EXERCISES

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(1) If X and Y are topological spaces and $A \subseteq X$, the show that "homotopy rel A" is an equivalence relation on the set of maps $X \longrightarrow Y$.
(2) Show that for any space X, any two maps $f, g : X \longrightarrow \mathbb{R}^n$, $n \geq 1$, are homotopic.
(3) Let $x, y \in X$. Let $c_x, c_y : X \longrightarrow X$ be the constant maps *at* x and y respectively. Show that $c_x \simeq c_y$ if and only if x and y lie in the same path component of X.
(4) Let $R_\alpha : S^1 \longrightarrow S^1$ be a rotation by α radians. Show that $R_\alpha \simeq 1_{S^1}$. Conclude that every continuous map $f : S^1 \longrightarrow S^1$ is homotopic to a map g with $g(1) = 1 = (1, 0)$.
(5) Show that contractible spaces are path connected.
(6) Let $X = \{a, b\}$ with a topology given by $\{\emptyset, X, \{a\}\}$. Is X contractible?
(7) Show by an example that a continuous image of a contractible space need not be contractible.
(8) Show that a retract of a contractible space is contractible.
(9) Show that S^n is path connected for all $n \geq 1$.
(10) For any space X, the cone CX over X is contractible.
(11) Give an example to show that the notions "deformation retract" and "strong deformation retract" are different notions.
(12) Let $A \subseteq B \subseteq X$. If B is a deformation retract of X and A that of B, then show that A is a deformation retract of X.
(13) Let $j : D^2 \longrightarrow \mathbb{R}^2$ be the inclusions map. Show that j is a homotopy equivalenve by constructing its homotopy inverse.
(14) Let $X = \{0\} \cup \{1/n \mid n \geq 1\}$ and $Y = \{1/n \mid n \geq 1\}$. Show that X and Y do not have the same homotopy type.
(15) Given a space X let $\pi_0(X)$ denote the set of path components of X. If two spaces X and Y are homotopically equivalent, then show that there is a set theoretic bijection $\pi_0(X) \longleftrightarrow \pi_0(Y)$.
(16) Show that the closed half plane is not homeomorphic to the whole plane.
(17) In each the following examples find a "nice" geometric object to which each is homotopically equivalent:
   - $\mathbb{R}^2$ minus a point in the plane.
   - $\mathbb{R}^2$ minus k points in the plane.
   - $\mathbb{R}^3$ minus a point.
   - $\mathbb{R}^3$ minus a line through the origin.
   - $\mathbb{R}^3$ minus k lines through the origin.
   - S^2 minus k points.

(18) Is the torus minus a point homotopically equivalent to the cylinder minus a point.

(19) Let $f : X \longrightarrow Y$ be a continuous map. The *mapping cylinder* M_f of f is the space

$$M_f = (Y \cup (X \times I))/\sim$$

where $f(x) \sim (x, 0)$. The mapping cylinder is equipped with the quotient topology of the disjoint union $Y \cup (X \times I)$. The disjoint union has the obvious topology. The spaces X and Y can be regarded as subspaces of the mapping cylinder as the subsets $X \times \{1\}$ and $\{[y] \mid y \in Y\}$ respectively. Show that Y is a deformation retract of M_f. Thus Y and M_f have the same homotopy type. Also note that the map f can now be factored as $X \hookrightarrow M_f \xrightarrow{r} Y$ where $r : M_f \longrightarrow Y$ is the retraction $r([x, t]) = f(x)$ and $r([y]) = y$. This is a trick that allows us to pretend that, up to homotopy, every map is an inclusion.

(20) Let $f : S^1 \longrightarrow S^1$ be the square map $z \mapsto z^2$. Describe the mapping cylinder of this map.

(21) Let $f : X \longrightarrow Y$ be a map. The *mapping cone* C_f of f is a quotient of the mapping cylinder defined by

$$C_f = M_f/(X \times \{1\}).$$

(22) Let X be path connected and $x_0, x_1 \in X$. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair of paths α, β from x_0 to x_1 we have $\widehat{\alpha} = \widehat{\beta}$.

(23) Let $A \subseteq X$ and $r : X \longrightarrow A$ a retraction. If $a_0 \in A$, show that $r_* : \pi_1(X, a_0) \longrightarrow \pi_1(A, a_0)$ is surjective.

(24) Show that if X is path connected, then the homorphism induced by a continuous map is independent of the base point up to isomorphisms of the groups involved. More precisely, let $h : X \longrightarrow Y$ be continuous with $h(x_0) = y_0$ and $h(x_1 0 = y - 1$. Let α be a path from x_0 to x_1 and let $\beta = h \circ \alpha$. Show that there is a commutative diagram

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{h^*} & \pi_1(Y, y_0) \\ \widehat{\alpha} \downarrow & & \downarrow \widehat{\beta} \\ \pi_1(X, x_1) & \xrightarrow{h^*} & \pi_1(Y, y_1). \end{array}$$

(25) Show that the following conditions on a map $f : S^1 \longrightarrow X$ are equivalent.
- f is nullhomotopic.
- f extends to a map $f : D^2 \longrightarrow X$.
- f_* is the trivial homomorphism.

(26) Show that the inclusion map $S^1 \longrightarrow \mathbb{R}^2 - 0$ is not null homotopic.

(27) Prove the Brouwer fixed point theorem: Every continuous map $D^2 \longrightarrow D^2$ has a fixed point.
(28) If A is a retract of D^2 show that every self map of A has a fixed point.
(29) If $f : S^1 \longrightarrow S^1$ is null homotopic, then show that there exist $x, y \in S^1$ such that $f(x) = x$ and $f(y) = -y$.
(30) Show that if A is a non singular 3×3 real matrix, then A has a positive real eigen value.
(31) Is the map $\exp : [0, 1] \longrightarrow S^1$ a covering?
(32) If $p : E \longrightarrow B$ and $p' : E' \longrightarrow B'$ are coverings, show that $p \times p' : E \times E' \longrightarrow B \times B'$ is also a covering.
(33) Show that every covering is a local homeomorphism. Is every local homeomorphism a covering?
(34) Show that every covering is an open map.
(35) If $p : E \longrightarrow B$ is a covering with B path connected and $|p^{-1}(b_0)| = k$, then show that the cardinality of every fiber is k.
(36) Let $p : E \longrightarrow B$ and $q : B \longrightarrow X$ be two coverings. If every fiber of q is finite show that $q \circ p$ is a covering.
(37) Let $p : E \longrightarrow B$ be a covering. If B is compact and every finber is finite, show that E is compact.
(38) If X is simply connected show that every map $f : X \longrightarrow S^1$ is null homotopic.
(39) Show that every map $\mathbb{RP}^n \longrightarrow S^1$ is null homotopic.