Structure of Block Quantum Dynamical Semigroups and their Product Systems

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Introduction

Abbreviations:

CP	Completely positive
CB	Completely Bounded
UCP	Unital Completely Positive
UNCP	Unital Normal Completely Positive
QDS	Quantum Dynamical Semigroup
QMS	Quantum Markov Semigroup.

Outline

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2. Structure of block quantum dynamical semigroups
   - Introduction
   - Hilbert C^*-modules
   - Structure of block QDS

3. References

CP maps

Definitions

Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a C^*-algebra.

For $n \in \mathbb{N}$, $\quad M_n(\mathcal{A}) \subseteq M_n(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^{\oplus^n})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a linear map.
For $n \in \mathbb{N}$, define $\phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ by

$$\phi_n([a_{ij}]_{i,j=1}^n) = [\phi(a_{ij})]_{i,j=1}^n, \quad \text{for } [a_{ij}]_{i,j=1}^n \in M_n(\mathcal{A}).$$

$$\mathcal{A} \otimes M_n \simeq M_n(\mathcal{A}) \quad \Longrightarrow \quad \phi_n = \phi \otimes I_n : \mathcal{A} \otimes M_n \to \mathcal{B} \otimes M_n.$$

ϕ is said to be n-positive if ϕ_n is positive.

ϕ is said to be completely positive (CP) if ϕ is n-positive for all $n \in \mathbb{N}$.
ϕ is said to be completely bounded (CB) if $sup_n \|\phi_n\| < \infty$.

Theorem (Stinespring's dilation for CP maps. 1955)

$$\phi : \mathcal{A} \xrightarrow{CP} \mathcal{B}(\mathcal{H}) \implies \exists(\pi, V, \mathcal{K}) \sim \begin{cases} \mathcal{K} - \textit{Hilbert space} \\ \pi : \mathcal{A} \to \mathcal{B}(\mathcal{K}) \quad \textit{repn.} \\ V \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \end{cases}$$

such that

$$\phi(a) = V^*\pi(a)V, \quad a \in \mathcal{A}.$$

$$\begin{array}{ccc} & & \mathcal{B}(\mathcal{K}) \\ & \overset{\pi}{\nearrow} & \big\downarrow V^*(\cdot)V \\ \mathcal{A} & \xrightarrow{\phi} & \mathcal{B}(\mathcal{H}) \end{array}$$

Such a triple $(\pi, V, \mathcal{K})$ is called a Stinespring's dilation for ϕ.

Let $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{Z}_+$.

Definition

Let $\mathcal{A}$ be a unital C^*-algebra. A family $\phi = (\phi_t)_{t \in \mathbb{T}}$ of CP maps on $\mathcal{A}$ is said to be a quantum dynamical semigroup (QDS) or one-parameter CP-semigroup if

1. $\phi_{s+t} = \phi_s \circ \phi_t$ for all $t \in \mathbb{T}$,
2. $\phi_0(a) = a$ for all $a \in \mathcal{A}$,
3. $\phi_t(\mathbf{1}) \leq \mathbf{1}$ for all $t \in \mathbb{T}$, (contractivity)
4. The map $t \mapsto \phi_t(a)$ is continuous for all $a \in \mathcal{A}$. (strong continuity)

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It is said to be conservative QDS or Quantum Markov semigroup (QMS) if ϕ_t is unital for all $t \in \mathbb{T}$.
If ϕ is a semigroup of CP maps on a von Neumann algebra $\mathcal{A}$, we assume every ϕ_t to be normal. (τ is normal $\iff$ $a_\lambda \uparrow a \implies \tau(a_\lambda) \uparrow \tau(a)$)

Let $\mathcal{A}$ be a unital C^*-algebra. Let $p \in \mathcal{A}$ be a projection. Set $p' = \mathbf{1} - p$.

$$x = \begin{pmatrix} pxp & pxp' \\ p'xp & p'xp' \end{pmatrix} \in \begin{pmatrix} p\mathcal{A}p & p\mathcal{A}p' \\ p'\mathcal{A}p & p'\mathcal{A}p' \end{pmatrix}. \tag{1}$$

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Let $p \in \mathcal{A}$ and $q \in \mathcal{B}$ be projections. We say that a map $\Phi : \mathcal{A} \to \mathcal{B}$ is a *block map* (with respect to p and q) if Φ respects the above block decomposition. i.e., for all $x \in \mathcal{A}$ we have

$$\Phi(x) = \begin{pmatrix} \Phi(pxp) & \Phi(pxp') \\ \Phi(p'xp) & \Phi(p'xp') \end{pmatrix} \in \begin{pmatrix} q\mathcal{B}q & q\mathcal{B}q' \\ q'\mathcal{B}q & q'\mathcal{B}q' \end{pmatrix}. \tag{2}$$

If $\Phi : \mathcal{A} \to \mathcal{B}$ is a block map, then we get:
$\phi_{11} : p\mathcal{A}p \to q\mathcal{B}q, \quad \phi_{12} : p\mathcal{A}p' \to q\mathcal{B}q',$
$\phi_{21} : p'\mathcal{A}p \to q'\mathcal{B}q, \quad \phi_{22} : p'\mathcal{A}p' \to q'\mathcal{B}q'.$

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We will look at BLOCK CP MAPS and their SEMIGROUPS!

Block CP maps

Introduction

$$\mathcal{B}(\mathcal{H}\oplus\mathcal{K}) \ni \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \geq 0 \iff \begin{cases} A, D \geq 0 \text{ and} \\ B = A^{\frac{1}{2}} T D^{\frac{1}{2}} \text{ for some contraction } T. \end{cases}$$

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Suppose $\Phi : M_2(\mathcal{A}) \to M_2(\mathcal{B})$ is a CP map, where $\mathcal{A}$ is a unital C^*-algebra.

$$\Phi = \begin{pmatrix} \phi_1 & \psi \\ \psi^* & \phi_2 \end{pmatrix} \text{ is block CP } \implies \begin{cases} \phi_1, \phi_2 \text{ are CP and} \\ \psi \text{ is CB, where } \psi^*(a) = \psi(a^*)^*, a \in \mathcal{A}. \end{cases}$$

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Theorem (Paulsen and Suen [PS85])

Let $\mathcal{A}$ be a unital C^*-algebra. Suppose $\Phi : M_2(\mathcal{A}) \to M_2(\mathcal{B}(\mathcal{H}))$ defined by $\Phi = \begin{pmatrix} \phi & \psi \\ \psi^* & \phi \end{pmatrix}$ is completely positive, and $(\mathcal{K}, \eta, V)$ is a Stinespring representation for ϕ. Then there is a contraction $T : \mathcal{K} \to \mathcal{K}$ with $\eta(a)T = T\eta(a)$ for all $a \in \mathcal{A}$ such that $\psi(a) = V^*T\eta(a)V$ for all $a \in \mathcal{A}$.

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Bhat and Mukherjee studied semigroups of block CP mas on $\mathcal{B}(\mathcal{H} \oplus \mathcal{K})$.

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Definition (Hilbert C^*-module)

E-complex vector space, $\mathcal{B}$- a C^*-algebra

$$E\text{-Hilbert } \mathcal{B}\text{-module} \iff \begin{cases} E \text{ is a right } \mathcal{B}\text{-module}, \\ E \text{ has a } \mathcal{B}\text{-valued inner product } \langle \cdot, \cdot \rangle, \\ E \text{ is complete in the norm: } \|x\| = \sqrt{\|\langle x, x \rangle\|}. \end{cases}$$

Cauchy-Schwarz inequality

$E-$ semi inner product $\mathcal{B}$-module,

$$\langle x, y \rangle \langle y, x \rangle \leq \|\langle y, y \rangle\| \langle x, x \rangle, \quad \text{for all } x, y \in E.$$

Consider $N = \{x \in E : \langle x, x \rangle = 0\}$ is a $\mathcal{B}$-submodule. Now E/N is a $\mathcal{B}$-mudule with the natural inner product.

Significant difference from Hilbert spaces?

self-duality, adjointability, complementability

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Definition (two-sided)

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. A Hilbert $\mathcal{B}$-module E with a non-degenerate representation $\pi : \mathcal{A} \to \mathfrak{B}^a(E)$ is said to be a *Hilbert $\mathcal{A}$-$\mathcal{B}$-module* or *$\mathcal{A}$-$\mathcal{B}$-correspondence*.
(π is non-degenerate if $\overline{\operatorname{span}}\, \pi(\mathcal{A})E = E$)

Definition (tensor product)

Let E be a Hilbert $\mathcal{A}$-$\mathcal{B}$-module and F be a Hilbert $\mathcal{B}$-$\mathcal{C}$-module. Then

$$\langle x \otimes y, x' \otimes y' \rangle = \langle y, \langle x, x' \rangle y' \rangle$$

defines a semi inner product on (the algebraic tensor product) $E \otimes F$ with the natural right $\mathcal{C}$-action. Let

$$N = \{w \in E \otimes F : \langle w, w \rangle = 0\}.$$

The *interior tensor product* of E and F is defined as

$$E \odot F = \overline{E \otimes F/N}$$

Note that $E \odot F$ is a Hilbert $\mathcal{A}$-$\mathcal{C}$-module with the natural left action of $\mathcal{A}$.

Let E be a Hilbert $\mathcal{A}$-$\mathcal{B}$-module. (**Notation:** ${}_{\mathcal{A}}E_{\mathcal{B}}$)
Let $\mathcal{B} \subseteq \mathcal{B}(\mathcal{G})$, ($\mathcal{G}$ can be viewed as ${}_{\mathcal{B}}\mathcal{G}_{\mathbb{C}}$).

$${}_{\mathcal{A}}\mathcal{H}_{\mathbb{C}} := {}_{\mathcal{A}}E_{\mathcal{B}} \odot {}_{\mathcal{B}}\mathcal{G}_{\mathbb{C}}$$

That is, $\mathcal{H}$ is a Hil. sp. with a rep. $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$.
For $x \in E$ let $L_x : \mathcal{G} \to \mathcal{H}$ be defined by $L_x(g) = x \odot g$, then $L_x \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ with $L_x^* : x' \odot g \mapsto \langle x, x' \rangle g$. Define

$$\eta : E \to \mathcal{B}(\mathcal{G}, \mathcal{H}) \quad \text{by } \eta(x) = L_x.$$

Then

$$L_x^* L_y = \langle x, y \rangle \in \mathcal{B} \subseteq \mathcal{B}(\mathcal{G}) \quad \text{and} \quad L_{axb} = \rho(a) L_x b.$$

$${}_{\mathcal{A}}E_{\mathcal{B}} \subseteq {}_{\mathcal{B}(\mathcal{H})}\mathcal{B}(\mathcal{G}, \mathcal{H})_{\mathcal{B}(\mathcal{G})}.$$

Definition

Let $\mathcal{B}$ be a von Neumann algebra on a Hilbert space $\mathcal{G}$. A Hilbert $\mathcal{B}$-module E is a von Neumann $\mathcal{B}$-module if E is strongly closed in $\mathcal{B}(\mathcal{G}, E \odot \mathcal{G})$.

Definition

Let $\mathcal{A}$ be a von Neumann algebra. A von Neumann $\mathcal{B}$-module E is said to be von Neumann $\mathcal{A}$-$\mathcal{B}$-module if it is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module such that the representation $\rho : \mathcal{A} \to \mathcal{B}(E \odot \mathcal{G})$ is normal.

Lemma

Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{B}$ be a von Neumann algebra on a Hilbert space $\mathcal{G}$. Let E be a Hilbert $\mathcal{A}$-$\mathcal{B}$-module. Then the operations $x \mapsto xb$, $x \mapsto \langle y, x \rangle$ and $x \mapsto ax$ are strongly continuous. Hence $\overline{E}^s$ is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module and a von Neumann $\mathcal{B}$-module.

Results

If E is a von Neumann $\mathcal{B}$-module, then $\mathcal{B}^a(E)$ is a von Neumann subalgebra of $\mathcal{B}(E \odot \mathcal{G})$. von Neumann modules are self-dual and hence any bounded right linear map between von Neumann module is adjointable. If F is avon Nuemann submodule of E then there exists a projection p $(p = p^2 = p^*)$ in $\mathcal{B}^a(E)$ onto F. (complementary)

GNS-construction (Paschke [7])

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a CP map. Then, there exists a pair (E, ξ) of a Hilbert Hilbert $\mathcal{A}$-$\mathcal{B}$-module E and a cyclic vector $\xi \in E$ (i.e., $E = \overline{\text{span}}(\mathcal{A}\xi\mathcal{B})$) such that

$$\phi(a) = \langle \xi, a\xi \rangle, \quad a \in \mathcal{A}.$$

The pair (E, ξ) is called the GNS-construction of ϕ and E is called the GNS-module for ϕ. Obviously ϕ is unital if and only if $\langle \xi, \xi \rangle = \mathbf{1}$.

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Definition

Let $\phi : \mathcal{A} \to \mathcal{B}$ be a CP map. Let E be a Hilbert $\mathcal{A}$-$\mathcal{B}$-module and $\xi \in E$, We call (E, ξ) as a GNS-representation for ϕ if $\phi(a) = \langle \xi, a\xi \rangle$ for all $a \in \mathcal{A}$. It is said to be minimal if $E = \overline{\text{span}}(\mathcal{A}\xi\mathcal{B})$. (uniqueness!)

Proposition 1

If E is the GNS-module of a normal completely positive map $\phi : \mathcal{A} \to \mathcal{B}$ between von Neumann algebras, then $\overline{E}^s$ is a von Neumann $\mathcal{A}$-$\mathcal{B}$-module.

Proposition 2

Let E be a von Neumann $\mathcal{A}$-$\mathcal{B}$-module. Let $\pi : \mathcal{B} \to \mathcal{B}(\mathcal{G})$ be a normal representation. Then $\rho : \mathcal{A} \to \mathcal{B}(E \odot G)$ is normal.

Proposition 3

If E be a von Neumann $\mathcal{A}$-$\mathcal{B}$-module and let F be a von Neumann $\mathcal{B}$-$\mathcal{C}$-module where $\mathcal{C}$ acts on a Hilbert space $\mathcal{G}$. Then the strong closure $E \bar{\odot}^s F$ of the tensor product $E \odot F$ in $\mathcal{B}(\mathcal{G}, E \odot F \odot \mathcal{G})$, is a von Neumann $\mathcal{A}$-$\mathcal{C}$-module.

Definition (conventions)

Due to Propositions 1, 2, 3 we make the following conventions:

1. Whenever $\mathcal{B}$ is a von Neumann algebra and $\phi : \mathcal{A} \to \mathcal{B}$ is a CP map, by GNS-module we always mean $\overline{E}^s$, where E is the GNS-module, constructed above.
2. If E and F are von Neumann modules, by tensor product of E and F we mean the strong closure $\overline{E \odot F}^s$ of $E \odot F$ and we still write $E \odot F$.

Observation

Let F be a Hilbert(von Neumann) $M_2(\mathcal{B})$-$M_2(\mathcal{B})$-module. Then F can be treated as a Hilbert(von Neumann) $\mathcal{B}$-$\mathcal{B}$-module with right and left $\mathcal{B}$-module action on F given by

$$wb := w \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \quad bw := \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} w, \quad w \in F, b \in \mathcal{B} \tag{3}$$

and with the $\mathcal{B}$-valued semi-inner product $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ on F given by

$$\langle z, w \rangle_{\mathcal{B}} := \sum_{i,j=1}^{2} \langle z, w \rangle_{i,j}, z, w \in F. \tag{4}$$

(Indeed, we consider $\overline{F/N}$, where $N = \{w : \langle w, w \rangle_{\mathcal{B}} = 0\}$, and we still write F instead of $\overline{F/N}$).

Stucture of block CP maps

Theorem (for a single block CP map)

Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{B}$ be a von Neumann algebra on a Hilbert space $\mathcal{G}$. Let $\Phi : M_2(\mathcal{A}) \to M_2(\mathcal{B})$ be the block CP map $\Phi = \begin{pmatrix} \phi_1 & \psi \\ \psi^* & \phi_2 \end{pmatrix}$, and let (E_i, ξ_i) be the GNS-construction for $\phi_i, i = 1, 2$. Then there is a unique adjointable bilinear contraction $T : E_2 \to E_1$ such that $\psi(a) = \langle \xi_1, Ta\xi_2 \rangle$ for all $a \in \mathcal{A}$.

Stucture of block CP maps

Proof.

Let (E, ξ) be the GNS-construction for Φ. Let $\hat{E}_i = \mathbb{E}_{ii}E, i = 1, 2$, ($\mathcal{B}$-$\mathcal{B}$-modules) where $\mathbb{E}_{ij} = \mathbf{1} \odot E_{ij}$. $(\hat{E}_i, \mathbb{E}_{ii}\xi\mathbb{E}_{ii})-$GNS for $\phi_i, i = 1, 2.$ Define $U : \hat{E}_2 \to \hat{E}_1$ by $Ux = \mathbb{E}_{12}x$ (U is a bilinear unitary). Let $V_i : E_i \to \hat{E}_i$ by $V_i(a\xi_i b) = a\mathbb{E}_{ii}\xi\mathbb{E}_{ii}b$. Take $T = V_1^* U V_2$.

$$\begin{aligned}
\langle \xi_1, Ta\xi_2 \rangle &= \langle \xi_1, V_1^* U V_2 a\xi_2 \rangle = \langle V_1\xi_1, UV_2\xi_2 \rangle \\
&= \langle \mathbb{E}_{11}\xi\mathbb{E}_{11}, a\mathbb{E}_{12}\mathbb{E}_{22}\xi\mathbb{E}_{22} \rangle = \left\langle \xi\mathbb{E}_{11}, \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \xi\mathbb{E}_{22} \right\rangle \\
&= \sum_{i,j=1}^{2} \left(\mathbb{E}_{11} \left\langle \xi, \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \xi \right\rangle \mathbb{E}_{22} \right)_{i,j} = \sum_{i,j=1}^{2} \left(\mathbb{E}_{11} \begin{pmatrix} 0 & \psi(a) \\ 0 & 0 \end{pmatrix} \mathbb{E}_{22} \right)_{i,j} \\
&= \sum_{i,j=1}^{2} \begin{pmatrix} 0 & \psi(a) \\ 0 & 0 \end{pmatrix}_{i,j} = \psi(a).
\end{aligned}$$

□

Stucture of block CP maps

von Neumann algebras

Example

Let $\mathcal{A} = \mathcal{B} = C([0,1])$, Let

$$h_1(t) = t, \quad h_2(t) = 1 \quad \text{for } t \in [0,1]. \tag{5}$$

Consider the CP map $\Phi : M_2(\mathcal{A}) \to M_2(\mathcal{B})$ defined by

$$\Phi \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} h_1^* & 0 \\ 0 & h_2^* \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} = \begin{pmatrix} h_1^* f_{11} h_1 & h_1^* f_{12} h_2 \\ h_2^* f_{21} h_1 & h_2^* f_{22} h_2 \end{pmatrix}.$$

Note that $E_1 = \{f \in C([0,1]) : f(0) = 0\} \subseteq C([0,1])$ and $E_2 = C([0,1])$. There is no bilinear adjointable contraction $T : E_2 \to E_1$ such that $\langle h_1, f h_2 \rangle = \langle h_1, T f h_2 \rangle$ for all $f \in C([0,1])$.

Definition

Let $\mathcal{B}$ be a C^*-algebra. An inclusion system (E, β) is a family $E = (E_t)_{t \in \mathbb{T}}$ of Hilbert $\mathcal{B}$-$\mathcal{B}$-modules with $E_0 = \mathcal{B}$ and a family $\beta = (\beta_{s,t})_{s,t \in \mathbb{T}}$ of two-sided isometries $\beta_{s,t} : E_{s+t} \to E_s \odot E_t$ such that, for all $r, s, t \in \mathbb{T}$,

$$(\beta_{r,s} \odot \mathrm{id}_{E_t})\beta_{r+s,t} = (\mathrm{id}_{E_r} \odot \beta_{s,t})\beta_{r,s+t}.$$

It is said to be a product system if every β_{st} is unitary.

$$\begin{array}{ccc} E_{r+s+t} & \xrightarrow{\beta_{r+s,t}} & E_{r+s} \odot E_t \\ \big\downarrow{\scriptstyle \beta_{r,s+t}} & & \big\downarrow{\scriptstyle \beta_{r,s} \odot \mathrm{id}_{E_t}} \\ E_r \odot E_{s+t} & \xrightarrow{\mathrm{id}_{E_r} \odot \beta_{s,t}} & E_r \odot E_s \odot E_t \end{array}$$

Remark

If $\mathcal{B}$ is von Neumann algebra in the above definition, then we consider incusion system of von Neumann $\mathcal{B}$-$\mathcal{B}$-modules.

Definition

Let (E, β) be an inclusion system. A family $\xi^{\odot} = (\xi_t)_{t \in \mathbb{T}}$ of vectors $\xi_t \in E_t$ is called a unit for the inclusion system, if $\beta_{s,t}(\xi_{s+t}) = \xi_s \odot \xi_t$. A unit is called unital, if $\langle \xi_t, \xi_t \rangle = 1$ for all $t \in \mathbb{T}$. A unit is called generating, if E_t is spanned by images of elements $b_n \xi_{t_n} \odot \cdots \odot b_1 \xi_{t_1} b_0$ $(t_i \in \mathbb{T}, \sum t_i = t, b_i \in \mathcal{B})$ under successive applications of appropriate mappings id $\odot \beta^*_{s,s'} \odot$ id .

Observation

Suppose (E, β) an inclusion system with a unit (unital) $\xi^\odot$. Consider $\phi_t : \mathcal{B} \to \mathcal{B}$ defined by

$$\phi_t(b) = \langle \xi_t, b\xi_t \rangle \text{ for } b \in \mathcal{B}.$$

Then as $\beta_{s,t}$'s are two-sided isometries and $\xi^\odot$ is a unit, for $b \in \mathcal{B}$ we have

$$\begin{aligned} \phi_t \circ \phi_s(b) &= \phi_t(\langle \xi_s, b\xi_s \rangle) = \langle \xi_t, \langle \xi_s, b\xi_s \rangle \xi_t \rangle \\ &= \langle \xi_s \odot \xi_t, b(\xi_s \odot \xi_t) \rangle = \langle \xi_{t+s}, b\xi_{t+s} \rangle \\ &= \phi_{t+s}(b). \end{aligned}$$

That is, $(\phi_t)_{t \in \mathbb{T}}$ is a QDS (QMS).

Converse?

Observation

Let $\phi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{C}$ be CP maps

$$\phi \rightsquigarrow (E, \xi), \quad \psi \rightsquigarrow (F, \zeta), \qquad \psi \circ \phi \rightsquigarrow (K, \kappa)$$

$$\psi \circ \phi(a) = \psi(\langle \xi, a\xi \rangle) = \langle \zeta, \langle \xi, a\xi \rangle \zeta \rangle = \langle \xi \odot \zeta, a\xi \odot \zeta \rangle.$$

$$(\psi \circ \phi \rightsquigarrow (E \odot F, \xi \odot \zeta)) \quad \text{(need not be minimal)}$$

Thus $\kappa \mapsto \xi \odot \zeta$ extends to a unique two-sided isometry $K \to E \odot F$.
So $K \hookrightarrow E \odot F$; $\quad K = \overline{\text{span}}(\mathcal{A}\xi \odot \zeta\mathcal{C})$;
$E \odot F = \overline{\text{span}}(\mathcal{A}\xi\mathcal{B} \odot \mathcal{B}\zeta\mathcal{C}) = \overline{\text{span}}(\mathcal{A}\xi \odot \mathcal{B}\zeta\mathcal{C}) = \overline{\text{span}}(\mathcal{A}\xi\mathcal{B} \odot \zeta\mathcal{C})$.

Stinespring representation?

Observation

Let $\phi = (\phi_t)_{t\in\mathbb{T}}$ be a QDS on a unital C^*-algebra $\mathcal{B}$.
Let (E_t, ξ_t) be the GNS-construction for ϕ_t.
(ξ_t-cyclic in E_t such that $\phi_t(b) = \langle \xi_t, b\xi_t \rangle$, $E_0 = \mathcal{B}$ and $\xi_0 = \mathbf{1}$.) Define

$$\beta_{s,t} : E_{s+t} \to E_s \odot E_t : \qquad \xi_{t+s} \mapsto \xi_s \odot \xi_t.$$

Then $\beta_{s,t}$'s are two-sided isometries. Now

$$\begin{aligned}(\beta_{r,s} \odot I_{E_t})\beta_{r+s,t}(\xi_{r+s+t}) &= (\beta_{r,s} \odot I_{E_t})(\xi_{r+s} \odot \xi_t) = (\xi_r \odot \xi_s) \odot \xi_t \\ &= \xi_r \odot (\xi_s \odot \xi_t) = (I_{E_r} \odot \beta_{s,t})(\xi_r \odot \xi_{s+t}) \\ &= (I_{E_r} \odot \beta_{s,t})\beta_{r,s+t}(\xi_{r+s+t})\end{aligned}$$

shows that (E, β) is an inclusion system of Hilbert $\mathcal{B}$-$\mathcal{B}$-module. It is obvious to see that $\xi^{\odot} = (\xi_t)$ is a generating unit for (E, β).

Definition

For a QDS $\phi = (\phi_t)_{t\geq 0}$ on $\mathcal{B}$, the inclusion system with the generating unit $(E, \beta, \xi^{\odot})$ given in the previous observation is called the inclusion system associated to ϕ.

Definition

Let (E, β) and (F, γ) be two inclusion systems. Let $T = (T_t)_{t\in\mathbb{T}}$ be a family of two-sided (bilinear) maps $T_t : E_t \to F_t$, satisfying $\|T_t\| \leq e^{tk}$ for some $k \in \mathbb{R}$. Then T is said to be a morphism or a *weak morphism* from (E, β) to (F, γ) if $\gamma_{s,t}$'s are adjointable and

$$T_{s+t} = \gamma^*_{s,t}(T_s \odot T_t)\beta_{s,t} \text{ for all } s, t \in \mathbb{T}. \tag{6}$$

It is said to be a *strong morphism* if

$$\gamma_{s,t}T_{s+t} = (T_s \odot T_t)\beta_{s,t} \text{ for all } s, t \in \mathbb{T}. \tag{7}$$

Hilbert C^*-modules

Product Systems: morphism

$$T_t : E_t \to F_t, \quad t \geq 0$$

weak

$$\begin{array}{ccc} E_{s+t} & \xrightarrow{T_{s+t}} & F_{s+t} \\ \downarrow \beta_{s,t} & & \uparrow \gamma^*_{s,t} \\ E_s \odot E_t & \xrightarrow{T_s \odot T_t} & F_s \odot F_t \end{array}$$

strong

$$\begin{array}{ccc} E_{s+t} & \xrightarrow{T_{s+t}} & F_{s+t} \\ \downarrow \beta_{s,t} & & \downarrow \gamma_{s,t} \\ E_s \odot E_t & \xrightarrow{T_s \odot T_t} & F_s \odot F_t \end{array}$$

Stucture of block CP maps

Problem

Let $\mathcal{A}, \mathcal{B}$ *be unital* C^**-algebras and let* $p \in \mathcal{A}, q \in \mathcal{B}$ *be projections. Let* $\Phi = \begin{pmatrix} \phi_1 & \psi \\ \psi^* & \phi_2 \end{pmatrix}$ *be a block CP map with respect to* p *and* q*. Let* (E_i, ξ_i) *be GNS-representation of* $\phi_i, i = 1, 2$*. Can we prove a theorem similar to the above theorem ? or What is the structure of* ψ *in terms of* (E_i, ξ_i)*?*

Stucture of block QDS

Lemma

Let $\mathcal{B}$ be a unital C^*-algebra. Given two inclusion systems (E^i, β^i, ξ^i) associated to the CP semigroups $\phi^i = (\phi^i_t), i = 1, 2$ on $\mathcal{B}$ and a contractive morphism $T : E^2 \to E^1$, there is a block CP semigroup $\Phi = (\Phi_t)_{t \geq 0}$ on $M_2(\mathcal{B})$ such that $\Phi_t = \begin{pmatrix} \phi^1_t & \psi_t \\ \psi^*_t & \phi^2_t \end{pmatrix}$ and $\psi_t(a) = \langle \xi^1_t, T_t(a\xi^2_t) \rangle$.

Stucture of block QDS

Proof.

Let $\Phi_t := \begin{pmatrix} \phi_t^1 & \psi_t \\ \psi_t^* & \phi_t^2 \end{pmatrix}$, where $\psi_t(a) := \langle \xi_t^1, T_t(a\xi_t^2)\rangle$. Then ϕ_t is CP.

Consider

$$\Phi_s \circ \Phi_t \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Phi_s \begin{pmatrix} \phi_t^1(a) & \psi_t(b) \\ \psi_t^*(c) & \phi_t^2(d) \end{pmatrix} = \begin{pmatrix} \phi_{s+t}^1(a) & \psi_s(\psi_t(b)) \\ \psi_s^*(\psi_t^*(c)) & \phi_{s+t}^2(d) \end{pmatrix}.$$

$$\begin{aligned}
\psi_s(\psi_t(b)) &= \langle \xi_s^1, T_s\psi_t(b)\xi_s^2\rangle = \langle \xi_s^1, \psi_t(b)T_s\xi_s^2\rangle = \langle \xi_s^1, \langle \xi_t^1, T_t b\xi_t^2\rangle T_s\xi_s^2\rangle \\
&= \langle \xi_t^1 \odot \xi_s^1, T_t b\xi_t^2 \odot T_s\xi_s^2\rangle = \langle \xi_t^1 \odot \xi_s^1, b(T_t \odot T_s)(\xi_t^2 \odot \xi_s^2)\rangle \\
&= \langle \beta_{t,s}^1(\xi_{t+s}^1), b(T_t \odot T_s)\beta_{t,s}^2(\xi_{t,s}^2)\rangle \\
&= \langle (\xi_{t+s}^1), b\beta_{t,s}^{1*}(T_t \odot T_s)\beta_{t,s}^2(\xi_{t,s}^2)\rangle \\
&= \langle \xi_{t+s}^1, bT_{t+s}\xi_{t+s}^2\rangle \\
&= \psi_{t+s}(b).
\end{aligned}$$

Stucture of block QDS

Theorem (for a block QDS on a vN-alg $\mathcal{B}$)

Let $\Phi_t = \begin{pmatrix} \phi_t^1 & \psi_t \\ \psi_t^* & \phi_t^2 \end{pmatrix} : M_2(\mathcal{B}) \to M_2(\mathcal{B})$ *and* $\Phi = (\Phi_t)_{t \geq 0}$ *be a semigroup (on $M_2(\mathcal{B})$) of block normal CP maps. Then there is a unique contractive morphism* $T : E^2 \to E^1$ *such that* $\psi_t(a) = \langle \xi_t^1, T_t(a\xi_t^2) \rangle$ *for all* $a \in \mathcal{B}$, $t \geq 0$, *where* $(E^i, \beta^i, \xi^{\odot i})$, *is the inclusion system associated to* $\phi^i, i = 1, 2$.

Proof.

For all $t \geq 0$ we have $T_t : E_2 \to E_1$, $\psi_t(a) = \langle \xi_t^1, T_t(a\xi_t^2) \rangle$, $\forall\ a \in \mathcal{A}$. □

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THANK YOU