Quantum Galois Groups of Subfactors

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Outline

Introduction

Our aim is to present a notion of quantum symmetry of subfactors. In particular, given a II_1 subfactor $N \subset M$ of finite index, we prove the existence of universal Hopf algebras of suitable type acting on M leaving N fixed. This is a natural quantum analogue of the Galois group. We also compute this universal Hopf algebra for several examples, including a generic depth 2 subfactor.

Basics

II_1 factors

A von Neumann algebra M is called a factor of type II_1 if it has trivial centre and there is a (unique up to a scalar multiple) faithful finite trace $\tau : M \to \mathbb{C}$, i.e., $\tau(xy) = \tau(yx)$ for all $x, y \in M$.

Definition

An inclusion $N \subset M$ of II_1 factors is called irreducible if the relative commutant is trivial, i.e., $N' \cap M = \mathbb{C}1$.

Notation

Let H be a Hilbert space and $M \subset B(H)$ be a nondegenerate embedding. Fix $0 \neq \xi \in H$. We denote by $P_\xi \in M$ and $P'_\xi \in M'$ the projections on the closed linear span of $\{x\xi \mid x \in M'\}$, and of $\{x\xi \mid x \in M\}$, respectively.

The Coupling Constant

Let H be a Hilbert space and $M \subset B(H)$ be a nondegenerate embedding such that M' is again of type II_1. Let τ' be the trace on M'.

Definition

The coupling constant or the dimension of H over M, denoted $\dim_M(H)$, is given by the ratio

$$\dim_M(H) := \frac{\tau(P_\xi)}{\tau'(P'_\xi)},$$

where we use the notation of the previous slide.

Fact

The above ratio does not depend on the choice of ξ.

Index and the Basic Construction

Definition

Given an inclusion $N \subset M$ of II_1 factors, Vaughan Jones defined its index to be

$$[M : N] := \dim_N(L^2(M)),$$

where $L^2(M)$ is the GNS space w.r.t. the unique tracial state on M.

One of the fundamental tools in the theory of subfactors is the Jones projection and the associated Jones tower construction.

Definition

Let e denote the projection in $L^2(M)$ on the closed subspace spanned by N, and is called the Jones projection. The von Neumann algebra M_1 generated by M and e yields a triple $N \subset M \subset M_1$ and is called the basic construction.

Depth of a Subfactor

The basic construction can be iterated to get the Jones tower $N \subset M \subset M_1 \subset M_2 \ldots$. Here $e_k \in M_k$, $k \geq 1$, is the projection onto M_{k-1}, so that M_k is the von Neumann algebra generated by M_{k-1} and e_k. There is a complete invariant given by Sorin Popa for a large class of finite index subfactors called 'amenable'. However, we do not want to go into further details. We just recall the definition of depth, in particular subfactors of depth 2.

Definition

Let $A_k = N' \bigcap M_k$. We call the subfactor to be of finite depth if there is some k such that the central support of e_{k-1} in A_k equals 1. The smallest such k is called the depth of the subfactor.

A Theorem of Pimsner and Popa

We recall the following theorem due to Pimsner and Popa.

Theorem

Let $N \subset M$ be type II_1 von Neumann algebras with finite dimensional centres and let τ_M be a faithful normal trace on M for which N' is finite on $L^2(M, \tau_M)$. Then

- As a right module over N, the algebra M is projective of finite type.
- $M_1 = \{\sum_{j=1}^{n} a_j e_N b_j \mid n \geq 1, a_j, b_j \in M\}$.
- If $\alpha : M \to M$ is a right N-module map, then α extends uniquely to an element of M_1 on $L^2(M, \tau_M)$.
- If $x \in M_1$ then $x(M) \subset M$, where M is viewed as a dense subspace of $L^2(M, \tau_M)$.

Corollary

Let $N \subset M$ be a pair of von Neumann algebras of type II_1 having finite dimensional centres and suppose that N is of finite index in M. Let τ_M be a faithful normal trace on M with e_N and E_N defined via τ_M. Then

$$\mathrm{End}(M_N) \cong M_1 \quad \text{as } \mathbb{C}\text{-algebras.}$$

By the above Corollary, $\mathrm{End}({}_N M_N) \cong N' \cap M_1$. The fact that $N \subset M_1$ is a finite index pair yields

Proposition

Let $N \subset M$ be a pair of finite index II_1 factors. Then $\mathrm{End}({}_N M_N)$ is finite dimensional.

Quantum Symmetry of Finite Spaces

Let us now recall Wang's result (in a dual picture) regarding universal Hopf action on a finite dimensional semisimple algebra.

Theorem

Let H be a Hopf $*$-algebra and $B = \oplus_{k=1}^{m} M_{n_k}(\mathbb{C})$. Suppose

- B is an H-module $*$-algebra;
- H preserves a faithful positive functional ψ on B, i.e., for all $x \in B$, $\psi(h \cdot x) = \varepsilon(h)\psi(x)$.

Then there exists a unique pairing $\langle , \rangle : H \otimes Q_{aut}(B, \psi) \to \mathbb{C}$ such that for all $h \in H$, $h \cdot e_{rs,j} = \sum_{i=1}^{m} \sum_{k,l=1}^{n_i} e_{kl,i} \langle h, a_{rs,ij}^{kl} \rangle$ holds.

Here, $e_{rs,j}$ form a complete set of idempotents of B and $a_{rs,ij}^{kl}$ generates $Q_{aut}(B, \psi)$ as a CQG Hopf algebra. Thus, the dual of $Q_{aut}(B, \psi)$, say $Q_{aut}^{*}(B, \psi)$, is the universal Hopf $*$-algebra inner faithfully acting on B and preserves ψ.

The Setup

Definition

Let $N \subset M$ be a pair of finite factors. Let $\mathrm{C}(N \subset M)$ be the category whose

- objects are Hopf $*$-algebras Q admitting an action on M making it a module $*$-algebra such that $N \subset M^Q$, where M^Q is the invariant subalgebra;
- morphisms between two objects, say Q and Q', are Hopf $*$-algebra morphisms $\phi : Q \to Q'$ such that the following diagram commutes:

$$\begin{array}{ccc} Q \otimes M & \xrightarrow{\phi \otimes \mathrm{id}} & Q' \otimes M \\ & \searrow & \downarrow \\ & & M \end{array}$$

where the unadorned arrows are the respective actions.

(contd.)

Keeping the notations from the previous slide, we define

Definition

We define the quantum Galois group of the inclusion $N \subset M$ denoted $\mathrm{QGal}(N \subset M)$ to be a terminal object of the category $\mathrm{C}(N \subset M)$.

Definition

Let $\mathrm{C}_\tau(N \subset M)$ be the full subcategory of $\mathrm{C}(N \subset M)$ consisting of Hopf $*$-algebras admitting a τ-preserving action on M. A terminal object in this category is denoted as $\mathrm{QGal}_\tau(N \subset M)$.

Existence of the Quantum Galois Group

Let H be a Hopf $*$-algebra such that M is an H-module algebra, with the action being $*$-compatible and $N \subset M^H$.

Here M^H denotes the fixed point or invariant subalgebra.

Since H leaves N invariant, we get a $*$-representation of H in the algebra $\mathrm{End}({}_N M_N)$ which is a finite dimensional semisimple algebra. To apply Wang's result we need the faithful trace to be preserved.

Recall that $\mathrm{End}({}_N M_N)$ is a finite dimensional subalgebra of M_1, namely, $N' \cap M_1$.

Invariance of the Trace

We recall the following.

Lemma

The canonical trace on M_1, say τ_1, has the Markov property:

$$\tau_1(e_N x) = \frac{1}{[M : N]} \tau_M(x) \quad \forall x \in M.$$

Proposition

If τ_M is preserved under the H-action then so is τ_1, i.e., $\tau_1(h \cdot x) = \varepsilon(h)\tau_1(x)$, for all $x \in M_1$.

Proof of the Proposition, Step 1

First observe that

$$\tau_1(\sum_j a_j e_N b_j) = \sum_j \tau_1(e_N b_j a_j) = \sum_j \frac{1}{[M:N]} \tau_M(b_j a_j).$$

The first equality follows from the traciality of τ_1. The second from the Markov property above.

Proof of the Proposition, Step 2

Now, for $x, y \in M$,

$$\begin{aligned}
\tau_1(h \cdot (y e_N x)) &= \tau_1((h_{(1)} \cdot y) e_N (h_{(2)} \cdot x)) \\
&= \tau_1(e_N (h_{(2)} \cdot x)(h_{(1)} \cdot y)) \\
&= \frac{1}{[M : N]} \tau_M((h_{(2)} \cdot x)(h_{(1)} \cdot y)) \\
&= \frac{1}{[M : N]} \tau_M((h_{(1)} \cdot y)(h_{(2)} \cdot x)) \\
&= \frac{1}{[M : N]} \tau_M(h \cdot (yx)) \\
&= \frac{1}{[M : N]} \varepsilon(h) \tau_M(yx) \\
&= \varepsilon(h) \tau_1(y e_N x).
\end{aligned}$$

We have used the fact that H acts trivially on e_N.

The Main Result

Applying Wang's result, we obtain

Theorem

Let H be a Hopf $*$-algebra and $N \subset M$ is a pair of finite index II_1 factors such that

- M is an H-module algebra through a $*$-compatible action;
- $N \subset M^H$, where M^H is the invariant subalgebra;
- H preserves τ_M, where τ_M is the unique normal trace.

Then

- the H-action factors through the dual action of a Hopf $*$-subalgebra of the dual $Q^*_{aut}(\mathrm{End}({}_N M_N), \tau_1)$ of $Q_{aut}(\mathrm{End}({}_N M_N), \tau_1)$;

The Main Result, contd.

Theorem, contd.

- there exists a universal Hopf $*$-algebra, to be denoted by $Q = \mathrm{QGal}_\tau(N \subset M)$, which has a $*$-compatible action on M such that N is in the invariant subalgebra M^Q
- this universal Hopf $*$-algebra consists of those elements $h \in Q^*_{aut}(\mathrm{End}({}_N M_N), \tau_1)$ such that

$$h \cdot (xy) = (h_{(1)} \cdot x)(h_{(2)} \cdot y)$$

for all $x, y \in M$.

Explicit Computations

Let us now make some computations of this universal Hopf algebra. A rich class of the candidates for calculation is the subfactors obtained by smashed or crossed product by Hopf algebras. In fact, such examples are essentially generic for depth 2 inclusions.

More precisely, we are interested in action of a finite dimensional Hopf C^*-algebra H on a type II_1 factor A which is outer in the sense that the centralizer is trivial, i.e.,

$$A' \cap (A \rtimes H) = \mathbb{C}.$$

In this case, $A \subset M = A \rtimes H$ is a finite index type II_1 subfactor, which is of depth 2. In fact, a generic (irreducible) depth 2 subfactor arises in this way.

(contd.)

We are able to show that

- $\mathrm{QGal}(A \subset A \rtimes H) = H^*$, i.e., the dual of H.
- $\mathrm{QGal}(A^H \subset A) = H$, where A^H is the invariant subalgebra w.r.t. the action of H.

We will prove the first one only (later), the other one being similar, by in some sense a dual argument.

Remark

As the H^*-action preserves the canonical trace of $A \rtimes H$, it follows that $\mathrm{QGal}(N \subset M) = \mathrm{QGal}_\tau(N \subset M)$ for an irreducible, depth 2, finite index subfactor.

Connection with Liu's Work

Theorem

Let $N \subset M$ be an irreducible pair of finite factors with $[M : N] < \infty$. Then the action of $\mathrm{QGal}(N \subset M)$ on M is outer. Furthermore, the invariant subalgebra $M^{\mathrm{QGal}(N \subset M)}$ is a factor with $[M : M^{\mathrm{QGal}(N \subset M)}] < \infty$.

Proof.

Denote by P the invariant subalgebra $M^{\mathrm{QGal}(N \subset M)}$. Thus $N \subset P \subset M$ and therefore $P' \cap P \subset P' \cap M \subset N' \cap M = \mathbb{C}1_M$, whence the result follows. □

This helps to connect our universal Hopf algebras to those associated with the maximal/minimal intermediate depth 2 subfactors considered by Liu.

(contd.)

Theorem

Let P be the smallest von Neumann algebra s.t. $N \subseteq P \subseteq M$ and $P \subseteq M$ is depth 2. Then $P = M^Q$, where $Q = \mathrm{QGal}(N \subset M)$.

Proof.

Clearly, $N \subset M^Q \subset M$ realizes M^Q as an intermediate subalgebra giving depth 2 inclusion. For any such intermediate subfactor $N \subset K \subset M$ with $K \subset M$ depth 2 and also irreducible and finite index, so we can write it as $K = M^H$ for a suitable (finite dimensional) Hopf $*$-algebra H acting outerly on M. But then, $N \subset M^H$ means H is an object in the category of Galois actions, hence $H \subseteq Q$, or, $M^Q \subseteq M^H = K$. □

Corollary

Let $N \subset M$ be an irreducible pair of finite factors with $[M : N] < \infty$. Then $\mathrm{QGal}(N \subset M)$ exists and is isomorphic to $\mathrm{QGal}_\tau(N \subset M)$.

Some Remarks

In general, $\mathrm{QGal}_\tau(N \subset M)$ will be smaller than $\mathrm{QGal}(N \subset M)$.

To see this, we consider $N \subset N \otimes M_n(\mathbb{C})$, where $n \geq 2$. The universal Hopf $*$-algebra of "quantum automorphisms" of $M_n(\mathbb{C})$ is much larger than the corresponding trace-preserving quantum automorphism group.

This shows that the $\mathrm{QGal}(N \subset N \otimes M_n(\mathbb{C}))$ will be strictly bigger than the trace-preserving quantum Galois group $\mathrm{QGal}_\tau(N \subset N \otimes M_n(\mathbb{C}))$.

Details of $\mathrm{QGal}(A \subset A \rtimes H) = H^*$

Let H be a finite dimensional Hopf C^*-algebra and A be a II_1 factor which is also an H-module algebra. The following is well-known.

Lemma

Let $V \in \mathrm{Hom}_{\mathbb{C}}(H, A \rtimes H)$ be the map

$$V(h) = 1 \rtimes h.$$

Then V is convolution invertible and "innerifies" the H-action, i.e.,

$$h \cdot x \rtimes 1 = V(h_1)(x \rtimes 1)V^{-1}(h_2),$$

where $h \in H$, $x \in A$, $\Delta h = h_1 \otimes h_2$.

(contd.)

Let Q be a Hopf $*$-algebra such that $A \rtimes H$ is Q-module algebra and $A \subset (A \rtimes H)^Q$, where $(A \rtimes H)^Q$ is the invariant subalgebra.

Such a Hopf algebra exists; for example, let H^* be a Hopf algebra dual to H. By this we mean, H^* is a Hopf algebra and there is a nondegenerate pairing

$$\langle , \rangle : H^* \otimes H \to \mathbb{C}$$

satisfying the usual compatibility conditions. For $u \in H^*$, $x \in A$ and $h \in H$, define

$$u \cdot (x \rtimes h) = x \rtimes (u \rightharpoonup h),$$

where $u \rightharpoonup h = h_1 \langle u, h_2 \rangle$. Then it is clear that the H^*-action is one such example.

(contd.)

What we show below is that this example is the universal example, under certain conditions. Recall that by universality, we mean that there should exist a Hopf algebra morphism $\phi : Q \to H^*$ such that the following diagram commutes:

$$\begin{array}{ccc} Q \otimes (A \rtimes H) & \xrightarrow{\phi \otimes 1} & H^* \otimes (A \rtimes H) \\ & \searrow & \downarrow \\ & & A \rtimes H \end{array}$$

Observe that, a necessary condition for this to happen is that for $q \in Q$, $h \in H$,

$$q \cdot (1 \rtimes h) = \phi(q) \cdot (1 \rtimes h) = 1 \rtimes h_1 \langle \phi(q), h_2 \rangle.$$

That is Q takes H into H in a very special way. We first achieve this.

(contd.)

Keeping the above notations, we have the following proposition.

Proposition

Let $q \in Q$, thought of as a map from $H \to A \rtimes H$, $h \mapsto q \cdot (1 \rtimes h)$. Then for each $h \in H$,

$$V^{-1}q(h) \in A' \cap (A \rtimes H),$$

where $V^{-1}q$ is the convolution product, $A' \cap (A \rtimes H)$ is the centralizer of A in $A \rtimes H$.

Proof of the Proposition

For the proof, let $x \in A$ and $h \in H$. We compute

$$\begin{aligned}
(x \rtimes 1)V^{-1}(h_1)q(h_2) &= V^{-1}(h_1)V(h_2)(x \rtimes 1)V^{-1}(h_3)q(h_4) \\
&= V^{-1}(h_1)(h_2 \cdot x \rtimes 1)q(h_3) \\
&= V^{-1}(h_1)q \cdot ((h_2 \cdot x \rtimes 1)(1 \rtimes h_3)) \\
&= V^{-1}(h_1)q \cdot ((1 \rtimes h_2)(x \rtimes 1)) \\
&= V^{-1}(h_1)q(h_2)(x \rtimes 1).
\end{aligned}$$

Therefore, we are done.

We have the following

Corollary

Let the extension $A \to A \rtimes H$ be irreducible, i.e., $A' \cap (A \rtimes H) = \mathbb{C}$ (outer action of H). Then for each $q \in Q$, there exists unique $\lambda_q \in \mathrm{Hom}_{\mathbb{C}}(H, \mathbb{C})$ such that

$$q \cdot (1 \rtimes h) = 1 \rtimes h_1 \lambda_q(h_2).$$

Therefore, Q actually takes H inside H.

Proof of the Corollary

By the previous Proposition, for each $q \in Q$ and $h \in H$ there exists $\lambda_q(h) \in \mathbb{C}$ such that $V^{-1}q(h) = \lambda_q(h)(1 \rtimes 1)$. Let $\Lambda_q \in \mathrm{Hom}_{\mathbb{C}}(H, A \rtimes H)$ be defined as

$$\Lambda_q(h) = 1 \rtimes \lambda_q(h)1.$$

Then $V^{-1}q = \Lambda_q$ which implies $q = V\Lambda_q$. So for each $h \in H$,

$$q \cdot (1 \rtimes h) = V(h_1)\Lambda_q(h_2) = (1 \rtimes h_1)(1 \rtimes \lambda_q(h_2)1) = 1 \rtimes h_1\lambda_q(h_2),$$

which was to be obtained. Uniqueness follows from applying ε.

Now using this λ_q, we define a dual pairing between Q and H, from which universality follows automatically. Define

$$\langle , \rangle : Q \otimes H \to \mathbb{C}$$

by

$$\langle q, h \rangle = \lambda_q(h) = (1 \rtimes \varepsilon)(q \cdot (1 \rtimes h)).$$

We show that this defines a dual pairing. We break the proof into several steps.

Step 1

$$\langle qq', h\rangle = \langle q \otimes q, \Delta h\rangle = \langle q, h_1\rangle\langle q', h_2\rangle$$

holds. For, by associativity,

$$qq' \cdot (1 \rtimes h) = q \cdot (1 \rtimes h_1\lambda_{q'}(h_2)) = 1 \rtimes h_1\lambda_q(h_2)\lambda_{q'}(h_3).$$

So

$$\langle qq', h\rangle = \varepsilon(h_1)\lambda_q(h_2)\lambda_{q'}(h_3) = \lambda_q(h_1)\lambda_{q'}(h_2) = \langle q, h_1\rangle\langle q', h_2\rangle.$$

Step 2

$$\langle q, hh' \rangle = \langle q_1, h \rangle \langle q_2, h' \rangle$$

holds. For, since $A \rtimes H$ is a Q-module algebra, we have

$$q \cdot (1 \rtimes hh') = q_1 \cdot (1 \rtimes h) q_2 \cdot (1 \rtimes h').$$

Now

$$q \cdot (1 \rtimes hh') = 1 \rtimes h_1 h'_1 \lambda_q(h_2 h'_2)$$

and

$$q_1 \cdot (1 \rtimes h) q_2 \cdot (1 \rtimes h') = h_1 \lambda_{q_1}(h_2) h'_1 \lambda_{q_2}(h'_2) = h_1 h'_1 \lambda_{q_1}(h_2) \lambda_{q_2}(h'_2).$$

Applying ε yields the result.

Step 3

$$\langle 1, h \rangle = \varepsilon(h), \quad \langle q, 1 \rangle = \varepsilon(q)$$

hold which can be seen easily.
The pairing thus defines a bialgebra morphism from $Q \to H^*$. Since a bialgebra morphism is in fact a Hopf algebra morphism,

$$\langle q, S(h) \rangle = \langle S(q), h \rangle$$

holds.

Summarizing all these, we get

Theorem

Let H be a finite dimensional Hopf C^*-algebra acting outerly on a II_1 factor A. Then $\mathrm{QGal}(A \subset A \rtimes H) = H^*$.

In the above computation, we investigated inclusions arising from crossed products by Hopf algebras. As mentioned above, by a result of Szymański, these are the irreducible depth 2 finite index inclusions. For a general depth 2 finite index inclusion, a result of Nikshych-Vainerman says that these arise as crossed products by weak Hopf algebras. Our techniques need to be modified to cover this case.

The next example is dual to the previous one in some sense (we omit the proof).

Invariant Subalgebra

Let H be a finite dimensional Hopf C^*-algebra acting outerly on a II_1 factor A. Let $A \rtimes H$ and A^H be the crossed product and the invariant subalgebra, respectively.

Theorem

Suppose $A^H \subset A \subset A \rtimes H$ is a Jones triple, i.e., $A \subset A \rtimes H$ is the basic construction of $A^H \subset A$. Then $\mathrm{QGal}(A^H \subset A) = H$.

Banica's Fixed Point Algebras

According to Banica, commuting squares having $\mathbb{C}$ in the lower left corner, i.e., of the form

$$\begin{array}{ccc} S & \subset & X \\ \cup & & \cup \\ \mathbb{C} & \subset & P, \end{array}$$

where S and P are finite dimensional von Neumann algebras, are isomorphic to one of the following forms

$$\begin{array}{ccc} S & \subset & (P \otimes (S \rtimes \hat{G}))^G \\ \cup & & \cup \\ \mathbb{C} & \subset & P, \end{array}$$

and the vertical subfactor associated to the first commuting square is of the form $\mathcal{R} \subset (P \otimes (\mathcal{R} \rtimes \hat{G}))^G$.

In the above description, G is a compact quantum group of Kac type, $\mathcal{R}$ is the hyperfinite II_1-factor. The action of $\hat{G}$ on $\mathcal{R}$ is outer and is a product-type action built from the action on S. The action of G on P is ergodic on the center. Both algebras $(P \otimes (\mathcal{R} \rtimes \hat{G}))^G$ and $(P \otimes (S \rtimes \hat{G}))^G$ are fixed point algebras in the sense of Banica. The outerness of the $\hat{G}$-action on $\mathcal{R}$ enables us to compute explictly the quantum Galois group of the inclusion $\mathcal{R} \subset (P \otimes (\mathcal{R} \rtimes \hat{G}))^G$.

Remark

We also note that although $\mathcal{R}$ and G are infinite-dimensional, the finite-dimensionality of P enables us to use the algebraic smash product rather than the von Neumann crossed product in defining $(P \otimes (\mathcal{R} \rtimes \hat{G}))^G$.

Notation

Let H denote the dense Hopf $*$-algebra $\mathcal{O}(G)$ inside $C(G)$ and τ be the Haar state. Then there exists a conditional expectation E of $P \otimes \mathcal{R} \rtimes H^{cop}$ onto $(P \otimes \mathcal{R} \rtimes H^{cop})^{H^{cop}}$ given by

$$E(b \otimes a \rtimes h) = b_0 \otimes a \rtimes h_2 \tau(h_1 S(b_1)).$$

A first description of the quantum Galois group of the Banica subfactor is the following.

Theorem

$\mathrm{QGal}(\mathcal{R} \subset (P \otimes \mathcal{R} \rtimes H^{cop})^{H^{cop}})$ is isomorphic with the universal Hopf $*$-algebra which acts on $E(P \otimes 1 \rtimes H^{cop})$ and maps each of the subspaces $E(P \otimes 1 \rtimes h)$ into itself, for $h \in H$.

We write $\widehat{H}$ for the subspace of H^* consisting of functionals of the form $\tau(\cdot h)$ for some $h \in H$. It is well-known that $\widehat{H}$ is a Hopf algebra in duality with H. We also recall that $\widehat{H^{cop}}$ is $\widehat{H}^{op}$. Therefore we identify $\widehat{H^{cop}}$ with the space consisting of linear functionals of the form $\tau(\cdot S(h))$ for some $h \in H$.

There is also the canonical action of $\widehat{H^{cop}}$ on P given by $\omega \rightharpoonup b = b_0\omega(b_1)$, which takes the form $b_0\tau(b_1 S(h)) = b_0\tau(hS(b_1))$ if ω is given by $\tau(\cdot S(h))$. The above Theorem can then be rewritten as

Theorem

$\mathrm{QGal}(\mathcal{R} \subset (P \otimes \mathcal{R} \rtimes H^{cop})^{H^{cop}})$ is isomorphic with the universal Hopf $*$-algebra which acts on P such that for each $\omega \in \widehat{H^{cop}}$ elements of the form $\omega \rightharpoonup b$ are mapped to elements of the same form.

To be more explicit, we introduce some notation.

Lemma

Let Q be a Hopf $*$-algebra and $S = S^* \subset Q$ be a subset. Let $\mathrm{C}_Q(S) = \{q \in Q \mid qs = sq, \forall s \in S\}$. Then there exists a largest Hopf $*$-subalgebra of Q contained in $\mathrm{C}_Q(S)$.

Definition

We denote by $\mathcal{H}\mathrm{C}_Q(S)$ the largest Hopf $*$-subalgebra of Q contained in $\mathrm{C}_Q(S)$ and call it the Hopf centralizer of S.

For example, if G is a finite group and H is a subgroup, then $\mathcal{H}\mathrm{C}_{\mathbb{C}G}(\mathbb{C}H)$ is $\mathbb{C}C_G(H)$, the group algebra of the centralizer. Let us denote the $\widehat{H^{cop}}$ action on B by $\Lambda : \widehat{H^{cop}} \to \mathrm{End}(P)$ from now on. Then with these notations,

Theorem

$\mathrm{QGal}(\mathcal{R} \subset (P \otimes \mathcal{R} \rtimes H^{cop})^{H^{cop}}) \cong \mathcal{H}\mathrm{C}_{Q_{aut}(P)}(\Lambda(\widehat{H^{cop}}))$.

Thank you!