Advanced Foundational School - II
I.S.I. Bangalore, May 7 - June 2, 2007
Miscellaneous problems in commutative algebra
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The following problems are intended to supplement those which are to be discussed in the school. They are based on the material covered in the school but only a few are standard ones. They are of different levels and if some prove very tough, do not despair !

Q 1.
Let A be a PID with quotient field K. If $A \subset B \subset K$, for an intermediate subring B, show that B must be a PID as well.

Q 2.
Let n be a natual number. Prove that $\mathbf{Z}/n\mathbf{Z}$ is an Armendariz ring; that is, if $f, g \in (\mathbf{Z}/n\mathbf{Z})[X]$ satisfy $fg = 0$, then $a_i b_j = 0$ for all coefficients a_i of f and b_j of g. Try to give a proof which works for any PID in place of $\mathbf{Z}$.
Hint : Prove it for prime powers n and use CRT.

Q 3.
Let A be any commutative ring with unity. Prove that

$$JacA[X] = NilA[X].$$

Q 4.
Let M be a finitely generated module over a commutative ring A with unity. Using the NAK lemma or otherwise, prove that any surjective A-module homomorphism $\theta : M \to M$ is automatically an isomorphism.

Q 5.
Let $A \subset B$ be domains and let C denote the integral closure of A in B. Suppose $f, g \in B[X]$ are monic polynomials such that $fg \in C[X]$, prove that $f, g \in C[X]$.

Q 6.
Let A be a commutative ring with unity.
(i) Assume that every prime ideal is principal. Show that every ideal must be principal.
(ii) If A satisfies the ACC for finitely generated ideals, then prove that A is Noetherian.

Q 7.
Consider the ring $A = \mathbf{C}[[X]]$ of formal power series $\{\sum_{n=0}^{\infty} a_n X^n : a_n \in \mathbf{C}\}$.
(a) Find the units of A.
(b) Find the quotient field of A.
(c) Show that A is a Euclidean domain which is local; find its maximal ideal.

Q 8.
(a) Prove that if A is a UFD, then $S^{-1}A$ is also a UFD for any multiplicative set S.
(b) If A is a FD (that is a domain in which each element is a product of irreducible elements not necessarily uniquely) such that $S^{-1}A$ is a UFD for some S, then A must be a UFD.
(c) Deduce from (b) Gauss's theorem that A UFD implies $A[X]$ is UFD.
Remarks : The result (b) is due to Nagata and implies among other things results such as the real co-ordinate ring $\mathbf{R}[X,Y,Z]/(X^2+Y^2+Z^2-1)$ *of the sphere is a UFD. Contrast with the next problem whose ring can be identified with the real co-ordinate ring of the unit circle.*

Q 9.
Consider the ring

$$A = \{c_0 + \sum_{n=1}^{d}(a_n Cos(n\theta) + b_n Sin(n\theta)) : c_0, a_n, b_n \in \mathbf{R}, d \in \mathbf{N}\}$$

of real, trigonometric polynomials under the (obvious) multiplication coming from multiplication of functions. Prove that A is a domain but that it is not a UFD.
Hint : Use an evident notion of degree to prove A is a domain. For the second assertion, use relations like $Cos^2\theta + Sin^2\theta = 1$.

Q 10.
Prove that the ring A of entire functions on $\mathbf{C}$ is an integral domain but is not a Euclidean domain. Use this to deduce that A is not Noetherian.

Q 11.
Let A be a domain in which each non-zero prime ideal contains a prime element. Prove A must be a UFD.

Q 12.
Let I, J be proper ideals of a ring A such that each prime ideal of A contains the product IJ but no prime ideal contains both I and J. Show that there are proper ideals I_1, I_2 of A such that $A \cong A/I_1 \times A/I_2$ as rings.
Hint : The hypothesis implies $IJ \subseteq Nil(A)$ *and* $I + J = A$.

Q 13.
Use the following steps to give a new proof of Hilbert's basis theorem :
Suppose A is Noetherian but $B = A[X]$ is not.
(i) Let J be an ideal in B maximal with respect to the property of not being finitely generated. Show $X \notin J$.
(ii) Show there is a finitely generated ideal $I = (f_1, \cdots, f_n) \subset J$ such that $J = I + (J : X)X$.
(iii) Deduce that $(J : X) = J$, and so $J = I + JX^r$ for all $r \geq 1$.
(iv) If $k = \max$ (deg $f_1, \cdots$,deg f_n), then show that the elements of J of degrees $< k$ form a finitely generated A-module C.

(v) Use (iii) to show that any $f \in J$ (of degree m, say) can be expressed as $\sum_{i=1}^{n} g_i f_i + hX^{m+1}$ where $g_i \in B$, deg $g_i \leq m$ and $h \in C$.
(vi) Deduce that $J = I + CB$ and that this gives a contradiction.

Q 14.
Let $A \subset B \subset C$ be commutative rings with 1. Suppose A is Noetherian and that C is finitely generated as a A-algebra (that is, C is generated as a ring by A and finitely many elements of C). Assume also that C is integral over B. Prove that B is finitely generated as an A-algebra.

Q 15.
Let A be a domain in which every finitely generated ideal is principal. Prove that a A-module M is flat if and only if it is torsion-free.

Q 16.
Let (A, M_A) and (B, M_B) be local rings and $\theta : A \to B$ be a ring homomorphism such that $\theta(M_A) \subset M_B$. If $N \neq 0$ is a finitely generated A-module, show that $N \otimes_A B \neq 0$.
Hint : Consider $B \otimes_A A/M_A$ and $N \otimes_A A/M_A$ first.

Q 17.
Prove that a ring A is absolutely flat, if and only if every ideal I is idempotent (that is, satisfies $I = I^2$).

Q 18.
Let $A = K[x_1, X_2, \cdots]$ and $B = A/(X_1^2 - X_1, X_2^2 - X_2, \cdots)$. Find all prime ideals of B and show that B is reduced and absolutely flat. What is Ass B ?

Q 19.
If P is a finitely generated projective A-module and $\mathcal{M} \subset A$ is a maximal ideal, show that $P_{\mathcal{M}}$ is a free $A_{\mathcal{M}}$-module.

Q 20.
Let $S \subset A$ be a multiplicative subset and M, N be A-modules. Show that the canonical homomorphism

$$S^{-1}Hom_A(M, N) \to Hom_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

is injective if M is finitely generated.

Q 21.
Prove that $\mathbf{Q} \otimes_{\mathbf{Z}} \prod_{p\ prime} \mathbf{Z}/p\mathbf{Z} \neq 0$ whereas $\prod_{p\ prime} (\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}/p\mathbf{Z}) = 0$.

Q 22.
(i) Find the primary decomposition of the ideal $(x^2 - y^3, y^2 - z^3, z^2 - x^3)$ in $\mathbf{C}[x, y, z]$.
(ii) Do the same when the field is $\mathbf{Q}$ instead of $\mathbf{C}$.

Q 23.
If A is Noetherian, M is a finitely generated A-module, and N is any A-module, prove that

$$Ass \;\; Hom_A(M, N) \subseteq Ass(N) \cap Supp(M).$$

Q 24.
In $A = K[X, Y]$, show that $(X) \cap (X, Y)^2 = (X) \cap (X^2, Y)$ are both reduced primary decompositions for $I = (X^2, XY)$. Find $Ass(A/I)$.

Q 25.
Let I be an ideal in a commutative ring A. If $I \cap ann(I) = 0$, then prove that $\text{End}_A(I)$ is commutative.

Q 26.
If A is a domain and M is a torsion-free A-module, prove that $\text{End}_A(M)$ commutative if and only if M is A-isomorphic to an ideal of A.

Q 27.
If A is Noetherian, show that $A[[X]]$ is a flat A-module.

Q 28.
Prove that if A is a Noetherian ring, I is an ideal, $a \in A$ is not a zero divisor, then there exists $n_0 > 0$ so that for all $n \geq n_0$, the relation $ab \in I^n$ implies $b \in I^{n-n_0}$.

Q 29.
Let M be a finitely generated module over a Noetherian local ring A. Prove that M is free if it is flat.

Q 30.
Show that an A-module M is injective if, for each ideal I of A, any A-module homomorphism from I to M extends to the whole of A.

Q 31.
If A is a ring over which every projective module is free, show that the only idempotents (that is, elements $e \in A$ satisfying $e^2 = e$) are 0 and 1.

Q 32.
Let A be a domain. Prove that it is integrally closed if and only if $A[X]/(f)$ is a domain for each monic irreducible polynomial $f \in A[X]$.

Q 33.
In the ring $A = \prod_{\mathbf{N}} \mathbf{R}$, consider for each $n \in \mathbf{N}$,

$$m_n := \{f : \mathbf{N} \to \mathbf{R}, f(n) = 0\}.$$

Show that m_n is a maximal ideal for every n and that $I := \oplus_{\mathbf{N}} \mathbf{R}$ is an ideal contained in $\bigcup_{n \geq 1} m_n$ but not contained in m_n for any n.

Q 34.
Show that in the ring $A = \mathbf{Z}[X]$, the ideal $I = (X, 4)$ is $\mathcal{M}$-primary, where $\mathcal{M} = (X, 2)$, but that I is not a power of $\mathcal{M}$.

Q 35.
Find all the prime ideals of $\mathbf{Z}[i]$ which lie over : (i) 2, (ii) 3, (iii) 5 in $\mathbf{Z}$.

Q 36.
For a field K, consider the subalgebra A of $K[X, Y]$ generated by the monomials

$X, X^2Y, X^3Y^2, \cdots$ Prove that $A[XY]$ is contained in a finitely generated A-module but that XY is not integral over A.

Q 37.
Let A be the ring of infinitely differentiable functions from $\mathbf{R}$ to itself. Let $\mathcal{M}$ be the maximal ideal consisting of all the functions which vanish at 0. Find (with proof) a non-zero function which belongs to the intersection $\bigcap_n \mathcal{M}^n$.

Q 38.
Let K be any field of characteristic zero. Consider the formal power series $e(X) := \sum_{r=0}^{\infty} \frac{X^r}{r!}$ as an element of the quotient field $K((X))$. Prove that X and $e(X)$ are algebraically independent transcendental elements over K.
Hint : You may use the fact that the formal derivative f' of an element $f = \sum_{r=-\infty}^{\infty} a_r X^r$ of $K((X))$ cannot be zero if $f \notin K$.

Q 39.
Prove that a Noetherian ring satisfies the descending chain condition on prime ideals.

Q 40.
Prove that if A is Artinian, then A is isomorphic to a finite direct product of Artinian local rings.
Hint : Prove and use the fact that A is semilocal.

Q 41.
Prove that A is Artinian if and only if it is Noetherian and all prime ideals are maximal.

Q 42.
If ACC holds for all principal ideals in a domain A, then prove that ACC also holds for principal ideals in $A[[X]]$.

Q 43.
Use the above result to show that if A is a UFD then $A[[X]]$ is a factorial ring if and only if the intersection of any two principal ideals in $A[[X]]$ is again principal.

Q 44.
If M, N are faithfully flat, then $M \otimes_A N$ is faithfully flat.

Q 45.
Let $\theta : A \to B$ be a ring homomorphism. Suppose there is a B-module N such that $\theta^*(N)$ is a faithfully flat A-module. Then, show :
(i) θ is injective,
(ii) for each ideal I of A, $\theta^{-1}(IB) = I$, and
(iii) if B is Noetherian, then A is Noetherian.

Q 46.
Given a submodule N of an A-module M, and a multiplicative subset S of A, recall that the *saturation of N in M with respect to S* is defined to be the kernel

of the composite

$$M \to M/N \to S^{-1}M/S^{-1}N.$$

(i) For $A = \mathbf{Z}S = \mathbf{Z} - p\mathbf{Z}$, find the prime ideals which are saturated with respect to S.
(ii) For $A = K[X, Y], P = (X), Q = (Y), S = A - (P \cup Q)$, prove that the saturation of $P + Q$ in A with respect to S is not equal to the sum of the saturations of P and Q.
(iii) For $A = K[X, Y, Z], P = (X, Z), Q = (Y, Z), S = A - (P \cup Q)$, prove that the saturation of PQ in A with respect to S is not equal to the product of the saturations of P and Q.

Q 47.
(i) If I is an ideal in a Noetherian ring A, prove that there are finitely many prime ideals $P_1, \cdots, P_n$ such that $P_1 P_2 \cdots P_n \subset I$.
(ii) Show that the above result is false for the ring $C([0,1], \mathbf{R})$ (and, therefore, $C([0,1], \mathbf{R})$ is not Noetherian).

Q 48.
Let $A \subset B$ be a subring. For every minimal prime ideal P of A prove that there exists a minimal prime ideal Q of B such that $Q \cap A = P$.

Q 49.
Show that every finitely presented, flat A-module is projective.

Q 50.
(i) If $M = \sum_{i \in I} M_i$, prove that $Supp(M) = \cup_{i \in I} Supp(M_i)$.
(ii) If $M = < x_1, \cdots, x_n >$, and $I_i = ann(x_i)$ for all i, then prove

$$Supp(M) = \cup_{i=1}^{n} V(I_i) = V(ann(M)).$$