Towards a classification of multi-faced independences

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17 January 2022

QP42

Bengaluru – Moeckow

Overview

1. Universal Products
2. Representing universal products
3. Multi-faced independences
4. Products of rep's and universal lifts
5. Classification

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1 Universal Products

2 Representing universal products

3 Multi-faced independences

4 Products of rep's and universal lifts

5 Classification

Non-commutative independence

Fix product operation for linear functionals on $*$-algebras

$$\bigtimes_i B_i{}' \ni (\varphi_i)_i \mapsto \bigodot_i \varphi_i \in \Big(\bigsqcup_i B_i\Big)'$$

Def: $\odot$-independence of random variables $j_i \colon B_i \to \mathcal{A}$

$$\Phi \circ \bigsqcup_i j_i = \bigodot_i (\Phi \circ j_i)$$

joint distribution = product of marginals

uau-products

Def: universal product, Ben Ghorbal & Schürmann '05

$$B_1' \times B_2' \ni (\varphi_1, \varphi_2) \mapsto \varphi_1 \odot \varphi_2 \in (B_1 \sqcup B_2)'$$

product operation (for arbitrary algebras B_1, B_2) which is

- **u**nital in the sense that $0 \odot \varphi = \varphi = \varphi \odot 0$
- **a**ssociative
- **u**niversal in the sense that (for hom's $j_i \colon B_i \to A_i$)

$$(\varphi_1 \odot \varphi_2) \circ (j_1 \sqcup j_2) = (\varphi_1 \circ j_1) \odot (\varphi_2 \circ j_2)$$

Examples

tensor, free, monotone, anti-monotone, Boolean

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$$(\varphi_1 \odot \varphi_2) \circ (j_1 \sqcup j_2) = (\varphi_1 \circ j_1) \odot (\varphi_2 \circ j_2)$$

Theorem: classification (Muraki 2002, 2013)

positivity or *double normalization* $\Longrightarrow$ no other examples

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Joint representations on $H_1 \otimes H_2$

Example: monotone product

Observation

For $\varphi_k = \langle \Omega, \pi_k(\cdot)\Omega \rangle$, $\pi_k : A_k \to L(H)$, $H_k = \mathbb{C}\Omega \oplus \hat{H}_k$:

$$\varphi_1 \odot \varphi_2 = \langle \Omega, (\pi_1 \odot \pi_2)(\cdot)\Omega \rangle, \quad \text{where}$$

$$\pi_1 \odot \pi_2(a) = \begin{cases} \pi_1(a) \otimes \mathrm{id} & a \in A_1, \odot \in \{\otimes, \triangleleft\} \\ \pi_1(a) \otimes P_\Omega & a \in A_1, \odot \in \{\diamond, \triangleright\} \\ \mathrm{id} \otimes \pi_2(a) & a \in A_2, \odot \in \{\otimes, \triangleright\} \\ P_\Omega \otimes \pi_2(a) & a \in A_2, \odot \in \{\diamond, \triangleleft\} \end{cases}$$

Joint representations on $H_1 \star H_2$

Def: **free product of spaces**

$$H_1 \star H_2 := \bigoplus_{\varepsilon_1 \neq \ldots \neq \varepsilon_n} \hat{H}_{\varepsilon_1} \otimes \cdots \otimes \hat{H}_{\varepsilon_n}$$

Observation

For $\varphi_k = \langle \Omega, \pi_k(\cdot)\Omega \rangle$, $\pi_k \colon A_k \to L(H)$, $H_k = \mathbb{C}\Omega \oplus \hat{H}_k$:

$$\varphi_1 \star \varphi_2 = \langle \Omega, (\pi_1 \star_\ell \pi_2)(\cdot)\Omega \rangle = \langle \Omega, (\pi_1 \star_r \pi_2)(\cdot)\Omega \rangle$$
$$\varphi_1 \diamond \varphi_2 = \langle \Omega, (\pi_1 \diamond \pi_2)(\cdot)\Omega \rangle$$

where

$$\pi_1 \star_\ell \pi_2(a) = \begin{cases} \pi_1(a) \otimes \mathrm{id} & a \in A_1 \\ \pi_2(a) \otimes \mathrm{id} & a \in A_2 \end{cases}, \quad \pi_1 \star_r \pi_2(a) = \begin{cases} \mathrm{id} \otimes \pi_1(a) & a \in A_1 \\ \mathrm{id} \otimes \pi_2(a) & a \in A_2 \end{cases}$$

$$\pi_1 \diamond \pi_2(a) = \pi_k(a) \otimes P_\Omega \quad a \in A_k$$

Benefits of a joint representation

Observation

- tensor, boolean, monotone and antimonotone product can be represented on $H_1 \otimes H_2$
- free and boolean product can be represented on $H_1 \star H_2$
- joint representation helps to prove **positivity**
- joint representation helps to prove **associativity**

Associativity of monotone product (Franz)

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Independence

Fix product operation for *m*-faced algebras $B_i = B_i^{(1)} \sqcup \cdots \sqcup B_i^{(m)}$

$$\bigtimes_i {B_i}' \ni (\varphi_i)_i \mapsto \bigodot_i \varphi_i \in \Big(\bigsqcup_i B_i\Big)'$$

Def: $\odot$-independence of *m*-faced rv's $j_i\colon B_i \to \mathcal{A}$

$$\Phi \circ \bigsqcup_i j_i = \bigodot_i (\Phi \circ j_i)$$

joint distribution = product of marginals

Multi-faced uau-products

Def: *m*-faced universal product (Manzel & Schürmann 2017)

$$B_1' \times B_2' \ni (\varphi_1, \varphi_2) \mapsto \varphi_1 \odot \varphi_2 \in (B_1 \sqcup B_2)'$$

product operation (for arbitrary *m*-faced algebras B_1, B_2) which is

- **u**nital in the sense that $0 \odot \varphi = \varphi = \varphi \odot 0$
- **a**ssociative
- **u**niversal in the sense that (for *m*-faced hom's $j_i\colon B_i \to A_i$)

$$(\varphi_1 \odot \varphi_2) \circ (j_1 \sqcup j_2) = (\varphi_1 \circ j_1) \odot (\varphi_2 \circ j_2)$$

Positivity

Def: restricted state

$\varphi \colon B \to \mathbb{C}$ on a $*$-algebra B whose unitalization is a state

Def: positive universal product

universal product $\odot$ s.t.

$$\varphi_1, \varphi_2 \text{ rest. states} \implies \varphi_1 \odot \varphi_2 \text{ rest. state}$$

Examples from joint representations

Observation

1. $\exists$ positive 4-faced universal product represented on $H_1 \otimes H_2$
2. $\exists$ positive 3-faced universal product represented on $H_1 \star H_2$

All previous positive examples included

- bimonotone II (G; Gu, Skoufranis & Hasebe): restriction of 1
- bifreeness (Voiculescu): restriction of case 2
- free-boolean, free-free-boolean (Liu): (restriction of) 2
- biboolean, bimonotone I (Gu & Skoufranis + Hasebe): $\ngeq$

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Universal products of representations

Def: universal product of $*$-rep's for $\boxtimes \in \{\otimes, \star\}$

$$*\text{-rep}(B_1) \times *\text{-rep}(B_2) \ni (\pi_1, \pi_2) \mapsto \pi_1 \odot \pi_2 \in *\text{-rep}(B_1 \sqcup B_2)$$

product operation which is unital, associative, algebraically universal and **spatially universal**, i.e.

$$\begin{tikzcd} H_k \arrow[r, "\pi_k(\cdot)"] \arrow[d, "W_k"'] & H_k \arrow[d, "W_k"] \\ G_k \arrow[r, "\sigma_k(\cdot)"'] & G_k \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} H_1 \boxtimes H_2 \arrow[r, "(\pi_1 \odot \pi_2)(\cdot)"] \arrow[d, "W_1 \boxtimes W_2"'] & H_1 \boxtimes H_2 \arrow[d, "W_1 \boxtimes W_2"] \\ G_1 \boxtimes G_2 \arrow[r, "(\sigma_1 \odot \sigma_2)(\cdot)"'] & G_1 \boxtimes G_2 \end{tikzcd}$$

Theorem (G, Hasebe & Ulrich)

m-faced UP of $*$-rep's $\rightsquigarrow$ positive m-faced UP of functionals

Universal lifts I

Def: left universal lift

family of $*$-hom's $\lambda_{H_1,H_2}\colon L(H_1) \to L(H_1 \boxtimes H_2)$ s.t.

- left restriction property: $\lambda_{H_1,H_2}(T) \upharpoonright H_1 = T$
- left associativity: $\lambda_{H_1,H_2\boxtimes H_3} = \lambda_{H_1\boxtimes H_2,H_3} \circ \lambda_{H_1,H_2}$
- left spatial universality:

$$\begin{tikzcd} H_1 \arrow[r, "T^{(*)}"] \arrow[d, "W_1"'] & H_1 \arrow[d, "W_1"] \\ G_1 \arrow[r, "S^{(*)}"'] & G_1 \end{tikzcd} \Longrightarrow \begin{tikzcd} H_1 \boxtimes H_2 \arrow[r, "\lambda_{H_1,H_2}(T)"] \arrow[d, "W_1\boxtimes W_2"'] & H_1 \boxtimes H_2 \arrow[d, "W_1\boxtimes W_2"] \\ G_1 \boxtimes G_2 \arrow[r, "\lambda_{G_1,G_2}(S)"'] & G_1 \boxtimes G_2. \end{tikzcd}$$

Universal lifts II

Def: **right universal lift**

family of $*$-hom's $\rho_{H_1,H_2}\colon L(H_2) \to L(H_1 \boxtimes H_2)$ s.t.

- right restriction property: $\rho_{H_1,H_2}(T) \restriction H_2 = T$
- right associativity: $\rho_{H_1 \boxtimes H_2, H_3} = \rho_{H_1, H_2 \boxtimes H_3} \circ \rho_{H_2,H_3}$
- right spatial universality:

$$\begin{tikzcd} H_2 \arrow[r, "T^{(*)}"] \arrow[d, "W_2"'] & H_2 \arrow[d, "W_2"] \\ G_2 \arrow[r, "S^{(*)}"'] & G_2 \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} H_1 \boxtimes H_2 \arrow[r, "\rho_{H_1,H_2}(T)"] \arrow[d, "W_1 \boxtimes W_2"'] & H_1 \boxtimes H_2 \arrow[d, "W_1 \boxtimes W_2"] \\ G_1 \boxtimes G_2 \arrow[r, "\rho_{G_1,G_2}(S)"'] & G_1 \boxtimes G_2. \end{tikzcd}$$

Universal lifts III

Def: universal lift

pair (λ, ρ) of left and right universal lift to $\boxtimes$ s.t.

- middle associativity: $\rho_{H_1,H_2\boxtimes H_3} \circ \lambda_{H_2,H_3} = \lambda_{H_1\boxtimes H_2,H_3} \circ \rho_{H_1,H_2}$

Theorem (G, Hasebe & Ulrich)

The following objects are in 1-to-1 correspondence:

1. m-faced UP's of $*$-rep's for $\boxtimes$
2. m-tuples of 1-faced UP's of $*$-rep's for $\boxtimes$
3. m-tuples of universal lifts to $\boxtimes$

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A universal deformation

Definition

For $T \in L(H)$, $H = \mathbb{C}\Omega \oplus \hat{H}$ and $\gamma \in \mathbb{T} \cup \{0\}$:

- decompose $T = \begin{pmatrix} \tau & (t')^* \\ t & \hat{T} \end{pmatrix}$ with $\tau \in \mathbb{C}$; $t, t' \in \hat{H}$; $\hat{T} \in L(\hat{H})$
- define $T_\gamma := \begin{pmatrix} |\gamma|\tau & \overline{\gamma}(t')^* \\ \gamma t & |\gamma|\hat{T} \end{pmatrix}$

Lemma

$T \mapsto T_\gamma$ is a $*$-homomorphism

Lifts to the tensor product

Theorem (G, Hasebe & Ulrich)

Every universal lift to $\otimes$ is of the form $(\lambda^\gamma, \rho^\delta)$ where

- $\lambda^\gamma(T) = T \otimes P_\Omega + T_\gamma \otimes P_{\Omega^\perp}$
- $\rho^\delta(S) = P_\Omega \otimes S + P_{\Omega^\perp} \otimes S_\delta$
- either $\gamma = \delta \in \mathbb{T}$, or at least one parameter equals zero.

Corollary

$\exists$ uncountably many 2-faced independences represented on $\otimes$

Lifts to the tensor product – example

2-faced independences represented on $H_1 \otimes H_2$

Parameters that determine the 2-faced universal product ${}_{\gamma_1}^{\gamma_2}\otimes\otimes_{\delta_1}^{\delta_2}$

face 2 / face 1	tensor $(\gamma_2 = \delta_2 \in \mathbb{T})$	antimonotone $(\gamma_2 \in \mathbb{T}, \delta_2 = 0)$	monotone $(\gamma_2 = 0, \delta_2 \in \mathbb{T})$	boolean $(\gamma_2 = \delta_2 = 0)$
tensor $(\gamma_1 = \delta_1 \in \mathbb{T})$	$\gamma_1\overline{\gamma_2}$	$\gamma_1\overline{\gamma_2}$	$\delta_1\overline{\delta_2}$	$\varnothing$
antimonotone $(\gamma_1 \in \mathbb{T}, \delta_1 = 0)$	$\gamma_1\overline{\gamma_2}$	$\gamma_1\overline{\gamma_2}$	$\gamma_1\overline{\delta_2}$	$\varnothing$
monotone $(\gamma_1 = 0, \delta_1 \in \mathbb{T})$	$\delta_1\overline{\delta_2}$	$\delta_1\overline{\gamma_2}$	$\delta_1\overline{\delta_2}$	$\varnothing$
boolean $(\gamma_1 = \delta_1 = 0)$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

Lifts to the free product

Theorem (G, Hasebe & Ulrich)

Every universal lift to $\star$ is one of the following:

1. $(\overrightarrow{\lambda}^{\gamma}, \overrightarrow{\rho}^{\delta})$ with $\gamma, \delta \in \mathbb{T}$
2. $(\overleftarrow{\lambda}^{\gamma}, \overleftarrow{\rho}^{\delta})$ with $\gamma, \delta \in \mathbb{T}$
3. $(\overrightarrow{\lambda}^{0}, \overrightarrow{\rho}^{0}) = (\overleftarrow{\lambda}^{0}, \overleftarrow{\rho}^{0})$

where

$$\overrightarrow{\lambda}^{\gamma}_{H_1,H_2}(T) = T \otimes P_\Omega + T_\gamma \otimes P_{\Omega^\perp}$$
$$\overrightarrow{\rho}^{\delta}_{H_1,H_2}(S) = S \otimes P_\Omega + S_\delta \otimes P_{\Omega^\perp}$$

and $\overleftarrow{\lambda}^{\gamma}_{H_1,H_2}, \overleftarrow{\rho}^{\delta}_{H_1,H_2}$ analogously from the right.

2-faced independences represented on $H_1 \star H_2$

Parameters that determine the 2-faced UP's ${}_{\gamma_1}^{\gamma_2}\overset{\rightarrow\rightarrow}{*\,*}{}_{\delta_1}^{\delta_2}$ and ${}_{\gamma_1}^{\gamma_2}\overset{\rightarrow\leftarrow}{*\,*}{}_{\delta_1}^{\delta_2}$

face 1 \ face 2	left free $(\vec{\lambda}, \vec{\rho})$ $(\gamma_2, \delta_2 \in \mathbb{T})$	right free $(\overleftarrow{\lambda}, \overleftarrow{\rho})$ $(\gamma_2, \delta_2 \in \mathbb{T})$	boolean $(\gamma_2 = \delta_2 = 0)$
left free $(\vec{\lambda}, \vec{\rho})$ $(\gamma_1, \delta_1 \in \mathbb{T})$	$\gamma_1\overline{\gamma_2}, \delta_1\overline{\delta_2}$	$\gamma_1\overline{\delta_2}, \delta_1\overline{\gamma_2}$	$\varnothing$
boolean $(\gamma_1 = \delta_1 = 0)$	$\varnothing$	$\varnothing$	$\varnothing$

Open problems

Are there more positive 2-faced independences?

Varšo's combinatorial approach allows

- tensor-free independence
- parameters of modulus < 1

positivity unknown!

Positive UP $\rightsquigarrow$ UP of $*$-rep's?

- other representation spaces besides $H_1 \otimes H_2$, $H_1 \star H_2$?
- more general theory without fixing product of spaces?

Understanding the new independences

- concise mixed-moment and moment-cumulant formulae
- understand limit distributions

आपको धन्यवाद