Nice Error Basis and Study of Quantum Maps

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Brief Outline of the presentation

1st part

- We introduce the notion of nice error basis and index group.
- Characterisation of abelian index group and as a special case error basis coming from discrete Weyl relation.
- non-abelian index group from groups of central type.

2nd part

- We use the nice error basis to characterise CP maps in terms of positive Kernel.
- we discuss the characterisation of semigroup of CP maps in terms of Kernel
- Characterisation of semigroup of k-positive maps.

Nice Error Basis

- Let G be a group of order n^2. The set $\mathcal{E} = \{\pi(g) \in U(n); g \in G\}$ is called **nice error basis** if

  1. $\pi(1) = Id$,
  2. $\mathrm{Tr}(\pi(g)) = n\delta_{g,1}$,
  3. $\pi(g)\pi(h) = \omega(g,h)\pi_{gh}$, where $\omega : G \times G \longrightarrow \mathbb{C}$.

  G is called **index group** if such representation exists.

- W.r.t. the Hilbert-Schmidt inner product on $M_n(\mathbb{C})$

$$\langle A, B \rangle = \frac{1}{n}\mathrm{Tr}(\mathrm{A}^*\mathrm{B})$$

  they form an orthonormal basis of $M_n(\mathbb{C})$ upto a scalar multiplication.

- Recall G is called symmetric if $G \cong H \times H$.

Theorem (Klappenecker, Rötteler)

An error basis $\mathcal{E}$ has abelian index group G if G is of symmetric type and conversely any finite abelian group of symmetric type is an index group of nice error basis.

NEB from Discrete Weyl relation

- Let G be a group such that $G \cong \mathbb{Z}_N \times \mathbb{Z}_N$.
- Define a bicharacter $\varkappa : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}$ such that
  1. $|\varkappa(x, y)| = 1$,
  2. $\varkappa(x, y) = \varkappa(y, x)$,
  3. $\varkappa(x, y) = 1 \quad \forall y$ iff $x = 0$,
  4. $\varkappa(x, y + z) = \varkappa(x, y).\varkappa(x, z)$.
- Example:

$$\varkappa(k, l) = \exp\left(\frac{2\pi ikl}{N}\right), \quad \text{for} \quad k, l \in \mathbb{Z}_N$$

.

- Let $\{|x\rangle; x \in \mathbb{Z}_N\}$ be orthonormal basis of $\mathbb{C}^N$. Define

$$U_a|x\rangle = |x+a\rangle,$$
$$V_b|x\rangle = \varkappa(b,x)x.$$

- *Weyl commutation relations:*

$$U_aU_b = U_{a+b}, \quad V_aV_b = V_{a+b}, \quad U_aV_b = \varkappa(a,b)V_bU_a.$$

- Define *Weyl Operators*

$$W_{a,b} = U_aV_b.$$

- Properties:

1. Unitary operators

$$W_{a,b}^* = W_{a,b}^{-1} = \varkappa(a,b) W_{-a,-b},$$

2. Projective representation

$$W_{a,b} W_{x,y} = \varkappa(b,x) W_{a+x,b+y}$$

3. $$\mathrm{Tr}(W_{a,b}) = \sum_{x \in G} \langle x | W_{a,b} | x \rangle = \begin{cases} N & \text{if } (a,b) = (0,0) \\ 0 & \text{otherwise} \end{cases}$$

4. **Nice Error Basis:** $\{N^{-1/2} W_{a,b}; a, b \in \mathbb{Z}_N\}$ forms an orthonormal basis of $M_N(\mathbb{C})$.

Non-commutative index group coming from Group of Central type

- A group H is called **central type** if $\exists$ an irreducible character χ s.t. $\chi(1)^2 = [H : Z(H)]$.

Theorem

If H is a group of central type with irreducible character χ then $(H/\mathrm{Ker}\chi)/(Z(H)/\mathrm{Ker}\chi) \cong H/Z(H)$ is an index group. Here $\mathrm{Ker}_\chi = \{h \in H; \chi(h) = \chi(1)|\}$.

Convenient basis of $L(M_n, M_n)$

- For $x, y \in G$ a index group and $\{\pi_g; g \in G\}$ be corresponding nice error basis. let $T_{x,y} : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ be given by

$$T_{x,y}(X) = \pi_x X \pi_y^* \quad \forall X \in M_n$$

Proposition

$\{\frac{1}{n} T_{x,y}; x, y \in G\}$ forms an ONB of $L(M_n, M_n)$ w.r.t. the Hilbert-Schmidt inner product.

- It follows $\forall\alpha \in L(M_n, M_n)$ we can write

$$\alpha(X) = \sum_{x,y\in G} D_\alpha(x,y) T_{x,y}(X) = \sum_{x,y\in G} D_\alpha(x,y)\pi_x X \pi_y^*$$

.

- **Computation of D_α:** Using $\{\frac{1}{n}\pi_g; g \in G\}$ ONB of $M_n(\mathbb{C})$

$$D_\alpha(x,y) = \left\langle \frac{1}{n} T_{x,y}, \alpha \right\rangle_{H.S} = \frac{1}{n^2} \sum_{g\in G} \mathrm{Tr}(\pi_y \pi_g^* \pi_x^* \alpha(\pi_g)). \tag{1}$$

Characterisation of positive maps

Theorem

A linear map $\alpha \in L(M_N, M_N)$ is positive if and only if it is block positive i.e. for all $u, v \in \mathbb{C}^N$ we have

$$\langle u \otimes v | \tau \circ \tilde{\alpha}(u \otimes v) \rangle \geq 0$$

where $\tilde{\alpha} = \sum_{x,y \in G} D_\alpha(x,y) \pi_x \otimes \pi_y^*$ and $\tau(a \otimes b) = b \otimes a$ is the flip operator.

Proof.

α is positive iff it maps rank one projection to positive operators, therefore

$$\begin{aligned} 0 \leq \langle v, \alpha(|u\rangle\langle u|) v \rangle &= \left\langle v, \sum D_\alpha(x,y) \pi_x(|u\rangle)\langle u|) \pi_y v \right\rangle \\ &= \left\langle u \otimes v, \tau \circ (\sum D_\alpha(x,y) \pi_x \otimes \pi_y^*)(u \otimes v) \right\rangle. \end{aligned}$$

□

k-positive and completely positive maps

- A linear map $T \in L(M_N, M_N)$ is called k-positive if the augmented map

$$T^{(k)} := T \otimes Id_k : M_N \otimes M_k \longrightarrow M_N \otimes M_k$$

  is positive, where $k \in \mathbb{N}$.

- It is called completely positive if $T^{(k)}$ is positive for all $k \in \mathbb{N}$.

Positive Kernels and CP maps

Theorem

A linear map $\alpha \in L(M_n, M_n)$ is completely positive if and only if the corresponding matrix D_α is positive semi-definite.

Sketch of the proof

- Suppose α is CP. $\implies \exists L_j \in M_n(\mathbb{C}), 1 \leq j \leq k$ such that

$$\alpha(X) = \sum_{j=1}^{k} L_j X L_j^*.$$

- Expand L_j w.r.t the O.N.B. $\{\frac{1}{\sqrt{n}}\pi_x; x \in G\}$-

$$L_j = \sum_{z \in G} l_j(z)\pi_z.$$

- $$D_\alpha(x,y) = \frac{1}{n^2}\sum_{g\in G}\mathrm{Tr}\left(\pi_y\pi_g^*\pi_x^*\sum_{j=1}^{k}\left(\sum_{z\in G} l_j(z)\pi_z\right)\pi_g\left(\sum_{z'\in G} l_j(z')\pi_{z'}\right)^*\right)$$
$$= \frac{1}{n^2}\sum_{j=1}^{k}\sum_{z,z'\in G} l_j(z)\overline{l_j(z')}\sum_{g\in G}\mathrm{Tr}\left(\pi_y\pi_g^*\pi_x^*\pi_z\pi_g\pi_{z'}^*\right)$$
$$= \frac{1}{n^2}\sum_{j=1}^{k}\sum_{z,z'\in G} l_j(z)\overline{l_j(z')}\omega(x,x^{-1})^{-1}\omega(x^{-1},z)\omega(z',z'^{-1})^{-1}\omega(z'^{-1},y)\delta_{xz}\delta_{z'y}n^2,$$
$$= \frac{1}{n^2}\sum_{j=1}^{k} l_j(x)\overline{l_j(y)}$$

Conversely,

- Assume D_α is positive. $\implies \exists A_1, A_2, \ldots, A_k \in M_{n^2}(\mathbb{C})$ such that $D_\alpha = \sum_{j=1}^{k} A_j A_j^*$
- writing $A_j = (a_j(x,y))_{x,y \in G}$, we have

$$D_\alpha = \sum_{j=1}^{k} \sum_{v \in G} a_j(x,v)\overline{a_j(y,v)}.$$

- $\implies \alpha(X) = \sum_{x,y \in G} D_\alpha(x,y)\pi_x X \pi_y^* = \sum_{j=1}^{k} \sum_{v \in G} a_j(x,v)\overline{a_j(y,v)}\pi_x X \pi_y^*.$
-

$$\alpha(X) = \sum_{j=1, v \in G}^{k} L_{j,v} X L_{j,v}^*.$$

where $\sum_{x \in G} a_j(x,v)\pi_x = L_{j,v}$, $\sum_{y \in G} a_j(y,v)\pi_y = L_{j,v}^*$.

Coalgebra

- **Coalgebra:** A coalgebra over $\mathbb{C}$ is a vector space C together with linear maps $\Delta : C \longrightarrow C \otimes C$ and $\epsilon : C \longrightarrow \mathbb{C}$ such that
    1. $(id_C \otimes \Delta) \circ \Delta = (\Delta \otimes id_C) \circ \Delta$ [coassociative]
    2. $(id_C \otimes \epsilon) \circ \Delta = id_C = (\epsilon \otimes id_C) \circ \Delta$ [counit]
- **Subcoalgebra:** A linear subspace U of a coalgebra C is called subcoalgebra if $\Delta(U) \subset U \otimes U$.

Coalgebra structure of $M_n^*(\mathbb{C})$

- Let $A := \mathrm{span}\{\pi_x; x \in G\} \cong M_n(\mathbb{C})$
- There is a natural coalgebra structure on $C := A^* \cong M_n(\mathbb{C})$, denoted by (C, Δ, ϵ) where

$$\begin{array}{cc} \Delta : C \longrightarrow C \otimes C & \epsilon : C \longrightarrow \mathbb{C} \\ \Delta(\phi) = \phi \circ m_A & \epsilon(\phi) = \phi(1_A). \end{array}$$

  where m_A is the multiplication operator on A.
- Let $\{\mathbf{1}_x; x \in G\}$ be dual basis of $\{\pi_x; x \in G\}$, i.e.

$$\langle \mathbf{1}_x, \pi_y \rangle = \delta_{x,y}.$$

- In terms of the dual basis the coalgebra structure can be written

$$\epsilon(\mathbf{1}_x) = \delta_{x,1}$$
$$\Delta(\mathbf{1}_x) = \sum_{p \in G} \omega(p, xp^{-1})\mathbf{1}_p \otimes \mathbf{1}_{xp^{-1}}.$$

- We denote by $\overline{C}$, the conjugate coalgebra of C i.e. if $\overline{c} \in \overline{C}$ then

$$\lambda\overline{c} = \overline{\overline{\lambda}c}$$

where $\lambda \in \mathbb{C}$ and $c \in C$.

- $C \otimes \overline{C}$ inherits a natural coalgebra structure.

- We can identify $(C \otimes \overline{C})^*$ with sesquilinear forms on C, by

$$\langle v, w \rangle_\Phi = \Phi(v \otimes \overline{w}),$$

for $v, w \in C$ and $\Phi \in (C \otimes \overline{C})^*$.

- Dual C^* of the coalgebra (C, Δ, ϵ) has algebra structure with multiplication given by convolution product

$$\Phi \star \Psi = (\Phi \otimes \Psi) \circ \Delta.$$

- Next, $\forall \alpha \in L(M_n, M_n)$ identify the coefficient matrix D_α as a functional on $C \otimes \overline{C}$ via

$$D_\alpha(\mathbf{1}_x \otimes \overline{\mathbf{1}}_y) = D_\alpha(x, y).$$

Proposition

$D : L(M_n, M_n) \ni \alpha \mapsto D_\alpha \in (C \otimes \overline{C})^*$ *is an isomorphism, i.e.*

$$D_{\alpha \circ \beta} = D_\alpha \star D_\beta.$$

Characterisation of semigroup of CP maps

- Therefore for any functional $K : C \otimes \overline{C} \longrightarrow \mathbb{C}$ we can define one parameter group

$$\exp_{\star}(tK) = \sum_{0}^{\infty} \frac{t}{n!} K^{\star n} = \epsilon + tK + \frac{t^2}{2} K \star K + \ldots$$

.

- A sesquilinear form K on a coalgebra (C, Δ, ϵ) is called conditionally positive if $K(v|v) \geq 0$ for all $v \in C$ with $\delta(v) = 0$.

Theorem (Schürmann's Schoenberg correspondence for sesqui-linear forms on a coalgebra)

Let K be a sesqui-linear form on a coalgebra C and denote by $\exp_{\star}(tK)$, $t \geq 0$, the one-parameter convolution semigroup associated to it. Then $\exp_{\star}(tK)$ is positive for all $t \geq 0$ if and only if K is conditionally positive.

Theorem

Let $(\alpha_t)_{t\geq 0}$ be a semigroup of linear maps on $M_n(\mathbb{C})$. Then α_t is completely positive for all $t \geq 0$ if and only if the sesqui-linear form with coefficients

$$K(x,y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} D_{\alpha_t}(x,y)$$

is conditionally positive.

A version of GKLS theorem

Using this result, for the generator $\lambda = \frac{d}{dt}|_{t=0}\alpha_t = \sum_{x,y \in G} K(x,y)\pi_x(.)\pi_y^*$ of the semigroup we get the form

$$\lambda(X) = \kappa X + VX + XV^* + \Phi(X),$$

where $\kappa \in \mathbb{R}$, $V = \sum_{x \in G-\{0\}} = v(x)\pi_x$, where $v \in \mathbb{C}^{N^2-1}$,
$\Phi(X) = \sum_{x,y \in G-\{0\}} \phi(x,y)\pi_x X \pi_y^*$, $\phi \in M_{N^2-1}(\mathbb{C})$ is positive.

Theorem (Gorini, Kossakowsky, Lindblad, Sudarshan)

Let $\lambda \in L(M_n, M_n)$. Then the semigroup $\alpha_t := \exp t\lambda$ is completely positive iff λ is of this form

$$\lambda(X) = VX + XV^* + \Phi(X)$$

where Φ is a completely positive map and $V \in M_n(\mathbb{C})$.

Characterisation of semigroup of k-positive maps

- The cone $\mathcal{P}_k$ of k-positive maps are proper i.e solid, pointed closed convex cone in $L(M_n, M_n)$ and they are also closed under composition.

A linear functional ϕ on a coalgebra V is called conditionally positive on the cone $C \subset V$ if $\phi(v) \geq 0$ for all $v \in C \cap \ker\delta$.

Theorem

Let $(\alpha_t)_{t\geq 0}$ be a semigroup of linear maps on $M_n(\mathbb{C})$. Then α_t is k-positive if and only if the sesqui-linear form with coefficient

$$K(x,y) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} D_{\alpha_t}(x,y)$$

is conditionally positive on the cone $D(\mathcal{P}_k)^+$.

Thank you!