Quantum Extension of Transformations on White Noise Functionals

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White Noise Functionals and Operators

The mathematical framework of the white noise theory is a Gelfand triple:

$$(E) \subset L^2(E_{\mathbb{R}}^*, \mu) \subset (E)^*$$

based on a underline Gelfand triple $E \subset H \subset E^*$, where H is a separable Hilbert space and E is a nuclear space which is densely and continuously embedded into H.

Here μ is the standard Gaussian measure on $E_{\mathbb{R}}^*$ which is uniquely determined by its characteristic function given by

$$\exp\left(-\frac{1}{2}|\xi|_H^2\right) = \int_{E_{\mathbb{R}}^*} e^{i\langle x, \xi\rangle}\mu(dx), \qquad \xi \in E_{\mathbb{R}},$$

where $E_{\mathbb{R}}$ is a real nuclear space such that $E = E_{\mathbb{R}} + iE_{\mathbb{R}}$. The probability space $(E_{\mathbb{R}}^*, \mu)$ is called the *Gaussian space*. We denote by $L^2(E_{\mathbb{R}}^*, \mu)$ the complex Hilbert space of μ-square integrable functions on $E_{\mathbb{R}}^*$.

Theorem

The Hilbert space $L^2(E_{\mathbb{R}}^*, \mu)$ and the Boson Fock space $\Gamma(H)$ over H are unitarily equivalent by the Wiener–Itô–Segal isomorphism whose correspondence is given by

$$\Gamma(H) \ni \phi_\xi := \left(1, \xi, \cdots, \frac{\xi^{\otimes n}}{n!}, \cdots\right) \quad \leftrightarrow \quad \phi_\xi(x) := e^{\langle x, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle} \in (L^2)$$

for $\xi \in H$, where ϕ_ξ is called an exponential vector (or coherent vector) associated with $\xi \in E$.

Then we have a Gelfand triple:

$$(E) \subset \Gamma(H) \subset (E)^*$$

Let $\mathscr{L}(\mathfrak{X}, \mathfrak{Y})$ be the space of all continuous linear operators from a locally convex space $\mathfrak{X}$ into another locally convex space $\mathfrak{Y}$. An element of $\mathscr{L}((E), (E)^*)$ is called a *white noise operator*.

Let $\{a_t, a_t^*\}_t$ be the quantum white noise, i.e. the pair of pointwisely defined annihilation and creation operators. Then the annihilation operator $a(x)$ and creation operator $a^*(y)$ are presented by

$$a(x) = \int x(t) a_t \, dt, \quad a^*(y) = \int y(t) a_t^* \, dt.$$

The quadratic operators of quantum white noise:

$$\Delta_{\mathrm{G}} = \int a_t^2 \, dt, \quad N = \Lambda(I) = \int a_t^* a_t \, dt$$

are called the Gross Laplacian and the number (conservation) operator, respectively. Then the adjoint operator Δ_{G}^* of Δ_{G} with respect to the canonical bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$ on $(E)^* \times (E)$ is represented by

$$\Delta_{\mathrm{G}}^* = \int (a_t^*)^2 \, dt.$$

More generally, consider an operator $K \in \mathscr{L}(E, E^*)$, and then there exists a unique $\tau_K \in E^* \otimes E^*$ such that

$$\langle \tau_K, \eta \otimes \xi \rangle = \langle K\xi, \eta \rangle, \quad \eta, \xi \in E$$

(kernel theorem), and then we have the white noise operators:

$$\Delta_{\mathrm{G}}(K) = \int\int \tau_K(s,t) a_s a_t \, ds dt, \quad \Lambda(K) = \int\int \tau_K(s,t) a_s^* a_t \, ds dt,$$

which are called the generalized Gross Laplacian and the conservation operator, respectively, and then we have

$$\Delta_{\mathrm{G}}^*(K) = \int\int \tau_K(s,t) a_s^* a_t^* \, ds dt$$

Note Let $K, S \in \mathscr{L}(E^*, E^*)$. Then we have

$$\begin{aligned}\int_{E_{\mathbb{R}}^*} \phi_\xi(Ky + Sx) d\mu(y) &= e^{\frac{1}{2}\langle (SS^* + KK^* - I)\xi, \xi \rangle} \phi_{S^*\xi}(x) \\ &= \Gamma(S^*) e^{\frac{1}{2}\Delta_{\mathrm{G}}(SS^* + KK^* - I)} \phi_\xi(x).\end{aligned}$$

Introduction (Transforms and Weyl Operators)

- (Cameron & Martin (1945, 1947): Fourier-Wiener transform)

$$\mathscr{F}\mathscr{W}(f)(x) := \int_{\mathscr{C}} f(y+ix)d\mathbf{w}(y),$$
$$\mathscr{F}\mathscr{W}_{\sqrt{2}}f(x) := \int_{\mathscr{C}} f(\sqrt{2}y+ix)d\mathbf{w}(y),$$

and then we have

$$\mathscr{F}\mathscr{W} = \Gamma(iI)e^{-\frac{1}{2}\Delta_{\mathrm{G}}}, \quad \mathscr{F}\mathscr{W}_{\sqrt{2}} = \Gamma(iI).$$

- (Hida (1970): Gauss transform)

$$\mathscr{G}\mathscr{T}f(x) := \int_{\mathscr{C}} f(iy+x)d\mathbf{w}(y),$$

which is represented as in the operator form by

$$\mathscr{G}\mathscr{T} = e^{-\frac{1}{2}\Delta_{\mathrm{G}}}.$$

- (Cameron and Storvick (1976): analytic Fourier-Feynmann transform) For $q > 0$,

$$\mathscr{F}\mathscr{F} f(x) := \lim_{\lambda>0;\lambda\to -iq} \int_{\mathscr{C}} f(\lambda^{-1/2} y + x) d\mathbf{w}(y),$$

which has an operator representation:

$$\mathscr{F}\mathscr{F} = e^{\frac{i}{2q}\Delta_{\mathrm{G}}}.$$

- (Kubo & Takenaka (1980): S-transform)

$$(S\varphi)(\xi) = \int_{E^*_{\mathbb{R}^*}} \varphi(y+\xi) d\mu(y) = \int_{E^*_{\mathbb{R}^*}} \varphi(y) e^{\langle y, \xi \rangle - \frac{1}{2}|\xi|^2_H} d\mu(y)$$
$$= \langle\langle \varphi, \phi_\xi \rangle\rangle,$$

and then S-transform is represented by

$$S = e^{\frac{1}{2}\Delta_{\mathrm{G}}}.$$

- (Lee (1982): integral transforms)

$$\mathscr{I}\mathscr{T}_{a,b}f(x) = \int_B f(ay+bx)d\mu(y),$$

where $a, b \in \mathbb{C}$, which is represented by

$$\mathscr{I}\mathscr{T}_{a,b} = \Gamma(bI)e^{\frac{1}{2}\left(a^2+b^2-1\right)\Delta_{\mathrm{G}}}$$

and called the Fourier-Gauss transform.

- (Kuo (1982): Fourier transform) For $\Phi \in (E)^*$,

$$\mathscr{K}\mathscr{F}\Phi(\xi) = \int_{E_{\mathbb{R}}^*} e^{-i\langle y,\xi\rangle}\Phi(y)d\mu(y),$$

which is represented by

$$\mathscr{K}\mathscr{F} = \left(\Gamma(-iI)e^{-\frac{1}{2}\Delta_{\mathrm{G}}}\right)^*$$

and called the Kuo's Fourier transform.

- (Kuo (1983; 1991): Fourier-Mehler transform) For $\Phi \in (E)^*$,

$$\mathscr{F}_\theta \Phi(\xi) = S(\Phi)(e^{i\theta}\xi) e^{\frac{i}{2} e^{i\theta} \sin\theta \langle \xi, \xi \rangle} = \left\langle\left\langle \Phi, \phi_{e^{i\theta}\xi} \right\rangle\right\rangle e^{\frac{i}{2} e^{i\theta} \sin\theta \langle \xi, \xi \rangle},$$

which is represented by

$$\mathscr{F}_\theta = \left(\Gamma(e^{i\theta} I) e^{\frac{i}{2} e^{i\theta} \sin\theta \Delta_{\mathrm{G}}} \right)^*$$

and called the Kuo's Fourier-Mehler transform. Then it has been proved that $\{\mathscr{F}_\theta\}_{\theta\in\mathbb{R}}$ is a differentiable one-parameter group with the infinitesimal generator $\left(iN + \frac{i}{2}\Delta_{\mathrm{G}}\right)^*$.

- (Bogachev & Röckner & Schmuland (1996): generalized Mehler semigroups)

$$p_t f(x) = \int_B f(T_t x - y)\nu_t(dy) = (\nu_t * f)(T_t x), \quad t \geq 0,$$

where B is a Banach space, $\{T_t\}_{t\geq 0}$ is a strongly continuous semigroup on B and $\{\nu_t\}_{t\geq 0}$ are probability measures such that

$$\nu_{t+s} = \left(\nu_t \circ T_s^{-1}\right) * \nu_s \quad \text{for all } s,t \geq 0.$$

- (Chung & J (1998): Fourier -Gauss and -Mehler transforms)

$$\mathscr{G}_{\alpha,\beta} = \Gamma(\beta I)e^{\alpha\Delta_{\mathrm{G}}}, \quad \mathscr{F}_{a,b} = \mathscr{G}^*_{\alpha,\beta} = e^{\alpha\Delta^*_{\mathrm{G}}}\Gamma(\beta I), \quad \alpha,\beta \in \mathbb{C}$$

with the characterization of all one-parameter groups induced by the transforms.

- (Chung & J (1997): generalized Fourier -Gauss and -Mehler transforms)

$$\mathscr{G}_{K,S} = \Gamma(S)e^{\Delta_{\mathrm{G}}(K)}, \quad \mathscr{F}_{K,S} = \mathscr{G}_{K,S}^{*} = e^{\alpha \Delta_{\mathrm{G}}^{*}(K)}\Gamma(S^{*})$$

for $K \in \mathscr{L}(E, E^{*})$ and $S \in \mathscr{L}(E, E)$.

- (Chung & J (2000): transformation groups) For $\alpha = (\alpha_1, \cdots, \alpha_5) \in \mathbb{C}$,

$$\mathscr{G}_{\alpha} = \alpha_1 e^{\alpha_5 a^{*}(\zeta)}\Gamma(\alpha_4 I)e^{\alpha_3 \Delta_G} e^{\alpha_2 a(\zeta)},$$
$$\mathscr{F}_{\alpha} = \mathscr{G}_{\alpha}^{*} = \alpha_1 e^{\alpha_2 a^{*}(\zeta)} e^{\alpha_3 \Delta_G^{*}}\Gamma(\alpha_4 I)e^{\alpha_5 a(\zeta)}$$

with the characterization of all one-parameter groups induced by the transformations.

- (van Neerven (2000): Gaussian Mehler semigroups) With one-parameter family $\{\nu_t\}_{t\geq 0}$ of Gaussian measures ν_t with mean 0 and the covariance operator $Q_t \in \mathscr{L}(B^*, B)$ (where B is a reflexive real Banach space). Then it has been proved that Q_t is given by

$$Q_t = \int_0^t T_s Q T_s^* ds$$

for some positive, symmetric operator $Q \in \mathscr{L}(B^*, B)$.

Note. In the white noise theory, the Gaussian Mehler semigroup $\{p_t\}_{t\geq 0}$ is represented by

$$p_t f(x) = \int_E f(T_t x - y)\nu_t(dy) = (\nu_t * f)(T_t x)$$
$$= \Gamma(T_t) e^{\Delta_{\mathrm{G}}(Q_t)} f(x), \quad t \geq 0.$$

- (Lee & J (2021): shifted Gaussian Mehler semigroup)

$$p_t^{\mathrm{s}} = \Gamma(T_t) e^{\Delta_{\mathrm{G}}(Q_t)} e^{a(\eta_t)},$$
$$(p_t^{\mathrm{s}})^* = e^{a^*(\eta_t)} e^{\Delta_{\mathrm{G}}^*(Q_t)} \Gamma(T_t^*), \quad t \geq 0,$$

where $\{\eta_t\}_{t\geq 0} \subset E^*$.

- (Parthasarathy-book: Weyl Operators) For unitary operator U on H and $\zeta \in H$,

$$\mathscr{W}_{U,\zeta} \phi_\xi = e^{-\frac{1}{2}|\zeta|_H^2 - \langle \zeta | U\xi \rangle} \phi_{U\xi+\zeta}, \quad \phi_\xi \in \Gamma(H),$$

and then

$$\mathscr{W}_{U,\zeta} = e^{-\frac{1}{2}|\zeta|_H^2} e^{a^*(\zeta)} \Gamma(U) e^{a(U^* \overline{\zeta})}.$$

- Quantum Laplacians have been studied by several authors, e.g., Accardi & Barhoumi & Ouerdiane (2006), Obata, Saito, Ettaieb, Turki Khalifa, Rguigui and J.

Purposes of This Talk

We discuss transformations:

$$\mathscr{G}_{\alpha,\zeta,K,S,\eta} = \alpha e^{a^*(\zeta)}\Gamma(S)e^{\Delta_G(K)}e^{a(\eta)},$$
$$\mathscr{F}_{\alpha,\zeta,K,S,\eta} = \mathscr{G}^*_{\alpha,\zeta,K,S,\eta} = \alpha e^{a^*(\eta)}e^{\Delta^*_G(K)}\Gamma(S^*)e^{a^*(\zeta)}$$

with characterizing all differentiable one parameter semigroups induced by the transformations, and their quantum extensions:

$$\mathscr{G}^{\mathrm{Q}}_{\alpha,\zeta,K,S,\eta}(\Xi) = \mathscr{G}_{\alpha,\zeta,K,S,\eta}\Xi\mathscr{G}^*_{\alpha,\zeta,K,S,\eta}, \quad \Xi \in \mathscr{L}((E)^*,(E))$$
$$\mathscr{F}^{\mathrm{Q}}_{\alpha,\zeta,K,S,\eta}(\Xi) = \mathscr{F}_{\alpha,\zeta,K,S,\eta}\Xi\mathscr{F}^*_{\alpha,\zeta,K,S,\eta}, \quad \Xi \in \mathscr{L}((E),(E)^*)$$

since $\mathscr{G}_{\alpha,\zeta,K,S,\eta} \in \mathscr{L}((E),(E))$ and $\mathscr{G}^*_{\alpha,\zeta,K,S,\eta} \in \mathscr{L}((E)^*,(E)^*)$.
Then we discuss invariant white noise functionals for and invariant white noise operators for the quantum extensions of the transformation semigroups.

Transformation Groups for White Noise Functionals

We recall the Gelfand triples:

$$(E) \subset \Gamma(H) \cong L^2(E^*_{\mathbb{R}}, \mu) \subset (E)^*, \quad E \subset H \subset E^*.$$

For each $\alpha \in \mathbb{C}$, $\zeta \in E$, $\eta \in E^*$, $K \in \mathscr{L}_{\mathrm{sym}}(E, E^*)$ and $S \in \mathscr{L}(E, E)$, there exists a unique white noise operator, denoted by $\mathscr{G}_{\alpha,\zeta,K,S,\eta}$, such that

$$\mathscr{G}_{\alpha,\zeta,K,S,\eta} = \alpha e^{a^*(\zeta)} \Gamma(S) e^{\Delta_G(K)} e^{a(\eta)} \in \mathscr{L}((E),(E)).$$

For each $\alpha_i \in \mathbb{C}$, $\zeta_i \in E$, $\eta_i \in E^*$, $K_i \in \mathscr{L}_{\mathrm{sym}}(E, E^*)$ and $S_i \in \mathscr{L}(E, E)$ for $i = 1, 2$, we have

$$\begin{aligned} &\mathscr{G}_{\alpha_2,\zeta_2,K_2,S_2,\eta_2} \mathscr{G}_{\alpha_1,\zeta_1,K_1,S_1,\eta_1} \\ &\quad = \mathscr{G}_{\alpha_2\alpha_1 e^{\langle \eta_2, \zeta_1 \rangle + \langle K_2\zeta_1, \zeta_1 \rangle}, S_2\zeta_1 + \zeta_2, K_1 + S_1^* K_2 S_1, S_2 S_1, \eta_1 + S_1^*(\eta_2 + 2K_2\zeta_1)}. \end{aligned}$$

Consider a group $\boldsymbol{G}$ defined by

$$\boldsymbol{G} := \mathbb{C}^* \times E \times \mathscr{L}_{\mathrm{sym}}(E, E^*) \times GL(E) \times E^*,$$

and then

$$\{\mathscr{G}_{\alpha,\zeta,K,S,\eta} : (\alpha, \zeta, K, S, \eta) \in \boldsymbol{G}\}$$

is a subgroup of $GL((E))$.
The adjoint of $\mathscr{G}_{\alpha,\zeta,K,S,\eta}$ is denoted by $\mathscr{F}_{\alpha,\zeta,K,S,\eta}$, i.e.

$$\mathscr{F}_{\alpha,\zeta,K,S,\eta} := \left(\mathscr{G}_{\alpha,\zeta,K,S,\eta}\right)^* = \alpha e^{a^*(\eta)} e^{\Delta_G^*(K)} \Gamma(S^*) e^{a(\zeta)},$$

and then

$$\{\mathscr{F}_{\alpha,\zeta,K,S,\eta} : (\alpha, \zeta, K, S, \eta) \in \boldsymbol{G}\}$$

is a subgroup of $GL((E)^*)$.

One-parameter Semigroups

We now consider a one-parameter family $\{\mathscr{G}_t\}_{t\geq 0} \subset \mathscr{L}((E),(E))$ defined by

$$\mathscr{G}_t := \mathscr{G}_{\alpha(t),\zeta(t),K(t),S(t),\eta(t)},$$

where

$$\begin{aligned}(\alpha,\zeta,K,S,\eta) :&= \{(\alpha(t),\zeta(t),K(t),S(t),\eta(t))\}_{t\geq 0} \\ &\subset \mathbf{G} = \mathbb{C}^* \times E \times \mathscr{L}_{\mathrm{sym}}(E,E^*) \times GL(E) \times E^*.\end{aligned}$$

Then it is obvious that $\mathscr{G}_0 = I$ if and only if $\alpha(0) = 1$, $S(0) = I$, $\zeta(0) = \eta(0) = 0$ and $K(0) = 0$.
For each $t \geq 0$, the adjoint of $\mathscr{G}_t \in \mathscr{L}((E),(E))$ is denoted by $\mathscr{F}_t$, i.e $\mathscr{F}_t = \mathscr{G}_t^*$ and then we have $\mathscr{F}_t \in \mathscr{L}((E)^*,(E)^*)$. Then we have

$$\begin{aligned}\mathscr{F}_t = \mathscr{G}_t^* &= (\mathscr{G}_{\alpha(t),\zeta(t),K(t),S(t),\eta(t)})^* \\ &= \alpha(t) e^{a^*(\eta(t))} e^{\Delta_{\mathrm{G}}^*(K(t))} \Gamma(S(t)^*) e^{a(\zeta(t))}.\end{aligned}$$

Theorem

Let $(\alpha, \zeta, K, S, \eta) : [0,\infty) \to \mathbf{G}$ *be differentiable functions such that* $\{S_t\}_{t\geq 0}$ *has the infinitesimal generator* $S \in \mathscr{L}(E,E)$. *Then* $\{\mathscr{G}_t\}_{t\geq 0}$ *is a differentiable one-parameter semigroup of operators in* $\mathscr{L}((E),(E))$ *if and only if*

$$K(t) = \int_0^t S(s)^* K S(s) ds, \quad K := K'(0),$$

$$\zeta(t) = \int_0^t S(s)\zeta ds, \quad \zeta := \zeta'(0),$$

$$\eta(t) = \int_0^t S(s)^* (\eta + 2K\zeta(s))\, ds, \quad \eta := \eta'(0), \quad K = K'(0),$$

$$\alpha(t) = \exp\left\{\alpha t + \int_0^t \langle \eta(s), \zeta\rangle\, ds\right\}, \quad \alpha := \alpha'(0), \;\; \zeta = \zeta'(0).$$

In this case, the infinitesimal generator of $\{\mathscr{G}_t\}_{t\geq 0}$ *is given by*

$$\alpha I + a^*(\zeta) + \Lambda(S) + \Delta_{\mathrm{G}}(K) + a(\eta).$$

Theorem

Let the function $(\alpha,\zeta,K,S,\eta):[0,\infty)\to\mathbf{G}$ *be given as in previous theorem. Then* $\{\mathscr{F}_t\}_{t\geq 0}$ *is a differentiable one-parameter semigroup of operators in* $\mathscr{L}((E)^*,(E)^*)$ *with the infinitesimal generator*

$$\alpha I + a(\zeta) + \Lambda(S^*) + \Delta_{\mathrm{G}}^*(K) + a^*(\eta).$$

Theorem

Let $(\alpha,\zeta,K,S,\eta)\in\boldsymbol{G}$ *be given. Then the transformation* $\mathscr{G}_{\alpha,\zeta,K,S,\eta}$ *has a unitary extension to* $\Gamma(H)$ *if and only if* $K=0$*,* S *has a unitary extension to* H*,*

$$\alpha = e^{-\frac{1}{2}|\zeta|_0^2} \quad \text{and} \quad \eta = -S^*\overline{\zeta}.$$

We put

$$\mathscr{O}(E;H) = \{g \in GL(E) : |g\xi|_0 = |\xi|_0 \text{ for all } \xi \in E\},$$

which is called the infinite dimensional rotation group. For each $S \in \mathscr{O}(E;H)$, $\Gamma(S)$ can be extended to $\Gamma(H)$ as a unitary operator and then we identify $\Gamma(S)$ with its unitary extension to $\Gamma(H)$.
For each $(S,\zeta) \in \mathscr{O}(E;H) \times E$, we put

$$\mathscr{W}_{S,\zeta} = \mathscr{G}_{e^{-\frac{1}{2}|\zeta|_0^2},\zeta,0,S,-S^*\overline{\zeta}} = e^{-\frac{1}{2}|\zeta|_0^2} e^{a^*(\zeta)} \Gamma(S) e^{-a(S^*\overline{\zeta})},$$

which (its extension) is a Weyl operator.
For a one-parameter family $\{(S(t),\zeta(t))\}_{t\geq 0} \subset \mathscr{O}(E;H) \times E$, we put

$$\mathscr{W}_t := \mathscr{W}_{S(t),\zeta(t)}.$$

Theorem

Let $\{(S(t),\zeta(t))\}_{t\geq 0}\subset \mathscr{O}(E;H)\times E$ *be a family such that* $\{S(t)\}_{t\geq 0}$ *is differentiable with the infinitesimal generator* $S\in\mathscr{L}(E,E)$. *Suppose that for any* $s,t\geq 0$, $\left\langle S(s)^*\overline{\zeta(s)},\zeta(t)\right\rangle$ *is real. Then the one-parameter semigroup* $\{\mathscr{W}_t\}_{t\geq 0}$ *of unitary operators* $\mathscr{W}_t$ *on* $\Gamma(H)$ *is differentiable and whose infinitesimal generator is given by*

$$a^*(\zeta)+\Lambda(S)-a(\overline{\zeta}).$$

In this case, the one-parameter semigroup $\{\mathscr{W}_t^*\}_{t\geq 0}$ *is differentiable and whose infinitesimal generator is given by*

$$a(\zeta)+\Lambda(S^*)-a^*(\overline{\zeta}).$$

Invariant White Noise Distributions

A complex measure ν on $E_{\mathbb{R}}^*$ is called a Hida complex measure if $(E) \subset L^1(\nu)$ and the linear functional

$$\varphi \mapsto \int_{E_{\mathbb{R}}^*} \varphi(x) d\nu(x)$$

is continuous on (E). The Hida complex measure ν induces a white noise distribution $\Phi_\nu \in (E)^*$ such that

$$\langle\langle \Phi_\nu, \varphi \rangle\rangle = \int_{E_{\mathbb{R}}^*} \varphi(x) d\nu(x)$$

for any $\varphi \in (E)$. A Hida complex measure is called a Hida measure if it is a measure. A white noise distribution Φ in $(E)^*$ is said to be positive if $\langle\langle \Phi, \varphi \rangle\rangle \geq 0$ for all nonnegative test functions $\varphi \in (E)$. A distribution $\Phi \in (E)^*$ is induced by a Hida measure if and only if Φ is positive.

A (complex) measure ν on $E_{\mathbb{R}}^*$ is said to be invariant for a one-parameter semigroup $\{T_t\}_{t\geq 0} \subset \mathscr{L}((E),(E))$ if $(E) \subset L^1(\nu)$ and

$$\int_{E_{\mathbb{R}}^*} T_t\varphi(x)d\nu(x) = \int_{E_{\mathbb{R}}^*} \varphi(x)d\nu(x)$$

for all $t \geq 0$ and $\varphi \in (E)$. If $\Phi_\nu \in (E)^*$ is induced by a complex Hida measure ν which is invariant for $\{T_t\}_{t\geq 0}$, then we have

$$\langle\langle \Phi_\nu, \varphi \rangle\rangle = \int_{E_{\mathbb{R}}^*} \varphi(x)d\nu(x) = \int_{E_{\mathbb{R}}^*} T_t\varphi(x)d\nu(x) = \langle\langle \Phi_\nu, T_t\varphi \rangle\rangle = \langle\langle T_t^*\Phi_\nu, \varphi \rangle\rangle$$

for all $\varphi \in (E)$.

Theorem

Let ν be a Hida complex measure corresponding to $\Phi_\nu \in (E)^*$. Then ν is invariant for a one-parameter semigroup $\{T_t\}_{t\geq 0} \subset \mathscr{L}((E),(E))$ if and only if Φ_ν is invariant for the one-parameter semigroup $\{T_t^*\}_{t\geq 0} \subset \mathscr{L}((E)^*,(E)^*)$.

Theorem

Let $(\alpha, \zeta, K, S, \eta) : [0, \infty) \to \mathbf{G}$ *be differentiable functions such that* $\{S_t\}_{t \ge 0}$ *has the infinitesimal generator* $S \in \mathscr{L}(E, E)$*. Suppose that* (α, ζ, K, η) *are explicitly given such that* $\{\mathscr{G}_t\}_{t \ge 0}$ *is a differentiable one-parameter semigroup, and*

- **(α)** $\alpha_\infty := \lim_{t \to \infty} \alpha(t)$ *exists in* $\mathbb{C}$ *such that* $\alpha_\infty \neq 0$,
- **(K)** $K_\infty := \lim_{t \to \infty} K(t)$ *exists in* $\mathscr{L}(E, E^*)$,
- **(η)** $\eta_\infty := \lim_{t \to \infty} \eta(t)$ *exists in* E^*.

Then the following assertions are equivalent:

- (i) *there exists* $\Phi \in (E)^*$ *such that* Φ *is invariant for* $\{\mathscr{G}_t^*\}_{t \ge 0}$,
- (ii) *there exists* $\Psi \in (E)^*$ *such that the limit* $\overline{\Psi} := \lim_{t \to \infty} \Gamma(S(t)^*) e^{a(\zeta(t))} \Psi$ *exists in* $(E)^*$.

In this case, the invariant vector Φ *for* $\{\mathscr{G}_t^*\}_{t \ge 0}$ *is explicitly given by*

$$\Phi = \alpha_\infty \overline{\Psi} \diamond \Psi_{K_\infty} \diamond \phi_{\eta_\infty}, \quad \Psi_{K_\infty} = \left(\frac{\tau_{K_\infty}^{\widehat{\otimes} n}}{n!} \right), \quad \phi_{\eta_\infty} = \left(\frac{\eta_\infty^{\widehat{\otimes} n}}{n!} \right).$$

Quantum Extension of One Parameter Semigroups

We recall the Gelfand triples:

$$(E) \subset \Gamma(H) \cong L^2(E^*_{\mathbb{R}}, \mu) \subset (E)^*, \quad E \subset H \subset E^*.$$

Then $\mathscr{L}((E),(E)^*)$ is the space of all white noise operators, and then we can consider the subspaces:

$$\mathscr{L}(\mathfrak{X},\mathfrak{Y}), \quad \mathfrak{X},\mathfrak{Y} = (E),\Gamma(H),(E)^*.$$

The quantum extensions of $\mathscr{G}_{\alpha,\zeta,K,S,\eta}$-transform and $\mathscr{F}_{\alpha,\zeta,K,S,\eta}$-transform, denoted by $\mathscr{G}^{\mathbf{Q}}_{\alpha,\zeta,K,S,\eta}$ and $\mathscr{F}^{\mathbf{Q}}_{\alpha,\zeta,K,S,\eta}$, are defined by

$$\mathscr{G}^{\mathbf{Q}}_{\alpha,\zeta,K,S,\eta}(\Xi) = \mathscr{G}_{\alpha,\zeta,K,S,\eta}\Xi(\mathscr{G}_{\alpha,\zeta,K,S,\eta})^*, \quad \Xi \in \mathscr{L}((E)^*,(E)),$$
$$\mathscr{F}^{\mathbf{Q}}_{\alpha,\zeta,K,S,\eta}(\Xi) = \mathscr{F}_{\alpha,\zeta,K,S,\eta}\Xi(\mathscr{F}_{\alpha,\zeta,K,S,\eta})^*, \quad \Xi \in \mathscr{L}((E),(E)^*),$$

respectively.

Theorem

For each

$$(\alpha, \zeta, K, S, \eta) \in \mathbb{C} \times E \times \mathscr{L}_{\mathrm{sym}}(E, E^*) \times \mathscr{L}(E, E) \times E^*,$$

the operators $\mathscr{G}^{\mathrm{Q}}_{\alpha,\zeta,K,S,\eta}$ *and* $\mathscr{F}^{\mathrm{Q}}_{\alpha,\zeta,K,S,\eta}$ *are continuous linear operators acting on* $\mathscr{L}((E)^*, (E))$ *and* $\mathscr{L}((E), (E)^*)$*, respectively.*

For a given function

$$\begin{aligned}(\alpha, \zeta, K, S, \eta) :&= \{(\alpha(t), \zeta(t), K(t), S(t), \eta(t))\}_{t \geq 0} \\ &\subset \boldsymbol{G} := \mathbb{C}^* \times E \times \mathscr{L}_{\mathrm{sym}}(E, E^*) \times GL(E) \times E^*,\end{aligned}$$

the quantum extensions of $\mathscr{G}_t$ and $\mathscr{F}_t$ are denoted by $\mathscr{G}^{\mathrm{Q}}_t$ and $\mathscr{F}^{\mathrm{Q}}_t$, i.e., $\mathscr{G}^{\mathrm{Q}}_t$ and $\mathscr{F}^{\mathrm{Q}}_t$ are operators acting on $\mathscr{L}((E)^*, (E))$ and $\mathscr{L}((E), (E)^*)$, respectively, and defined by

$$\mathscr{G}^{\mathrm{Q}}_t(\Xi) = \mathscr{G}_t \Xi \mathscr{G}^*_t = \mathscr{G}_t \Xi \mathscr{F}_t, \quad \Xi \in \mathscr{L}((E)^*, (E)),$$

$$\mathscr{F}^{\mathrm{Q}}_t(\Xi) = \mathscr{F}_t \Xi \mathscr{F}^*_t = \mathscr{F}_t \Xi \mathscr{G}_t, \quad \Xi \in \mathscr{L}((E), (E)^*).$$

Theorem

Let $(\alpha,\zeta,K,S,\eta):[0,\infty)\to\mathbf{G}$ *be differentiable functions such that* $\{S_t\}_{t\geq 0}$ *has the infinitesimal generator* $S\in\mathscr{L}(E,E)$*. Suppose that* (α,ζ,K,η) *are explicitly given such that* $\{\mathscr{G}_t\}_{t\geq 0}$ *is a differentiable one-parameter semigroup. Then* $\left\{\mathscr{G}_t^{\mathrm{Q}}\right\}_{t\geq 0}$ *is a differentiable one-parameter semigroup of operators acting on* $\mathscr{L}((E)^*,(E))$ *with the infinitesimal generator* L_{Q}*, where* $L=\alpha I+a^*(\zeta)+\Lambda(S)+\Delta_{\mathrm{G}}(K)+a(\eta)$ *is the infinitesimal generator of* $\{\mathscr{G}_t\}_{t\geq 0}$ *and* L_{Q} *is the quantum extension of* L*, i.e.*

$$L_{\mathrm{Q}}(\Xi)=L\Xi+\Xi L^*,\quad \Xi\in\mathscr{L}((E)^*,(E)).$$

Also, $\left\{\mathscr{F}_t^{\mathrm{Q}}\right\}_{t\geq 0}$ *is a differentiable one-parameter semigroup of operators acting on* $\mathscr{L}((E),(E)^*)$ *with the infinitesimal generator* $(L^*)_{\mathrm{Q}}$*, where* $(L^*)_{\mathrm{Q}}$ *is the quantum extension of* L^**, i.e.*

$$(L^*)_{\mathrm{Q}}(\Xi)=L^*\Xi+\Xi L,\quad \Xi\in\mathscr{L}((E),(E)^*).$$

Invariant White Noise Operators

Theorem

Let $(\alpha,\zeta,K,S,\eta):[0,\infty)\to\mathbf{G}$ *be differentiable functions as in previous theorem. Suppose that*

- **(α)** $\alpha_\infty := \lim_{t\to\infty}\alpha(t)$ *exists in* $\mathbb{C}$ *such that* $\alpha_\infty \neq 0$,
- **(K)** $K_\infty := \lim_{t\to\infty}K(t)$ *exists in* $\mathscr{L}(E,E^*)$,
- **(η)** $\eta_\infty := \lim_{t\to\infty}\eta(t)$ *exists in* E^*.

There exists $\Xi\in\mathscr{L}((E),(E)^*)$ *such that* Ξ *is invariant for* $\{\mathscr{F}_t^{\mathrm{Q}}\}_{t\geq 0}$ *if and only if there exists* $\Upsilon\in\mathscr{L}((E),(E)^*)$ *such that the limit*

$$\overline{\Upsilon} := \lim_{t\to\infty}\Gamma(S(t)^*)e^{a(\zeta(t))}\Upsilon e^{a^*(\zeta(t))}\Gamma(S(t))\in\mathscr{L}((E),(E)^*).$$

In this case, an invariant operator Ξ *for* $\{\mathscr{F}_t^{\mathrm{Q}}\}_{t\geq 0}$ *is explicitly given by*

$$\Xi = \alpha_\infty\overline{\Upsilon}\diamond\mathfrak{G}_{K_\infty,\eta_\infty},\quad \mathfrak{G}_{K_\infty,\eta_\infty} = e^{a^*(\eta_\infty)+\Delta_{\mathrm{G}}^*(K_\infty)}e^{\Delta_{\mathrm{G}}(K_\infty)+a(\eta_\infty)}.$$

Consider the Gelfand triple:

$$E \subset H \subset E^*$$

for a Hilbert space H. It is well-known that

$$\Gamma(H \oplus H) \cong \Gamma(H) \otimes \Gamma(H) \quad \text{(unitarily isomorphic)},$$

and then we can see that $(E \oplus E)^* \cong (E)^* \otimes (E)^* \cong \mathscr{L}((E),(E)^*)$ and then by applying canonical topological isomorphisms ($\mathscr{U}$ and $\mathscr{V}$), we can study (general including entangled) transformations $\mathfrak{F}$ on white noise operators as the following diagram:

$$\begin{array}{ccccc}
\mathscr{L}((E),(E)^*) & \xrightarrow{\mathscr{U}} & (E)^* \otimes (E)^* & \xrightarrow{\mathscr{V}} & (E \oplus E)^* \\
\mathfrak{F} \downarrow & & \downarrow & & \downarrow \mathscr{F}=\mathscr{G}^* \\
\mathscr{L}((E),(E)^*) & \xleftarrow[\mathscr{U}^{-1}]{} & (E)^* \otimes (E)^* & \xleftarrow[\mathscr{V}^{-}]{} & (E \oplus E)^*
\end{array}$$

Thank you very much!