$ abelian? This question was posed in the Springer-Steinberg article
  Conjugacy classes, in Seminar on Algebraic Groups and Related Finite Groups, (1970).

It was known (Kostant, characteristic 0, and Springer, characteristic p), that for a regular unipotent element u, $C_G(u)^\circ$ is abelian. Then Lou showed (1968) that for regular elements the full centralizer $C_G(u)$ is abelian. Kurtzke (1983) showed that in good characteristic u is regular if and only if $C_G(u)^\circ$ is abelian.

Lawther extended these results to cover bad characteristics; he showed:

Theorem (Lawther, 2011)

For $u \in G$ unipotent, u is regular if and only if $C_G(u)$ is abelian.

The result of Kurtzke does not generalize however. Lawther shows:

Theorem

For $u \in G$ unipotent, with $C_G(u)^\circ$ abelian, then either

- *u is regular, or*
- *$u \in G = G_2$, $p = 3$ and u lies in the class of subregular elements.*

Again more or less, case-by-case considerations.