Sufficient Statistic, Rao-Blackwell Theorem in Quantum Probability

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Overview

Abraham Wald founded the Decision Theory in Classical Statistics:

Wald, Abraham.: " *Statistical Decision Functions* ", John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1950.

For a more recent account one can see the following:

Ferguson, Thomas S.: *"Mathematical Statistics-A Decision Theoretic Approach"*, Probability and Mathematical Statistics, Vol. 1, Academic Press, New York-London, 1967.

The theory (or equivalently von Neumann's theory of games) starts with three basic objects:

(i) Θ, the parameter space for the (joint-) probability distribution or state of the random variables or observables.

(ii) Ω (the set of decisions), a measure space with $D : \mathcal{X} \times (\text{mble subset of } \Omega) \mapsto \mathbb{R}_+$, a family of probability measures, with $\mathcal{X}$ being the sample(experiment) space.

(iii) $L : \Theta \times \Omega \to \mathbb{R}_+$, Loss function. Given $\{\Theta, (\Omega, D), L\}$, the Risk function

$$\mathcal{R}(\theta, D) \equiv \int_\Omega L(\theta, \omega) \int_\mathcal{X} \mu_\theta(dx) D(x, d\omega)$$
$$= \int_\Omega L(\theta, \omega)(\mu_\theta \circ D)(d\omega) \tag{1}$$
$$= \int_\mathcal{X} \mu_\theta(dx) \left(\int_\Omega L(\theta, \omega) D(x, d\omega) \right) \equiv \mathbb{E}_\theta \Big(L(\theta, D(X)) \Big), \tag{2}$$

where X stands for random variable in $\mathcal{X}$, and

$$\mathbb{E}_\theta(\cdot) = \int_\mathcal{X} (\cdot) \mu_\theta(dx).$$

The associated Baysian Risk function

$$\mathcal{R}(\pi, D) \equiv \int_{\Theta} \pi(d\theta)\mathcal{R}(\theta, D), \tag{3}$$

where π the 'prior' probability measure.

Goal of the Dicision theory:

Minimax Property, viz. Does there exist a " Decision rule D_0" such that

$$\inf_{D \in \{D\}} \sup_{\Theta} \mathcal{R}(\theta, D) = \sup_{\Theta} \mathcal{R}(\theta, D_0) \ ?$$

Among the set of random variables in $\mathcal{X}$, that is the set of real-valued measurable functions on $\mathcal{X}$, often one can find (in most cases due to existence of specific symmetries in the probability distributions involved) a small subset (often only one) called "sufficient statistic" T, such that the "conditional probability given $T = t$" is independent of the parameter $\theta \in \Theta$. Equivalently, the joint probability distribution factorizes into

$$\phi(t, \theta) \cdot \rho_t \tag{4}$$

- ρ_t = conditional probability given equals to $T = t$,
- ϕ : (range of T) $\times \Theta \mapsto \mathbb{R}_+$ is the rest.

Example (Binomial):

Two i.i.d. $\{1, 0\}$- valued Random variables X_1, X_2 having binomial p-distribution:

$$Prob\{X_j = x_j\} = p^{x_j}(1-p)^{1-x_j}$$

For $0 < p < 1$, the joint p.d.f. is given by

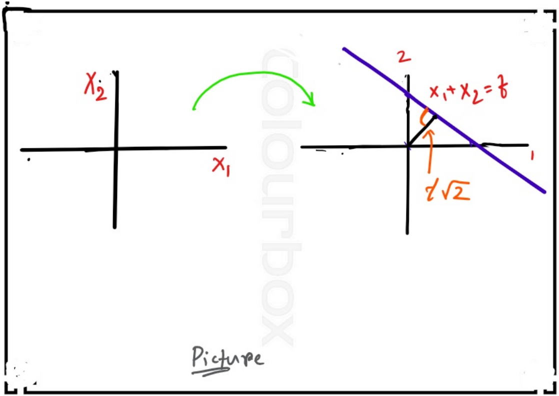
$$Prob\{X_1 = x_1, X_2 = x_2\} = p^{(x_1+x_2)}(1-p)^{1-(x_1+x_2)}$$

Choose $T = X_1 + X_2$, then $Prob\{X_1 = x_1, X_2 = x_2\} = \phi(t, p)\rho_t$, where

$$\phi(t, p) = \binom{2}{t} p^t (1-p)^t \text{ and } \rho_t = \binom{2}{t}^{-1}$$

is the probability conditional to $T = t$.

Here the symmetry is that of the transformation:



This gives two parallel definitions of sufficient statistic T, the first in terms of the conditional probability, given $T=t$, being independent of parameter θ or the joint probability distribution has the " factorization property (4)". It is the second which we adopt in Quantum case.

Theorem 1.1 (Classical Rao-Blackwell).

Let Ω be a convex subset of $\mathbb{R}^n$, $L(\theta, \cdot)$ be a convex function on Ω and let T be a sufficient statistic for μ_θ in (1). Then $\mathcal{R}(\theta, D) \geq \mathcal{R}(\theta, D_T)$, where $D_T(t) = \mathbb{E}\left(D(X) | T = t\right)$. Furthermore, $\mathbb{E}_\theta(D_T) = \mathbb{E}_\theta(D(X))$.

Proof.

Since $L(\theta, \cdot) : \Omega \to \mathbb{R}_+$ is convex, then by Jensen's inequality

$$\begin{aligned}\mathcal{R}(\theta, D) = \mathbb{E}_\theta\left(L(\theta, D(X))\right) =& \mathbb{E}_\theta\left\{\mathbb{E}\left(L(\theta, D(X)) | T = t\right)\right\} \\ \geq& \mathbb{E}_\theta\left\{L(\theta, \mathbb{E}(D(X) | T = t)\right\} \\ =& \mathbb{E}_\theta\left(L(\theta, D_T(\cdot))\right) = \mathcal{R}(\theta, D_T).\end{aligned}$$

$$\mathbb{E}_\theta(D_T) = \mathbb{E}_\theta\left(\mathbb{E}\left(D(X) \Big| T = t\right)\right) = \mathbb{E}_\theta\left(D(X)\right).$$

$\square$

$D(X)$ is an unbiased(random) estimator for $\theta \implies$ the same for non-random D_T.

As mentioned earlier, we adapt the factorization property of a 'state' as in (4) to be the definition/ starting point for the Quantum theory.

Let

- $\underline{A} = (A_1, A_2, \ldots, A_n)$ be a family of bounded commuting self-adjoint operators (simultaneously measurable observables) in $\mathcal{H}$.
- $T = T(\underline{A})$ be a self-adjoint operator function of $\underline{A}$ with the unique spectral transformation (or diagonalizing unitary map):

$$\mathcal{U} : \mathcal{H} \to \int_{sp(T)}^{\oplus} \mathfrak{h}_t \ \nu(dt),$$

1. $(\mathcal{U}f)(t) \equiv f_t \in \mathfrak{h}_t \qquad a.a.(\nu)t \in sp(T)$,
2. $(\mathcal{U}Tf)(t) = tf_t \in \mathfrak{h}_t \qquad a.a.(\nu)t \in sp(T)$,
3. $\langle f, g\rangle_{\mathcal{H}} = \int \nu(dt)\langle f_t, g_t\rangle_{\mathfrak{h}_t}, \qquad$ and
4. For $B \in \{\underline{A}\}'$,

   $(\mathcal{U}Bf)(t) = B(t)f_t, \quad a.a.(\nu)\ t \in sp(T)$ with $B(t) \in \mathcal{B}(\mathfrak{h}_t)$.

Definition 2.1.

Let T be as above in $\mathcal{H}$ and $\{\tau_\theta\}_{\theta\in\Theta}$ be a parameterized family of states on $\{\underline{A}\}'$. Then T is a "quantum sufficient statistic"(QSS) for $\{\tau_\theta\}$ if for every $B \in \{\underline{A}\}'$,

$$\tau_\theta(B) = \int \nu(dt)\phi(t,\theta)\,\mathrm{Tr}_t(B(t)\rho_{_t}), \tag{5}$$

where $\rho_{_t} \in \mathcal{B}_{1+}(\mathfrak{h}_t)$, Tr_t is the $\mathcal{B}(\mathfrak{h}_t)$-trace and $\phi : sp(T) \times \Theta \to \mathbb{R}_+$, a bounded measurable function, satisfying

$$\tau_\theta(I) = \int \nu(dt)\phi(t,\theta)\,\mathrm{Tr}_t(\rho_{_t}) = 1 \tag{6}$$

Remark 2.2.

τ_θ is implemented by a density Matrix $\rho_\theta \overset{(5)}{\Longrightarrow} \rho_\theta$ commutes with T. Moreover, $\rho_\theta \in \mathcal{B}_1(\mathcal{H}) \implies$ the measure ν in (5) have a non-trivial atomic part.

In the Definition 2.1, the operator $\rho_{_t} \in \mathcal{B}_{1+}(\mathfrak{h}_t)$ may be considered as the "conditional density matrix, given $T = t$," (except that $\mathrm{Tr}_t(\rho_{_t})$ non-negative, may not be equal to 1 for a.a.$(\nu)t$).

Note that, if $S = \{t \in sp(T) | \, \mathrm{Tr}_t(\rho_{_t}) = 0\}$, then in the RHS of (5) and (6), the integrals has non-zero contributions only from $S^c \cap sp(T)$. Therefore, WLOG, we can assume that $S \cap sp(T) = \emptyset$.

The earliest and simplest Quantum version of the decision theory (due to A.S. Holevo) the parameter space Θ, Decision space Ω remain classical (i.e. measure spaces), but

- the sampling/experiment space $\mathcal{X} \mapsto$ a separable Hilbert space $\mathfrak{h}_s$
- D: σ- algebra of measurable sets $\mapsto$ positive operator-valued measure (POVM) in $\mathfrak{h}_s$ satisfying

1. Countable additivity:

$$D\left(\bigcup_{1}^{\infty}{}^{\!\!\bullet}\, \Delta_j\right) = \sum_{j=1}^{\infty} D(\Delta_j), \text{ converging in WOT}$$

2. $D(\Omega) = I$.

$\Longrightarrow$ Partially Quantum Risk function

$$\mathcal{R}(\theta, D) = \int_{\Omega} L(\theta, \omega)\ \tau_{\theta}\big(D(d\omega)\big), \tag{7}$$

$\{\tau_\theta |\ \theta \in \Theta\}$ states on $\{\underline{A}\}'$, replaces the probability measure μ_θ in (1).

The associated partially quantum Baysian risk function (with prior probability π):

$$\begin{aligned} \mathcal{R}(\pi, D) &= \int_{\Theta} \pi(d\theta)\mathcal{R}(\theta, D) \\ &= \int_{\Theta\times\Omega} L(\theta, \omega)\ \{\pi(d\theta)\tau_{\theta}\big(D(d\omega)\big)\} \end{aligned} \tag{8}$$

Combining Definition 2.1 and equations (7)-(8), leads to partially quantum Rao-Blackwell theorem.

- Holevo, A. S.: "*Statistical decision theory for quantum systems*", J. Multivariate Anal., **3** (1973), 337–394.
- Petz, D.: "*Quantum information theory and quantum statistics*", Theoretical and Mathematical Physics. Springer-Verlag, Berlin, 2008.
- Sinha, K. B.: "*A Decision Theory in Non-Commutative Domain*", Statistics and Applications, **19** (1), 1–8, 2021.

Theorem 2.3.

Let $\{\Theta, \Omega, L, D\}$ be the objects introduced earlier leading to expressions (7)-(8). Let Ω be a bounded convex subset of $\mathbb{R}^n$, let T be a QSS relative to $\{\tau_\theta\}$ and let $\{D(\cdot)\} \in \{\underline{A}\}'$. If furthermore, $L(\theta, \cdot)$ is convex, then

$$\mathcal{R}(\theta, D) \geq \mathcal{R}(\theta, D_T) \equiv \tau_\theta(L(\theta, D_T)),$$

where D_T is a unique bounded self-adjoint operator function of the QSS T. Moreover,

$$\tau_\theta(D_T) = \tau_\theta\left(\int_\Omega \omega D(d\omega)\right).$$

To prove the above theorem we need following two lemmas:

Lemma 2.4.

Let A be a bounded self-adjoint operator in $\mathcal{H}$ and let σ be a density matrix. Then for $f : \mathbb{R} \to \mathbb{R}$ bounded measurable convex function,

$$\mathrm{Tr}\left(\sigma f(A)\right) \geq f\left(\mathrm{Tr}(\sigma A)\right).$$

The proof of the above lemma follows easily from the spectral theorem and Jensen's inequality.

Lemma 2.5.

Let $\{D(\Delta)\}$ be a POVM in $\mathcal{H}$ over Ω and let the family commute with a self-adjoint operator T. Then in the spectral representation of T given earlier, for a.a.$(\nu)t \in sp(T)$, the family $\{D_t(\Delta) | \Delta$ measuarble subset of $\Omega\}$ is a POVM in the decomposition

$$\mathfrak{h}_s = \int_{sp(T)}^{\oplus} \mathfrak{h}_t \nu(dt).$$

Sketch of proof of Lemma 2.5.

- Since $\langle f, D(\cdot)f\rangle$ is countably based, it suffices to look at the appropriate countable family $\{\Delta_k\}_{k=1}^{\infty}$ and

$$\langle f, D(\Delta_k)f\rangle = \int \nu(dt)\,\langle f_t, D_t(\Delta_k)f_t\rangle\,,$$

where $D_t(\Delta_k)$ is defined for $t \in N(\Delta_k)^c$ with $\nu\big(N(\Delta_k)\big) = 0$.

- Let $\{\Delta_j\}_1^{\infty}$ be a family of disjoint mble subsets of Ω and let δ be a ν- mble subset of $sp(T)$. Then

$$\left\langle f, D\left(\dot{\bigcup_{j=1}^{\infty}}\Delta_j\right)\mathcal{X}_\delta(T)g\right\rangle = \int_\delta \nu(dt)\sum_{j=1}^{\infty}\langle f_t, D_t(\Delta_j)g_t\rangle_t$$

$$= \int_\delta \nu(dt)\left\langle f_t, D_t\left(\dot{\bigcup_{j=1}^{\infty}}\Delta_j\right)g_t\right\rangle_t$$

$\Longrightarrow$ Countable additivity for a.a.(ν) t.

□

Set $N = \bigcup_{k=1}^{\infty} N(\Delta_k)$, then $\nu(N) = 0$ and $D_t(\cdot)$ is defined $a.a.(\nu)t$.

Sketch of proof of Theorem 2.3

Definition 2.1 and equation (6) implies that

$$\mathcal{R}(\theta, D) = \int_{sp(T)} \nu(dt)\phi(t,\theta) \int_{\Omega} L(\theta,\omega)\,\mathrm{Tr}_t \left(\rho_t D_t(d\omega)\right)$$

- Observe that $\mathrm{Tr}_t(\rho_t) = 0 \implies \rho_t = 0$ since $\rho_t \geq 0$.
- $\implies S \equiv \{t \in sp(T) | \mathrm{Tr}_t(\rho_t) = 0\}$
- $\implies$ the $t-$ integral in (6) and the above integral is restricted to $S^c \cap sp(T)$, in which case $\rho_t > 0$
- $\implies$ For $t \in S^c \cap sp(T)$, set $\sigma_t = \left(\mathrm{Tr}_t(\rho_t)\right)^{-1} \rho_t$, then σ_t is a density matrix in $\mathfrak{h}_t$ and by Lemma 2.4, since $L(\theta,\cdot)$ is convex

Continue...

- $\Longrightarrow$

$$\int_\Omega L(\theta,\omega)\,\mathrm{Tr}_t\left(\sigma_t D_t(d\omega)\right) \geq L\left(\theta, \int_\Omega \omega\,\mathrm{Tr}_t\left(\sigma_t D_t(d\omega)\right)\right)$$

$$\Longrightarrow \mathcal{R}(\theta, D) \geq \int_{sp(T)} \nu(dt)\phi(t,\theta)\,\mathrm{Tr}_t(\rho_{_t})L(\theta, D_T(t)),$$

where

$$D_T(t) = \begin{cases} \left(\mathrm{Tr}_t(\rho_{_t})\right)^{-1}\left(\int_\Omega \omega\,\mathrm{Tr}_t\left(\sigma_t D_t(d\omega)\right)\right), & t \in S^c \cap sp(T) \\ 0, & t \in S. \end{cases}$$

- $D_T(\cdot)$ is a bounded measurable function, defines a bounded operator, commuting with QSS T, and

$$\mathcal{R}(\theta, D) \geq \tau_\theta\Big(L(\theta, D_T)\Big) = \mathcal{R}(\theta, D_T).$$

Continue...

- Furthermore,

$$\theta \equiv \tau_\theta\left(\int_\Omega \omega D(d\omega)\right) = \int_{sp(T)} \nu(dt)\phi(t,\theta)\,\mathrm{Tr}_t\left(\rho_t\left(\int_\Omega \omega D_t(d\omega)\right)\right)$$
$$= \int_{sp(T)} \nu(dt)\phi(t,\theta)\,\mathrm{Tr}_t\left(\rho_t\right) D_T(t)$$
$$= \tau_\theta(D_T),$$

that is the expectation of "Rao-Blackwell observable" D_T in state τ_θ is an unbiased estimator for the parameter θ.

Corollary 2.6 (Partially quantum Baysian Rao-Blackwell theorem).

Under the set of hypothesis of Theorem 2.3, the risk function

$$\mathcal{R}(\pi, D) \geq \mathcal{R}(\pi, D_T) = \int_\Theta \pi(d\theta)\mathcal{R}(\theta, D_T).$$

Furthermore,

$$\int \pi(d\theta)\theta = \int \pi(d\theta)\tau_\theta\left(\int \omega D(d\omega)\right) = \int \pi(d\theta)\tau_\theta(D_T).$$

Proof of the above corollary immediate from Theorem 2.3.

Example 2.7.

Let $\mathcal{H} = L^2(\mathbb{R}^3), \underline{P} = (P_1, P_2, P_3)$-the momentum operators, $H_0 = \sum_{j=1}^{3} p_j^2 = -\Delta$ (as self-adjoint operator in $\mathcal{H}$), $\underline{L} \equiv (L_1, L_2, L_3)$-the angular momentum operators with $\underline{L}^2 = \sum_{j=1}^{3} L_j^2$, the total angular momentum operators.

For the operator H_0, the spectral representation is given by the unitary isomorphism via the Fourier transform : $\mathcal{H} \simeq L^2(\mathbb{R}_+; L^2(S^{(2)}), \frac{1}{2} t^{1/2} dt)$, with

$$(H_0 f)_t = t f_t \in \mathfrak{h}_t \simeq L^2(S^{(2)}) \ \forall \ t, \tag{9}$$

where $S^{(2)}$ is the unit sphere of 2-dimensions embedded in $\mathbb{R}^3$.

On the other hand, we set $\tilde{\rho}_t = (\underline{L}^2 + 1)^{-2}$ in $\mathfrak{h}_t \forall\ t$ and note that $\tilde{\rho}_t$ is a positive trace-class operator in $\mathfrak{h}_t \cong L^2(S^{(2)})$ and its trace:

$$Tr_t(\tilde{\rho}_t) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{(\ell(\ell+1)+1)^2} \equiv C < \infty.$$

If we set $\varphi(t,\theta) = (\pi/16\theta^3)^{-1/2} \exp(-\theta t)$ and normalise $\rho_t = C^{-1}\tilde{\rho}_t$, then it is an easy calculation to verify that

$$\int \varphi(t,\theta)(Tr_t \rho_t)\frac{1}{2}t^{1/2}dt = 1. \tag{10}$$

Therefore we define the state τ_θ on $\mathcal{A} = \{H_0'\}$, the von Neumann algebra driven by the constants of free motion, generated by the Hamiltonian H_0, as

$$\tau_\theta(B) = \int_0^\infty \varphi(t,\theta)\, Tr_t(\rho_t B(t)) \cdot \frac{1}{2} t^{1/2} dt, \tag{11}$$

setting the stage to recognize the self-adjoint operator $H_0 = T$, the sufficient statistic for this state τ_θ.

In this example, the symmetry group of rotations in 3-dimensions has played a role in the background.

Example 2.8.

The 2-dimensional (spin 1/2) representation of the proper rotation group $O^+(3)$ in $\mathbb{R}^3$ leads to the following standard operators in $\mathbb{C}^2$:

$\underline{S} = \{S_1, S_2, S_3\}$, $\underline{S}^2 = \sum_{j=1}^{3} S_j^2$ and the spanning eigenbases for $(\underline{S}^2, S_3)$ given by $\mid 3/4, \uparrow\rangle$ and $\mid 3/4, \downarrow\rangle$, in which $\underline{S}^2$ has $1/2(1/2+1) = 3/4$ and S_3 has $\pm 1/2$ as eigenvalues respectively with $\uparrow$ representing $+1/2$ and $\downarrow$ for $-1/2$.

- Next consider in $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$, two such independent spin systems $\underline{S}^{(j)}$ $(j = 1, 2)$ be given and we form the new observables: $\underline{S} = \underline{S}^{(1)} \otimes I_2 + I_1 \otimes \underline{S}^{(2)}$ and corresponding $\underline{S}^2$ in $\mathcal{H}$.
- Then we look for the decomposition of $\mathcal{H}$, corresponding to the irreducible representations of $O^+(3)$: $\mathcal{H} \simeq \mathfrak{h}_1 \oplus \mathfrak{h}_0$, where $\mathfrak{h}_1$ and $\mathfrak{h}_0$ are spanned by the eigenbases of $\{\underline{S}^2, S_3\}$ of the total system:

$$\text{for } \mathfrak{h}_1 : |2, \uparrow\uparrow\rangle, \ \left|2, \frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}}\right\rangle, \ |\, 2, \downarrow\downarrow\rangle \text{ and } \text{ for } \mathfrak{h}_0 : \left|0, \frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}}\right\rangle, \tag{12}$$

in which the normalized eigenstates corresponding to the eigenvalues $(1, 0, -1)$ in $\mathfrak{h}_1$ and (0) in $\mathfrak{h}_0$ of S_3 respectively are represented pictorially as above.

- Let the state be given by $\rho_\theta = Z^{-1}\exp(-\theta \underline{S}^{(1)} \cdot \underline{S}^{(2)})$, where Z is such that $Tr\rho_\theta = 1$.
- An easy calculation, using the identity: $\underline{S}^{(1)} \cdot \underline{S}^{(2)} = 1/2(\underline{S}^2 - \underline{S}^{(1)^2} - \underline{S}^{(2)^2})$, leads to the fact that the operator $(\underline{S}^{(1)} \cdot \underline{S}^{(2)})$ has eigenvalues $1/4$ and $-3/4$ respectively in each eigenfamilies in $\mathfrak{h}_1$ and $\mathfrak{h}_0$ respectively.
- Thus $\rho_\theta = Z^{-1}(e^{-\theta/4}P_1 + e^{3\theta/4}P_0)$, where P_1 and P_0 are the projections onto $\mathfrak{h}_1(3-\text{dim})$ and $\mathfrak{h}_0(1-\text{dim})$ respectively, in $\mathcal{H}$.
- This implies that $Z = Tr(e^{-\theta/4}P_1 + e^{3\theta/4}P_0) = 3e^{-\theta/4} + e^{3\theta/4}$, which leads to the following expression for ρ_θ:

$$\rho_\theta = (3+e^\theta)^{-1}P_1 + (1+3e^{-\theta})^{-1}P_0. \tag{13}$$

- Therefore, here $\underline{S}^2 = (\underline{S}^{(1)} + \underline{S}^{(2)})^2$ is the candidate for "quantum sufficient statistic" and

$$\rho_\theta(B) = \sum_{s=0}^{1} \varphi(s,\theta) Tr_s(\rho_s B(s)),$$

where $s(s+1)$ are the eigenvalues of $\underline{S}^2$, and

$$\varphi(0,\theta) = (1 + 3e^{-\theta})^{-1}, \varphi(1,\theta) = (3 + e^{\theta})^{-1};$$

$\rho_0 = P_0$, $\rho_1 = P_1$, the orthogonal projections respectively.

Fully Quantum Rao-Blackwell theorem:

For this part, we shall assume that the QSS T is a bounded self-adjoint operator in $\mathfrak{h}_s$ (the Hilbert space of observations) with only discrete spectrum.

Unlike in the previous section, here the Baysian part is quantized $\implies$ the Baysian Hilbert space $\mathfrak{h}_B$ and the theory is put in $\mathcal{H} = \mathfrak{h}_B \times \mathfrak{h}_s$, Φ is a density Matrix in $\mathcal{H}$, $\Omega = [a, b] \subseteq \mathbb{R}$, $L : \Omega \to \mathcal{B}_+(\mathfrak{h}_B)$ strongly continuous, $\{D(\cdot)\}$ POVM commuting with QSS T in $\mathfrak{h}_s$.

Then one has

$L \cdot D \equiv \int_a^b L(\omega)D(d\omega)$ exists as trong Riemann-Stiltje's integral on $\mathcal{H}$,

$$\|L \cdot D\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{\omega} \|L(\omega)\|_{\mathcal{B}(\mathfrak{h}_B)}.$$

The fully quantum risk function:

$$\mathcal{R}(\Phi, D) = \Phi(L \cdot D)$$

Here

$$\mathcal{H} \simeq \mathfrak{h}_B \otimes \int^{\oplus} \nu(dt)\mathfrak{h}_t \simeq \int^{\oplus} \nu(dt)(\mathfrak{h}_B \otimes \mathfrak{h}_t),$$

ν is only atomic and $\Phi \simeq \int^{\oplus} \nu(dt)(\Phi_B(t) \otimes \rho_t)$, $\Phi_B(t) \in \mathcal{B}_{1+}(\mathfrak{h}_B)$,

$$\operatorname{Tr}\Phi = 1 \int \nu(dt) \operatorname{Tr}_B(\Phi_B(t)) \operatorname{Tr}(\rho_t).$$

Fully quantum Rao-Blackwell Theorem II

Assume all that has gone before, and $S = \{t \in sp(T) | \rho_t = 0\}$ is a mble set and set $D_T(t)$ as before to get a mble bounded function. Let L be weakly convex, that is. $\omega \mapsto \langle f, L(\omega) f \rangle$ is a positive convex, continuous function for every $f \in \mathfrak{h}_B$. Then

$$\Phi(L \cdot D) \geq \Phi(L(D_T)),$$

where $L(D_T)(t) = L(D_T(t))$ and range of $D_T(\cdot) \subseteq \Omega$.

$$\Phi\left(\int_\Omega \omega D(d\omega)\right) = \int \nu(dt)\, \mathrm{Tr}_B(\Phi_B(t)).$$

$$\mathrm{Tr}\left(\rho_t \int \omega D_t(d\omega)\right) = \int \nu(dt)\, \mathrm{Tr}\left(\Phi_B(t) \otimes \rho_t\right) D_T(t) = \Phi(D_T).$$

More general theory needs the use of the central decomposition of von Neumann algebras and of states on them.


Thank
You