Composition operators on Hardy spaces of rooted trees

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Hardy Spaces

Definition

For $p \in (0, \infty)$, the Hardy space H_p consists of holomorphic functions on $\mathbb{D}$ with

$$\|f\|_p := \sup_{r \in [0,1)} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}}$$

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is finite.

- If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\|f\|_2^2 = \sum_{n=0}^{\infty} |a_n|^2$.
- For $p \geq 1$, H_p is a Banach space and H_2 is aHilbert space.
- If $0 < p < q$, then $H_q \subsetneq H_p$.

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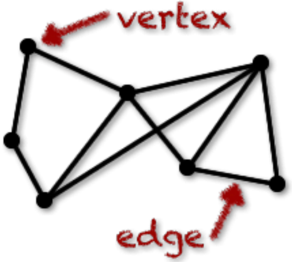
$h_p =$ Harmonic Hardy space

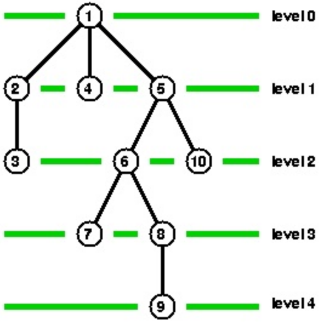
$H_p^g =$ generalized Hardy space .

Discrete function spaces

In recent years, there has been a considerable interest in the study of function spaces on discrete set such as tree (more generally on graphs). For example,

- Lipschitz space of a tree (discrete analogue of Bloch space) [5]
- Weighted Lipschitz space of a tree [3],
- Iterated logarithmic Lipschitz space of a tree [2],
- Weighted Banach spaces of an infinite tree [4] and so on.





Definition (Rooted Tree Graph)

A tree T is a locally finite connected graph without cycles. A rooted tree is a tree in which a special vertex (called root) is singled out.

Every tree graph can be thought of as a metric space under edge counting distance.

Rooted tree

Let T be a rooted tree with root o.

- $|v|$ denotes the distance between o and v.
- For $n \in \mathbb{N}_0, D_n$ denotes the set of all vertices v with $|v| = n$.
- c_n denotes the number of elements in D_n.

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- $|v|$ denotes the distance between o and v.
- For $n \in \mathbb{N}_0, D_n$ denotes the set of all vertices v with $|v| = n$.
- c_n denotes the number of elements in D_n.
- For example, if T is a $(q+1)-$homogeneous tree, then

$$c_n = \begin{cases} (q+1)q^{n-1} & \text{if } n \in \mathbb{N} \\ 1 & \text{if } n = 0. \end{cases}$$

- If T is a $2-$homogeneous tree, then $c_n = 2$ for all $n \in \mathbb{N}$.

Discrete Hardy space

For every $n \in \mathbb{N}$, we introduce

$$M_p(n,f) := \begin{cases} \left(\dfrac{1}{c_n} \displaystyle\sum_{|v|=n} |f(v)|^p \right)^{\frac{1}{p}} & \text{if } p \in (0,\infty) \\ \displaystyle\max_{|v|=n} |f(v)| & \text{if } p = \infty, \end{cases}$$

and $M_p(0,f) := |f(o)|$.

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Definition

The discrete analogue of the generalized Hardy space ($\mathbb{T}_p$) defined by

$$\mathbb{T}_p := \{f : T \to \mathbb{C} | \ \|f\|_p := \sup_{n \in \mathbb{N}_0} M_p(n,f) < \infty\}.$$

$$\mathbb{T}_{p,0} := \{f \in \mathbb{T}_p : \lim_{n \to \infty} M_p(n,f) = 0\}$$

- For $1 \leq p \leq \infty$, $\|\cdot\|_p$ induces a Banach space structure on the spaces $\mathbb{T}_p$ and $\mathbb{T}_{p,0}$.
- As a direct consequence of Holder's inequality, for $0 < p < q \leq \infty$, we have

$$M_p(n, f) \leq M_q(n, f) \text{ for all } n \in \mathbb{N}_0.$$

Inclusions

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- As a direct consequence of Holder's inequality, for $0 < p < q \leq \infty$, we have

$$M_p(n, f) \leq M_q(n, f) \text{ for all } n \in \mathbb{N}_0.$$

- For $0 < p < q \leq \infty$, $\mathbb{T}_q \subseteq \mathbb{T}_p$ and $\mathbb{T}_{q,0} \subseteq \mathbb{T}_{p,0}$.
- These inclusions are proper if and only if $\{c_n\}$ is a unbounded sequence.

- For $x = (x_0, x_1, x_2, \ldots) \in l^\infty$, define $f_x : T \to \mathbb{C}$ by $f_x(v) = x_n$ if $|v| = n$. Then,

$$M_p(n, f) = |x_n| \text{ for all } n \in \mathbb{N}_0 \text{ and } \|f_x\|_p = \|x\|_\infty.$$

Separability

- For $x = (x_0, x_1, x_2, \ldots) \in l^\infty$, define $f_x : T \to \mathbb{C}$ by $f_x(v) = x_n$ if $|v| = n$. Then,

$$M_p(n, f) = |x_n| \text{ for all } n \in \mathbb{N}_0 \text{ and } \quad \|f_x\|_p = \|x\|_\infty .$$

- The map $x \mapsto f_x$ is a linear isometry from l^∞ to $\mathbb{T}_p$.

Theorem

For $0 < p \leq \infty$, the space $\mathbb{T}_p$ is not separable, whereas $\mathbb{T}_{p,0}$ is a separable space as the span of $\{\chi_v : v \in T\}$ is dense in $\mathbb{T}_{p,0}$.

Growth estimate

- If $f \in \mathbb{T}_p$, then for $v \in T$ with $|v| = n$, we have

$$\frac{1}{c_n}|f(v)|^p \leq M_p^p(n, f) \leq \|f\|_p^p.$$

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- Norm convergence in $\mathbb{T}_p$ implies pointwise convergence:

$$\text{If } \lim_{n\to\infty} \|f_n - f\| = 0, \text{ then } \lim_{n\to\infty} f_n(v) = f(v) \text{ for each } v \in T.$$

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- Choose two vertices v_1 and v_2 such that $|v_1| = 1$ and $|v_2| = 2$. Take $f = \sqrt{c_1}\chi_{v_1}$ and $g = \sqrt{c_2}\chi_{v_2}$. Then,

$$\|f\|_2 = \|g\|_2 = \|f + g\|_2 = \|f - g\|_2 = 1$$

and hence the parallelogram law

$$\|f + g\|_2^2 + \|f - g\|_2^2 = 2(\|f\|_2^2 + \|g\|_2^2)$$

is not satisfied. Therefore, $\mathbb{T}_2$ cannot be a Hilbert space under $\|.\|_2$.

Natural operators

Definition

Let X be a linear space consisting of complex-valued functions defined on a set Ω and let ϕ be a self-map of Ω. The composition operator C_ϕ with symbol ϕ is defined as

$$C_\phi f = f \circ \phi \quad \text{for } f \in X$$

and for a given complex valued function ψ defined on Ω, the multiplication operator with symbol ψ is defined by

$$M_\psi f = \psi f \quad \text{for } f \in X.$$

- C_ϕ, M_ψ are linear maps (always).
- The class of these operators is not so narrow as it may look at a first glance.

- Consider the backward shift operator on the sequence space l^2 defined by

$$(x(0), x(1), x(2), \ldots) \mapsto (x(1), x(2), x(3), \ldots).$$

By viewing l^2 as square summable power series, this is a multiplication operator M_ψ induced by $\psi(z) = z$ or this can be viewed as a composition operator C_ϕ induced by $\phi(n) = n + 1$.

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- Consider the evaluation map on a function space (eg: C[0,1], dual of normed linear space, H^p, $\mathcal{B}, \ldots$)

$$ev_a(f) := f(a).$$

This is a composition operator C_ϕ induced by the constant function $\phi \equiv a$.

Multiplication operators

Let X denotes $\mathbb{T}_p$ or $\mathbb{T}_{p,0}$ with norm $\|\cdot\|_p$, where $1 \leq p \leq \infty$.

Theorem

Let M_ψ be multiplication operator on X defined on a homogeneous rooted tree T. Then,

- *M_ψ is a bounded linear operator on X if and only if ψ is a bounded function on T. Moreover, $\|M_\psi\| = \|\psi\|_\infty$.*
- *M_ψ is a compact operator on X if and only if $\psi(v) \to 0$ as $|v| \to \infty$.*
- *M_ψ is an isometry on X if and only if $|\psi(v)| = 1$ for all $v \in T$.*
- *M_ψ is invertible on X if and only if $0 < m \leq |\psi(v)| \leq M < \infty$ for all $v \in T$.*
- *The spectrum of M_ψ is given below:*
    1. $\sigma_e(M_\psi) = \text{Range of } \psi = \psi(T)$;
    2. $\sigma(M_\psi) = \overline{\psi(T)}$.

Questions

C_ϕ	$\mathbb{T}_\infty$ ($\mathbb{T}_{\infty,0}$)	$\mathbb{T}_p$ ($\mathbb{T}_{p,0}$) on $2-$homogeneous trees	$\mathbb{T}_p$ ($\mathbb{T}_{p,0}$) on $k-$homogeneous trees
Bounded			
Norm			
Compact			
Isometry			
Invertible			

Composition operators on $\mathbb{T}_\infty$

Theorem

- *Every self map ϕ of T induces bounded composition operator on $\mathbb{T}_\infty$ with $\|C_\phi\| = 1$.*
- *C_ϕ is compact on $\mathbb{T}_\infty$ if and only if ϕ is a bounded self map of T.*
- *C_ϕ is an isometry on $\mathbb{T}_\infty$ if and only if $\phi : T \to T$ is onto.*
- *The operator C_ϕ is invertible on $\mathbb{T}_\infty$ if and only if ϕ is bijective on T.*

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Theorem

- *The composition operator C_ϕ is bounded on $\mathbb{T}_{\infty,0}$ if and only if $|\phi(v)| \to \infty$ as $|v| \to \infty$. Moreover, $\|C_\phi\| = 1$.*
- *There are no compact composition operators on $\mathbb{T}_{\infty,0}$.*
- *C_ϕ is an isometry on $\mathbb{T}_{\infty,0}$ if and only if $\phi : T \to T$ is onto and $|\phi(v)| \to \infty$ as $|v| \to \infty$.*
- *The operator C_ϕ is invertible on $\mathbb{T}_{\infty,0}$ if and only if ϕ is bijective on T and $|\phi(v)| \to \infty$ as $|v| \to \infty$.*

2−HOMOGENEOUS TREES

THEOREM

Let T be a 2−homogeneous tree and $1 \leq p < \infty$.

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Theorem

Let T be a $2-$homogeneous tree and let $1 \leq p < \infty$. Then C_ϕ is an isometry on $\mathbb{T}_p$ if and only if the following properties hold:

1. $\phi(o) = o$
2. *ϕ is onto*
3. *$|\phi(v)| = |\phi(w)|$ whenever $|v| = |w|$*
4. *If $\phi(w) \neq o$ for some $w \in T$, then ϕ is injective on $D_{|w|}$.*

Theorem

Let T be a $2-$homogeneous tree with root o and let $D_n = \{a_n, b_n\}$ for each $n \in \mathbb{N}$ and ϕ be a self map of T, $1 \leq p < \infty$.

- *If $\phi(o) \neq o$, then $\|C_\phi\|^p = 2$.*
- *If $\phi(o) = o$, then any one of the following distinct cases must occur:*
  - (A) *Either $\phi \equiv o$ or for every $n \in \mathbb{N}$, $\phi(D_n) = D_m$ for some $m \in \mathbb{N}$ then $\|C_\phi\|^p = 1$.*
  - (B) *If ϕ maps exactly one element of D_n to o for each $n \in \mathbb{N}$ then $\|C_\phi\|^p = \frac{3}{2}$.*
  - (C) *Either there exist a $n \in \mathbb{N}$ such that $\phi(a_n) = \phi(b_n) \neq o$ or there exist a $n \in \mathbb{N}$ such that $|\phi(a_n)|$ and $|\phi(b_n)|$ are not equal and both are different from 0 then $\|C_\phi\|^p = 2$.*

Norm estimates

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Theorem

Let T be a $2-$homogeneous tree. Then, the composition operator C_ϕ is bounded on $\mathbb{T}_{p,0}$ if and only if $|\phi(v)| \to \infty$ as $|v| \to \infty$.

Bounded self map

Let ϕ be a self map of homogeneous rooted tree T.

- For $n \in \mathbb{N}_0$ and $w \in T$, let $N_\phi(n, w)$ denote the number of pre-images of w for ϕ in $|v| = n$. That is $N_\phi(n, w)$ is the number of elements in $\{\phi^{-1}(w)\} \bigcap D_n$.
- Finally, for each $m, n \in \mathbb{N}_0$,

$$N_{m,n} := \max_{|w|=m} N_\phi(n, w).$$

Theorem

If T is a $(q+1)-$homogeneous tree with $q \geq 2$ and ϕ is a self map of T such that $\sup_{v \in T} |\phi(v)| = M$, then $\|C_\phi\|^p \leq c_M$. Moreover, $\|C_\phi\|^p = c_M$ if and only if

$$\sup_{n \in \mathbb{N}_0} \frac{N_{M,n}}{c_n} = 1.$$

Norm estimates

Theorem

Let T be a $(q+1)-$homogeneous tree and $1 \leq p < \infty$. Then C_ϕ is bounded on $\mathbb{T}_p$ if and only if

$$\alpha = \sup_{n \in \mathbb{N}_0} \left\{ \frac{1}{c_n} \sum_{m=0}^{\infty} N_{m,n} c_m \right\} < \infty.$$

Moreover, $\|C_\phi\|^p = \alpha$.

Theorem

Let T be a $(q+1)-$homogeneous tree and consider C_ϕ on $\mathbb{T}_p$, where $1 \leq p < \infty$, $q \geq 1$ and ϕ be an automorphism of T. Then we have

(i) $\|C_\phi\| = 1$ *if* $\phi(o) = o$

(ii) $\|C_\phi\|^p = (q+1)q^{|\phi(o)|-1}$ *if* $\phi(o) \neq o$.

Isometry and Invertibility

Theorem

Let T be a $(q+1)-$homogeneous tree with $q \geq 2$ and let $1 \leq p < \infty$. Denote $\frac{c_k N_{k,n}}{c_n}$ by $\lambda_{k,n}$. Then, C_ϕ is an isometry on $\mathbb{T}_p$ if and only if the following properties hold:

1. $|\phi(v)| \leq |v|$. *In particular,* $\phi(o) = o$.
2. $\sum_{k=0}^{n} \lambda_{k,n} = 1$ *for all* $n \in \mathbb{N}_0$.
3. *For each* $k \in \mathbb{N}_0$, $N_\phi(n, w) = N_{k,n}$ *whenever* $|w| = k$.
4. $\sup_{n \in \mathbb{N}_0} \lambda_{k,n} = 1$ *for all* $k \in \mathbb{N}_0$. *In particular,* ϕ *is onto.*

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Theorem

Let T be a $(q+1)-$homogeneous tree with $q \geq 2$, and $1 \leq p < \infty$. C_ϕ is invertible on $\mathbb{T}_p$, if and only if ϕ is invertible and there exists an $M > 0$ such that $||\phi(v)| - |v|| \leq M$ for all $v \in T$.

Compact composition operators

Theorem

1. *Every bounded self map ϕ of T induces compact composition operator on $\mathbb{T}_p$.*
2. *Let T be a $(q+1)-$homogeneous tree. If C_ϕ is compact on $\mathbb{T}_p$, then*

$$\sup_{n\in\mathbb{N}_0}\left\{q^{|w|-n}N_\phi(n,w)\right\}\to 0 \quad as\ |w|\to\infty.$$

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$$\sup_{n\in\mathbb{N}_0}\left\{q^{|w|-n}N_\phi(n,w)\right\}\to 0 \ \text{ as } |w|\to\infty.$$

3. *If C_ϕ is compact on $\mathbb{T}_p$, then $|v|-|\phi(v)|\to\infty$ as $|v|\to\infty$.*
4. *C_ϕ is a compact operator on $\mathbb{T}_p$ whenever*

$$\frac{1}{c_n}\sum_{m=0}^{\infty}N_{m,n}c_m\to 0 \ \text{ as } \ n\to\infty.$$

5. *There are no compact composition operators on $\mathbb{T}_{p,0}$.*

Examples

- For each $n \in \mathbb{N}_0$, choose the vertex $v_n \in D_n$. Define $\phi_1(v) = v_n$ if $|v| = n$. Then, C_{ϕ_1} is not a bounded operator on $\mathbb{T}_p$.

Examples

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- $$\phi_2(v) = \begin{cases} o & \text{if } v = o \\ v^- & \text{otherwise} \end{cases}$$

  where v^- denotes the parent of v. Then, C_{ϕ_2} is bounded on $\mathbb{T}_p$ which is not compact.

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$$\phi_2(v) = \begin{cases} o & \text{if } v = o \\ v^- & \text{otherwise} \end{cases}$$

  where v^- denotes the parent of v. Then, C_{ϕ_2} is bounded on $\mathbb{T}_p$ which is not compact.
- For each $n \in \mathbb{N}_0$, choose a vertex v_n such that $|v_n| = n$. Define a self map ϕ_3 by

$$\phi_3(v) = \begin{cases} v_k & \text{if } v = v_{2k} \text{ for some } k \in \mathbb{N} \\ o & \text{otherwise.} \end{cases}$$

  Then, ϕ_3 is an unbounded self map of T which induces compact composition operators on $\mathbb{T}_p$.

Examples

- For each $n \in \mathbb{N}$ which is not of the form $n = 4k$, $k \in \mathbb{N}_0$, choose $v_n \in T$ such that $|v_n| = n$. Define

$$\phi(v) = \begin{cases} v_{4k+2} & \text{if } \ v = v_{2k+1} \ \text{ for some } \ k \in \mathbb{N}_0, \\ v_{2k+1} & \text{if } \ v = v_{4k+2} \ \text{ for some } \ k \in \mathbb{N}, \\ v & \text{elsewhere.} \end{cases}$$

Clearly, ϕ is bijective on T. But C_ϕ is an unbounded operator on $\mathbb{T}_p$ for every $(q+1)-$homogeneous trees with $q \geq 2$.

EXAMPLES

- For each $n \in \mathbb{N}$ which is not of the form $n = 4k$, $k \in \mathbb{N}_0$, choose $v_n \in T$ such that $|v_n| = n$. Define

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  Clearly, ϕ is bijective on T. But C_ϕ is an unbounded operator on $\mathbb{T}_p$ for every $(q+1)-$homogeneous trees with $q \geq 2$.

- There are bijective self maps ϕ of T which induce a bounded composition operator C_ϕ on $\mathbb{T}_p$ over $(q+1)-$homogeneous trees with $q \geq 2$, but ϕ^{-1} does not induce a bounded composition operator.

Questions

1. As with the l^p and the H_p spaces, whether $\mathbb{T}_p$ is not isomorphic to $\mathbb{T}_q$ when $p \neq q$? What can be said about the dual of $\mathbb{T}_p$?
2. What can be said about the spectrum of C_ϕ?

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2. What can be said about the spectrum of C_ϕ?
3. Necessary and sufficient conditions for C_ϕ to be compact operator on $\mathbb{T}_p$ over $(q+1)-$homogeneous trees with $q \geq 2$?
4. What about bounded and compact composition operators on $\mathbb{T}_{p,0}$ over $(q+1)-$homogeneous trees with $q \geq 2$?

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THANK YOU