A study of random variables in terms of their Number Operator

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Outline

- Background
- Representing the linear operators in terms of X and D
- Random variables whose number operator is quadratic in D
- Symmetric random variables whose number operator satisfy a quadratic equation, in which the "constant" term is quadratic in D

Background

Let X_1, X_2, ..., X_d be random variables on $(\Omega, \mathcal{F}, P)$.

Assume, for all $1 \leq i \leq d$ and $p > 0$, we have:

$$E\left[|X_i|^p\right] < \infty. \tag{1}$$

For all $n \in \mathbb{N} \cup \{0\}$, let:

$$F_n := \{P(X_1, X_2, \ldots, X_d) \mid P \text{ polyn., } \deg(P) \leq n\}.$$

$$\mathbb{C} \equiv F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq L^2(\Omega, P).$$

Background

We define $G_0 := F_0$, and for all $n \geq 1$:

$$G_n \quad := \quad F_n \ominus F_{n-1}.$$

Consider the multiplication operator:

$$P(X_1, X_2, \cdots, X_d) \quad \mapsto \quad X_i P(X_1, X_2, \cdots, X_d).$$

Lemma

For all $1 \leq i \leq d$ and $n \geq 0$, we have:

$$X_i G_n \quad \perp \quad G_k, \tag{2}$$

$\forall k \neq n-1,\ n,\ n+1.$

For all $n \geq 0$, let:

$$\overline{P}_n : L^2(\Omega, P) \to G_n, \quad \text{be orthogonal projection,}$$

and

$$P_n := \overline{P}_{n|F}.$$

Let $I : F \to F$ be the identity operator, where $F := \cup_{n \geq 0} F_n$.

For all $1 \leq i \leq d$, we have:

$$\begin{aligned} X_i &= I X_i I \\ &= \left(\sum_{m=0}^{\infty} P_m\right) X_i \left(\sum_{n=0}^{\infty} P_n\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_m X_i P_n \\ &= \sum_{|m-n| \leq 1} P_m X_i P_n \end{aligned}$$

Background

$$X_i = \sum_{n=0}^{\infty} P_{n+1}X_iP_n + \sum_{n=0}^{\infty} P_nX_iP_n + \sum_{n=1}^{\infty} P_{n-1}X_iP_n.$$

Let us define:

$$a^+(i) := \sum_{n=0}^{\infty} P_{n+1}X_iP_n, \tag{3}$$

$$a^0(i) := \sum_{n=0}^{\infty} P_nX_iP_n, \tag{4}$$

$$a^-(i) := \sum_{n=0}^{\infty} P_{n-1}X_iP_n. \tag{5}$$

Background

For all $1 \leq i \leq d$ and $n \geq 0$, we have:

$$a^{+}(i)G_n \subseteq G_{n+1}, \quad (6)$$

$a^{+}(i)$ – creation operator,

$$a^{0}(i)G_n \subseteq G_n, \quad (7)$$

$a^{0}(i)$ – preservation operator,

$$a^{-}(i)G_n \subseteq G_{n-1}, \quad (8)$$

$a^{-}(i)$ – annihilation operator

Background

Theorem

For all $1 \leq i \leq d$, we have:

$$X_i = a^+(i) + a^0(i) + a^-(i). \tag{9}$$

For all f and $g \in F$, and all $1 \leq i \leq d$:

$$\langle a^+(i)f, g\rangle = \langle f, a^-(i)g\rangle, \tag{10}$$

$$\langle a^0(i)f, g\rangle = \langle f, a^0(i)g\rangle. \tag{11}$$

Semi-Quantum Operators

Semi-annihilation operators:

$$U_i = a^-(i) + \frac{1}{2}a^0(i). \tag{12}$$

Semi-creation operators:

$$V_i = a^+(i) + \frac{1}{2}a^0(i). \tag{13}$$

Lemma

For all $i \in \{1, 2, \ldots, d\}$, we have:

$$X_i = U_i + V_i \tag{14}$$

and

$$U_i^* = V_i. \tag{15}$$

Number Operator

$$N : F \to F$$

For all $n \in \mathbb{N} \cup \{0\}$,

$$N_{|G_n} \quad = \quad nI_{|G_n}. \tag{16}$$

That means, if $\forall n \geq 0$, P_n denotes the restriction to F of the projection of L^2 on G_n, then:

$$N \quad = \quad \sum_{n=0}^{\infty} nP_n. \tag{17}$$

Universal Commutator Rules

For all $1 \leq i \leq d$, we have:

- $$[N, a^+(i)] = a^+(i) \quad (18)$$

- $$[a^-(i), N] = a^-(i) \quad (19)$$

- $$[N, a^0(i)] = 0 \quad (20)$$

- $$[N, X_i] = a^+(i) - a^-(i) \quad (21)$$
  $$= V_i - U_i \quad (22)$$

Recovering U_i and V_i from N

Solving the system:

$$\begin{cases} V_i + U_i = X_i \\ V_i - U_i = [N, X_i] \end{cases},$$

we get:

$$V_i = \frac{1}{2}(X_i + [N, X_i]) \tag{23}$$

and

$$U_i = \frac{1}{2}(X_i - [N, X_i]) \tag{24}$$

Recovering $a^{+}(i)$ and $a^{-}(i)$ from N

Since:

$$
\begin{aligned}
[N, V_i] &= [N, a^{+}(i)] + \frac{1}{2}[N, a^{0}(i)] \\
&= a^{+}(i) \qquad (25)
\end{aligned}
$$

and by duality:

$$
[U_i, N] = a^{-}(i), \qquad (26)
$$

we have:

$$
\begin{aligned}
a^{+}(i) &= [N, V_i] \\
&= \frac{1}{2}\left([N, X_i] + [N, [N, X_i]]\right) \qquad (27)
\end{aligned}
$$

and by duality:

$$
a^{-}(i) = \frac{1}{2}\left(-[N, X_i] + [N, [N, X_i]]\right). \qquad (28)
$$

Recovering $a^0(i)$ from N

We have:

$$a^0(i) = 2\left(V_i - a^+(i)\right) = X_i - [N,[N,X_i]]. \quad (29)$$

Corollary

The random vector $(X_1, X_2, \ldots, X_d)$ is polynomially symmetric, which means that for all $i_1 + i_2 + \cdots + i_d = 2n - 1$, for $n \in \mathbb{N}$,

$$E\left[X_1^{i_1} X_2^{i_2} \cdots X_d^{i_d}\right] = 0 \quad (30)$$

if and only if, for all $1 \leq i \leq d$, we have:

$$[N,[N,X_i]] = X_i. \quad (31)$$

Let $k \in \mathbb{Z}$ be fixed. Let $T : F \to F$ be a linear operator, such that, for all $n \in \mathbb{N} \cup \{0\}$, we have:

$$TF_n \quad \subseteq \quad F_{n+k}. \tag{32}$$

Then, there exist $\{A_n(X)\}_{n \geq 0} \subset F$, such that:

- $\forall n \geq 0$, A_n has degree at most $n + k$.
- For all $f(X) \in F$, we have:

$$Tf(X) \quad = \quad \sum_{n=0}^{\infty} A_n(X) D^n f(X), \tag{33}$$

where $D : F \to F$ is the *differentiation* operator, that means:

$$D\left(a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0\right) \tag{34}$$

$$= \quad n a_n X^{n-1} + (n-1) a_{n-1} X^{n-1} + \cdots + a_1, \tag{35}$$

for all $n \geq 0$, $a_1, a_2, \cdots, a_n \in \mathbb{C}$.

In particular:

- For $T = a^-$, $k = -1$. So, there exists $\{A_n(X)\}_{n \geq 1} \subset F$, such that, for all $n \geq 1$, $deg(A_n) \leq n - 1$, and:

$$a^- = \sum_{n=1}^{\infty} A_n(X) D^n. \tag{36}$$

- For $T = a^0$, $k = 0$. So, there exists $\{B_n(X)\}_{n \geq 0} \subset F$, such that, for all $n \geq 0$, $deg(B_n) \leq n$, and:

$$a^0 = \sum_{n=0}^{\infty} B_n(X) D^n. \tag{37}$$

- For $T = a^+$, $k = 1$. So, there exists $\{C_n(X)\}_{n \geq 0} \subset F$, such that, for all $n \geq 0$, $deg(C_n) \leq n + 1$, and:

$$a^+ = \sum_{n=0}^{\infty} C_n(X) D^n. \tag{38}$$

If the position-momentum decomposition of N is:

$$N = \sum_{n=1}^{\infty} A_n(X)D^n, \tag{39}$$

then:

$$\begin{aligned} U &= \frac{1}{2}(X - [N, X]) \\ &= \frac{1}{2}(X - A_1(X))\, I - \frac{1}{2}\sum_{n=2}^{\infty} nA_n(X)D^{n-1}. \end{aligned} \tag{40}$$

Since, for U, all the position left factors must have the degree less than or equal to the exponent of the momentum right factors, we conclude that:

$$A_1(X) = X - \mu, \tag{41}$$

and for all $n \geq 2$, we have:

$$A_n(X) = c_{n-1}X^{n-1} + c_{n-2}X^{n-2} + \cdots \tag{42}$$

Random variables for which the number operator is quadratic in D

Let us assume that X is a random variable whose number operator is:

$$N = (bX + c)\,D^2 + (X - \mu)\,D, \tag{43}$$

with a, b, and c real numbers. Then, we have:

$$\begin{aligned} U &= \frac{1}{2}(X - [N, X]) \\ &= \frac{1}{2}\{X - 2(bX + c)D - (X - \mu)I\} \\ &= -(bX + c)D + \frac{\mu}{2}I. \end{aligned} \tag{44}$$

For all t in a neighborhood of 0, if we denote the Laplace transform of X, by:

$$\varphi(t) := E\left[\exp(tX)\right]. \tag{45}$$

We have:

$$\begin{aligned}
\varphi'(t) &= E\left[X\exp(tX)\right] \\
&= \langle X\exp(tX)1, 1\rangle \\
&= \langle (U+V)\exp(tX)1, 1\rangle \\
&= \langle U\exp(tX)1, 1\rangle + \langle \exp(tX)1, U1\rangle \\
&= \langle (-(bX+c)D + (1/2)\mu)\exp(tX)1, 1\rangle \\
&\quad + \langle \exp(tX)1, (-(bX+C)D + (1/2)\mu)1\rangle \\
&= -t\langle (bX+c)\exp(tX)1, 1\rangle + \mu\langle \exp(tX)1, 1\rangle \\
&= -bt\varphi'(t) + (\mu - ct)\varphi(t).
\end{aligned}$$

This implies:

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{-ct+\mu}{bt+1}. \tag{46}$$

Thus, the derivative of the Laplace transform of X, on a neighborhood of 0, is a Möbius function.

It is not hard to see from here that up to a re-scaling and translation, X is a Gamma or Gaussian random variable.

Solving a quadratic equation in N

Let us consider a polynomially symmetric random variable X, whose number operator, N, satisfies the equation:

$$N^2 + 2\alpha N = \left(aX^2 + b\right) D^2 + 2cXD, \tag{47}$$

where α, a, b, and c are real numbers.

Let us compete the square in the left, by adding $\alpha^2 I$ to both sides, and obtaining:

$$(N + \alpha I)^2 = \left(aX^2 + b\right) D^2 + 2cXD + \alpha^2 I. \tag{48}$$

Let us commute both sides with X to the right:

$$\left[(N+\alpha I)^2, X\right] = \left[\left(aX^2+b\right)D^2+2cXD+\alpha^2 I, X\right]. \quad (49)$$

Using Leibniz commutator formula and $[D, X] = I$, we have:

$$[N,X](N+\alpha I)+(N+\alpha I)[N,X] = 2\left(aX^2+b\right)D+2cX. \quad (50)$$

Let us commute now both sides with N to the left:

$$\begin{aligned} & [N,[N,X]](N+\alpha I)+(N+\alpha I)[N,[N,X]] \\ = & \ 2\left[N,\left(aX^2+b\right)D\right]+2c\,[N,X]. \quad (51) \end{aligned}$$

Since X is polynomially symmetric, we have:

$$[N,[N,X]] = X. \quad (52)$$

Thus, we obtain:

$$\begin{aligned} & X(N+\alpha I)+(N+\alpha I)X \\ = \quad & 2\left[N,\left(aX^2+b\right)D\right]+2c\,[N,X]\,. \end{aligned} \tag{53}$$

Let the position-momentum decomposition of N be:

$$N \quad = \quad \sum_{n=1}^{\infty} A_n(X)D^n. \tag{54}$$

Then, the position-momentum decomposition of $N+\alpha I$ is:

$$N+\alpha I \quad = \quad \sum_{n=0}^{\infty} A_n(X)D^n, \tag{55}$$

where:

$$A_0(X) \quad = \quad \alpha. \tag{56}$$

Substituting N and $N+\alpha I$ into equation (53), applying Leibniz commutator formula, using the fact that $[D,\, f(X)] = f'(X)$, we get:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(2XA_n(X)+(n+1)A_n(X)\right)D^n \\
= &\sum_{n=0}^{\infty}\left(-2\left(aX^2+b\right)A_n'(X)+4anXA_n(X)+2an(n+1)A_{n+1}(X)\right. \\
&\left.\quad +2c(n+1)A_{n+1}(X)\right)D^n.
\end{aligned}
$$

Equating the left position coefficients of the corresponding D^n, we obtain (after moving terms from one side to another):

$$
(n+1)\left(an+c-\frac{1}{2}\right)A_{n+1}(X) \quad = \quad \left(aX^2+b\right)A_n'(X)+(-2an+1)XA_n(X),
$$

for all $n \geq 0$. Since $A_0(X)=\alpha$ and $A_1(X)=X$, we must have:

$$
c \quad = \quad \alpha+\frac{1}{2}. \tag{57}
$$

Since $A_2(X)$ must have the degree <u>strictly</u> less than 2, we must have:

$$
a \quad = \quad 1. \tag{58}
$$

Dividing both sides of the recursive equation by $aX^2 + b = X^2 + b$, we obtain:

$$(n+1)(n+\alpha)\frac{1}{X^2+b}A_{n+1}(X) \quad = \quad A_n'(X) + \frac{-2n+1}{X^2+b}XA_n(X).$$

Multiplying both side by the integrating factor $\rho(X) = (X^2 + b)^{-n+(1/2)}$, we get:

$$\begin{aligned} & (n+1)(n+\alpha)\left(X^2+b\right)^{-n-1+(1/2)}A_{n+1}(X) \\ = \quad & \left(\left(X^2+b\right)^{-n+(1/2)}A_n(X)\right)'. \end{aligned}$$

For all $n \geq 0$, let us define:

$$B_n(X) \quad := \quad \left(X^2+b\right)^{-n+(1/2)}A_n(X).$$

We have the recursive formula:

$$(n+1)(n+\alpha)B_{n+1}(X) \quad = \quad B_n'(X).$$

Iterating this recursive relation, we obtain in the end:

$$B_n(X) \quad = \quad \frac{1}{n!\alpha_{(n)}}\frac{d^n}{dX^n}B_0(X), \tag{59}$$

where $\alpha_{(n)} := \alpha(\alpha+1)\cdots(\alpha+n-1)$ is the Pochhammer symbol.

Since $B_0(X) = \alpha\sqrt{x^2+b}$, we obtain:

$$B_n(X) \quad = \quad \frac{\alpha}{n!\alpha_{(n)}}\frac{d^n}{dX^n}\left(\sqrt{X^2+b}\right), \tag{60}$$

which means, for all $n \geq 0$,

$$A_n(X) \quad = \quad \frac{\alpha}{n!\alpha_{(n)}}\left(X^2+b\right)^{n-(1/2)}\frac{d^n}{dX^n}\left(\sqrt{X^2+b}\right). \tag{61}$$

Thus, the position-momentum decomposition of N is:

$$N = \sum_{n=1}^{\infty} \frac{\alpha}{n!\alpha_{(n)}} \left(X^2 + b\right)^{n-(1/2)} \frac{d^n}{dX^n} \left(\sqrt{X^2 + b}\right) D^n. \qquad (62)$$

Using the formula:

$$a^- = \frac{1}{2}\left(X - [N, X]\right),$$

we obtain that the position-momentum decomposition of a^- is:

$$a^- = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{\alpha}{n!\alpha_{(n+1)}} \left(X^2 + b\right)^{n+(1/2)} \frac{d^{n+1}}{dX^{n+1}} \left(\sqrt{X^2 + b}\right) D^n.$$

Faà di Bruno's formula shows that:

$$\left(X^2 + b\right)^{n-(1/2)} \frac{d^n}{dX^n} \left(\sqrt{X^2 + b}\right)$$

is a polynomial of degree $n - 2$, for all $n \geq 2$.

THANK YOU VERY MUCH!