Harnack inequalities - from PDE to random graphs
Lecture 4: Stability of the EHI

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Ashok Maitra Memorial Lectures, March 2021

Stability

Let $(\mathbb{G}, a) = (\mathbb{V}, E, a)$ be a *weighted graph.*

Recall that the continuous time random walk $(X_t, t \geq 0)$ on $(\mathbb{V}, E, a)$ has generator

$$\mathcal{L}_a f(x) = \sum_{y \sim x} a_{xy}(f(y) - f(x)).$$

A property $\mathcal{P}$ (of X) is *stable* if when it holds for $(\mathbb{V}, E, a)$ and a' are weights with $a' \asymp a$, i.e. there exist $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 a_{xy} \leq a'_{xy} \leq c_2 a_{xy} \text{ for all edges } \{x, y\}\ ,$$

then $\mathcal{P}$ holds for $(\mathbb{V}, E, a')$.

Examples

1. Transience/recurrence is stable. (Lyons 1983, Doyle and Snell, 1984). (Consider $(\mathbb{G}, a)$ as an electric network: then X is transient if and only if the effective resistance from 0 to infinity is finite.)

2. Define for $D \subset \mathbb{V}$ the Dirichlet form

$$\mathcal{E}_{D,a}(f,g) = \tfrac{1}{2} \sum_{x \in D} \sum_{y \sim x} a_{xy}(f(y) - f(x))(g(x) - g(y)).$$

If $c_1 a_{xy} \le a'_{xy} \le c_2 a_{xy}$, then

$$c_1 \mathcal{E}_{D,a}(f,f) \le \mathcal{E}_{D,a'}(f,f) \le c_2 \mathcal{E}_{D,a}(f,f).$$

Typically, any property which can be characterized in terms of inequalities involving just the Dirichlet form $\mathcal{E}_D(f,f)$ is stable.

3. Volume doubling and the Poincaré inequality are stable.

4. By the ‘PHI theorem’ it follows that (PHI) and (GB) are stable.

Stability of the PI

Suppose $c_1 a_{xy} \leq a'_{xy} \leq c_2 a_{xy}$, let $B = B(x_0, R)$, $2B = B(x_0, 2R)$, and assume the PI holds for $(\mathbb{V}, E, a)$. Write

$$\bar{f}_B = \frac{\sum_{x \in B} f(x) a_x}{\mu_a(B)}, \qquad \bar{f}_{B,'} = \frac{\sum_{x \in B} f(x) a'_x}{\mu_{a'}(B)},$$

Then

$$\begin{aligned}
\sum_{x \in B} (f(x) - \bar{f}_{B,'})^2 a'_x &\leq \sum_{x \in B} (f(x) - \bar{f}_B)^2 a'_x \\
&\leq c_2 \sum_{x \in B} (f(x) - \bar{f}_B)^2 a_x \\
&\leq c_2 C_{PI,a} R^2 \tfrac{1}{2} \sum_{x,y \in 2B} a_{xy} (f(y) - f(x))^2 \\
&\leq c_2 c_1^{-1} C_{PI,a} R^2 \tfrac{1}{2} \sum_{x,y \in 2B} a'_{xy} (f(y) - f(x))^2,
\end{aligned}$$

which gives the PI for $(\mathbb{V}, E, a')$.

Liouville property

A function $h : \mathbb{V} \to \mathbb{R}$ is *harmonic* in a set D if

$$\mathcal{L}_a h(x) = 0 \text{ for all } x \in D.$$

(So $h(X_{t \wedge \tau_D})$ is a local martingale.)

$(\mathbb{G}, a)$ has the *Liouville property (LP)* if whenever h is a bounded harmonic function on $\mathbb{V}$ then h is constant. (Strong LP: all positive harmonic functions are constant).

LP relates to whether or not the process X has trivial tail behaviour.

Examples. 1. $\mathbb{Z}^d$ satisfies the LP.
2. The binary tree does not satisfy the LP.

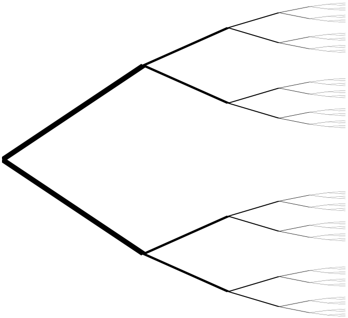
Theorem (T. Lyons 1983). The Liouville property (and the strong LP) are not stable.

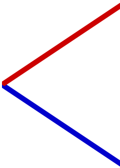
Basic idea of the proof

Look at the binary tree

$$\mathbb{B} = \bigcup_{n=0}^{\infty} \{0,1\}^n .$$

Natural edge weights:
$a_{xy} = 1$ for all edges.
Random walk X.





Call an edge from x to $(x, 0)$ a 0-edge, and from x to $(x, 1)$ a 1-edge.

Let a' be 1 on all 0-edges, and 2 on all 1-edges; associated random walk X'.

Then there exists a subset $A \subset \mathbb{B}$ such that with probability 1

$$X_t \in A \text{ for all large } t\ , \quad X'_t \in \mathbb{B} - A \text{ for all large } t\ .$$

LP is not stable

Connect each point in $\mathbb{B} - A$ to a point on the x_1-axis in $\mathbb{Z}^4$.

The RW X visits $\mathbb{B} - A$ only finitely often, so has a positive probability of remaining in $\mathbb{B}$ for all t.

The RW X' wants to visit $\mathbb{B} - A$ infinitely often, but (because of the links to $\mathbb{Z}^4$, and because the x_1-axis in $\mathbb{Z}^4$ is transient), it will ultimately stay in $\mathbb{Z}^4$. So it has trivial tail behaviour, and hence satisfies the LP.

The binary tree in this example satisfies exponential volume growth.

But this example still works if one replaces each edge from level n to $n+1$ of the binary tree by a sequence of b^n edges, where $b > 1$.
If $\alpha = \log 2 / \log b$,

$$|B_{\mathbb{G}}(0, b^n)| \simeq 2^n = (b^n)^\alpha,$$

so the new graph has polynomial volume growth.

Questions

After the work of Grigoryan and Saloff-Coste (around 1992) which characterized the PHI and hence proved that it is stable, the following questions arise naturally:

(1) Are EHI and PHI equivalent?

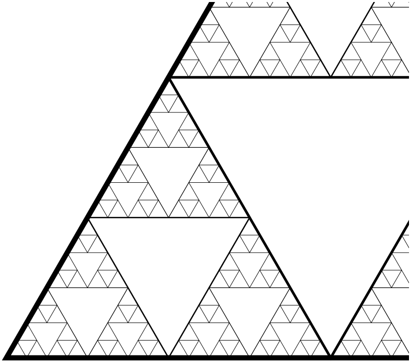
(2) If not, then is EHI stable?

Recall that PHI $\Rightarrow$ EHI $\Rightarrow$ (LP), and that (PHI) is stable, (LP) is not stable.

Attempts (by me and others) to use the idea behind the Lyons - Benjamini example to prove instability of the EHI were not successful.

Counterexamples

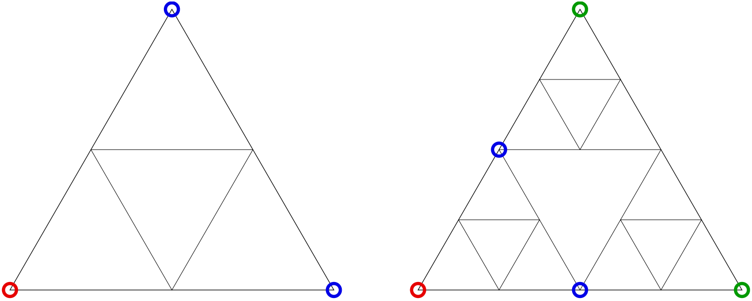
EHI and PHI are not equivalent. Example from diffusions on fractals.
Look the 'pre-Sierpinski gasket' graph $\mathbb{G}_{SG}$:



Geometry.
There are order 3^n vertices in each 'triangle' of side 2^n, so it follows that $|B(x,r)| \asymp r^{d_f}$ where the 'fractal dimension' $d_f = \log 3 / \log 2$.

It follows easily that volume doubling holds.

Key property for RW on the SG



Easy calculation for 6 state Markov chain (with symmetry): Mean time starting at 0 (red) to 'cross' the level 2 triangle and reach a blue vertex is $5 = 5^{2-1}$.

Mean time to cross level 3 triangle (reach green vertex) is 5^2.

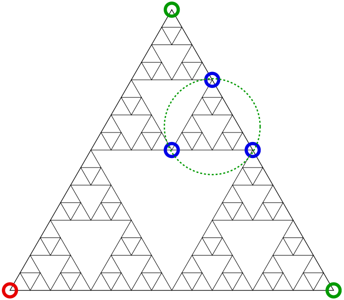
Mean time to cross level n triangle is 5^{n-1}.

Typically the SRW on the SG takes time R^{d_w} to move a distance R, where $d_w = \log 5 / \log 2$.

Compare with SRW on $\mathbb{Z}^d$ which takes time R^2 to move a distance R.

Harnack inequality on pre Sierpinski gasket

The special structure of the SG makes it easy to prove the EHI.



Let $h(x) = \mathbb{P}^x(X$ leaves the triangle at the red vertex $)$, and D be the region inside the green circle.

For each pair of blue vertices y, z we have $h(y) \geq \frac{1}{4} h(z)$, and it follows that

$$\sup_D h \leq 4 \inf_D h.$$

PHI for the SG

The standard PHI looks at space time cylinders $Q = [0, T] \times B(x, R)$ with $T = R^2$ – i.e. the usual space-time scaling for a RW.
For the SG one finds that a PHI still holds, but now with $T = R^{d_w} \gg R^2$, so the cylinder for the PHI is

$$Q = [0, R^{d_w}] \times B(x, R).$$

Call this PHI(d_w). The usual PHI is PHI(2).

The PHI cannot hold with two essentially different space-time scaling functions, so PHI(2) fails on the pre-SG.

So the pre-SG satisfies EHI but not PHI(2), and thus EHI and PHI(2) are not equivalent.

More generally we can consider spacetime scaling $T = \Psi(R)$ for functions $\Phi : [0,\infty) \to [0,\infty)$, and consider the associated PHI(Ψ) and PI(Ψ). The following result was proved by Moser's methods.

Theorem. (MB, Bass 2004) *Given suitable local regularity, the following are equivalent for a weighted graph:*
(1) PHI(Ψ) *holds,*
(2) (Sub-Gaussian heat kernel bounds hold),
(3) $\mathcal{X}$ *satisfies VD,* PI(Ψ) *and CS*(Ψ).

CS(Ψ) implies that there exists φ with $\varphi = 1$ in $B(x,R)$, $\mathrm{supp}(\varphi) \subset B(x,2R)$ and

$$\mathcal{E}_{B(x,2R)}(\varphi,\varphi) \le \frac{\mu(B(x,2R))}{\Psi(R)},$$

i.e. there exist 'low energy' cutoff functions in $B(x,2R) - B(x,R)$.

Remark. PI(Ψ) and CS(Ψ) are stable, so PHI(Ψ) is stable.

Delmotte's counterexample

One can find a pre-fractal graph $\mathbb{G}_V$ such that the join of $\mathbb{Z}$ and $\mathbb{G}_V$ satisfies EHI. (Join: connect one pair of vertices in the two graphs.) However, one has

$$|B_{\mathbb{Z}}(x,r)| \asymp r, \quad |B_{G_V}(x,r)| \asymp r^{\log 5/\log 3}$$

and it follows that this graph does not satisfy VD.

This example shows that any attempt to characterise EHI must deal with spaces which do not satisfy VD, and also with spaces with different space-time scaling regimes in different regions.

All de Giorgi, Moser, Nash arguments use VD in an essential way.

Bass, 2013: Stability of EHI for a graph/cable system which satisfies VD. Introduced PI with space dependent functions $\Psi(\cdot, R)$:

$$\int_{B(x,R)} (f - \bar{f}_B)^2 dm \le C_P \Psi(x,R) \mathcal{E}_{8B}(f,f).$$

Change of time/measure

Recall we have weights a_{xy} for $(x,y) \in \mathbb{V} \times \mathbb{V}$, and we defined a measure μ_a on $\mathbb{V}$ by $a_x = \sum_y a_{xy}$, $\mu_a(D) = \sum_{x \in D} a_x$. The generator of X is

$$\mathcal{L}_a f(x) = \frac{1}{a_x} \sum_y a_{xy}(f(y) - f(x)).$$

Recall that we defined the Dirichlet form

$$\mathcal{E}(f,g) = \tfrac{1}{2} \sum_x \sum_y a_{xy}(f(y) - f(x))(g(y) - g(x)).$$

Discrete Gauss-Green formula (easy to verify); for $f, g \in L^2(\mu_a)$,

$$\mathcal{E}(f,g) = \langle -\mathcal{L}_a f, g \rangle_{L^2(a)} = -\sum_x a_x(\mathcal{L}_a f(x)) g(x).$$

Given a measure m on $\mathbb{V}$ then we can define a new operator $\mathcal{L}_m$ by

$$\mathcal{E}(f,g) = \langle -\mathcal{L}_m f, g \rangle_{L^2(m)}.$$

Change of time/measure II

(Assume $m_x > 0$ for all x.) One finds

$$\mathcal{L}_m f(x) = \frac{a_x}{m_x} \mathcal{L}_a f(x) = \frac{1}{m_x} \sum_y a_{xy}(f(y) - f(x)).$$

The associated process $X^{(m)}$ is a time change of $X = X^{(a)}$; while $X^{(a)}$ jumps out of x at rate 1, $X^{(m)}$ jumps out at rate a_x/m_x.
Changing the measure does not change the jump probabilities or the set of harmonic functions.

We have an alternative definition of a harmonic function h in a set D:

$$\mathcal{E}(h,f) = 0 \text{ for all } f \text{ with } \mathrm{supp}(f) \subset D.$$

This definition depends only on the quadratic form $\mathcal{E}$ and not on m.

Overview: EHI on weighted graphs

Consider a weighted graph $(\mathbb{V}, E, a)$.

This has the natural graph metric d with $d(x, y)$ the smallest number of edges to connect x and y, and also has the natural measure μ_a. Weighted graphs are examples of MMD spaces - a metric space with a measure and a Dirichlet form.

The metric d plays no role in the definition of the SRW X, but is used to define balls, so is needed for EHI.

Measure μ_a: no role in EHI. But it does play an essential role in PI, CS inequalities.

General vague idea. Given a MMD space $(\mathcal{X}, d, m, \mathcal{E})$, change d or m to 'improve' the process, or the way we record information about it.

Example: Gurel-Gurevich/Nachmias on recurrence of UIPT replaced the natural graph metric with a 'resistance metric'.

Main theorem

Notation. We write $(\mathbb{V}, E, d', \mathcal{E}_a, m)$ for the MMD space given by the graph $(\mathbb{V}, E)$ with metric d' on $\mathbb{V}$, Dirichlet form $\mathcal{E}_a$ arising from the weights a, and random walk X^m with generator $\mathcal{L}^m$. d_G will denote the natural graph metric.

Theorem. (MB and M. Murugan 2018). On a locally regular weighted graph EHI is stable. Thus if $(\mathbb{V}, E, d_G, \mathcal{E}_a, a)$ satisfies the EHI, and a'_{xy} are weights with $a' \asymp a$, then $(\mathbb{V}, E, d_G, \mathcal{E}_{a'}, a')$ also satisfies the EHI.

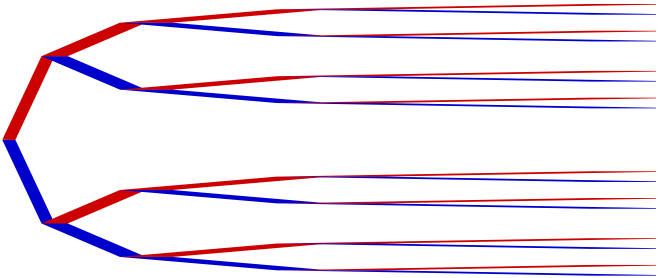
Note. We also have versions for diffusions on manifolds, and general metric measure spaces.
MB, Z. Chen, M. Murugan (2020) – good conditions for the necessary local regularity in the general metric measure space situation.

Metric doubling (MD)

Definition. A metric space $(\mathcal{X}, d)$ satisfies metric doubling if there exists $N < \infty$ such that any ball $B(x, R)$ can be covered by N balls $B(z_i, R/2)$.

It is easy to show that VD implies MD.

Example The binary tree with edges at stage n replaced by b^n edges has polynomial volume growth but does not satisfy metric doubling.



Outline of proof

Step 1. EHI for $(\mathbb{V}, E, d_G, \mathcal{E}_a, a)$ implies that the metric space $(\mathbb{V}, d_G)$ satisfies metric doubling (MD).

Theorem (Volberg, Konyagin 1987). If MD holds then there exists a measure μ which satisfies VD.

Step 2. Using Volberg-Konyagin (VK) method, construct a measure μ such that there exist scaling functions $\Psi = \Psi_\mu(x, R)$ and such that VD, PI(Ψ), CS(Ψ) hold for $(\mathbb{V}, E, d_G, \mathcal{E}_a, \mu)$.

Step 3. Prove that VD, PI(Ψ), CS(Ψ) on $(\mathbb{V}, E, d_G, \mathcal{E}_a, \mu)$ imply EHI.

Outline 2: Proof of stability given Steps 1 – 3

Let $(\mathbb{V}, E, a)$ be a weighted graph, $(\mathbb{V}, E, d_G, \mathcal{E}_a, a)$ be the associated MMD space, and assume $a' \asymp a$.

1. By Step 1 $(\mathbb{V}, d_G)$ satisfies MD. By Step 2 there exists a measure μ such that $(\mathbb{V}, E, d_G, \mathcal{E}_a, \mu)$ satisfies VD, PI(Ψ), CS(Ψ).

2. Since $a \asymp a'$ the PI holds for $(\mathbb{V}, E, d_G, \mathcal{E}_{a'}, \mu)$. Similarly CS holds.

3. By Step 3 EHI holds for $(\mathbb{V}, E, d_G, \mathcal{E}_{a'}, \mu)$ and hence for $(\mathbb{V}, E, d_G, \mathcal{E}_{a'}, \mu_{a'})$, i.e. for the weighted graph $(\mathbb{V}, E, a')$.

Step 1 - from EHI to metric doubling

Basic idea. Let $x_0 \in \mathbb{V}$. If metric doubling fails, then for some R we can find $N \gg 1$ disjoint balls $B_i = B(z_i, R/10)$ with $d(x_0, z_i) = R$. Let $B = B(x_0, R)$.

Let

$$h_i(y) = \mathbb{P}^y(\ X \text{ first leaves } B \text{ through } B_i).$$

(1) The EHI implies that $h_i \geq c_1 > 0$ in $B_i \cap B$.

(2) The EHI also implies that if $y \in B$ and $d(y, \mathbb{V} - B) = r$ then $h_i(y') \asymp h_i(y)$ for $y' \in B(y, r/2)$. Using this we can 'chain' the EHI for h_i from B_i to x_0 and that $h_i(x_0) > c_2$ for some $c_2 > 0$.

But then

$$1 \geq \sum_{i=1}^{N} h_i(x_0) \geq c_2 N.$$

Step 2 – capacities and measures

Let μ be a measure on $\mathbb{V}$, and $\mathbb{E}^x_\mu$ be the law of X^μ.
Define for $D_1 \subset D_2$,

$$\mathrm{Cap}(D_1|D_2) = \inf\{\mathcal{E}(f,f) : f \geq 1 \text{ on } D_1, \mathrm{supp}(f) \Subset D_2\}.$$

We have (if EHI, VD hold) that writing τ_D for the exit time of D, $B = B(x,R)$, $8B = B(x,8R)$,

$$\mathbb{E}^x_\mu \tau_{8B} \simeq \frac{\mu(8B)}{\mathrm{Cap}(B|8B)} = \Psi_\mu(x,R).$$

Thus the function Ψ_μ gives space-time scaling function for the time changed process X^μ.

We need our new measure μ to make a good connection between volume and capacity of balls.

Construction of new measure μ

To construct μ on $B = B(x, R)$ we decompose B into 'generalised dyadic cubes' $Q_{k,j}$, with $\mathrm{diam}(Q_k, j) \asymp 8^{-k}R$.
Using mass transport we define a sequence of measures ν_k with $\nu_k(x)/a(x)$ constant on each $Q_{k,j}$. At each stage we move the measure $\nu_k(Q_{k,j})$ onto the 'successor' cubes $\{Q_{k+1,i} : Q_{k+1,i} \subset Q_{k,j}\}$, and then following [VK] perform some adjustments.
Let $8Q_{k,j}$ denote a suitable expansion of $Q_{k,j}$. We can ensure that ν_k satisfies VD, and if $Q_{k+1,i} \subset Q_{k,j}$ then

$$\frac{\nu_{k+1}(Q_{k+1,i})}{\mathrm{Cap}(Q_{k+1,i}|8Q_{k+1,i})} \le \frac{\nu_k(Q_{k,j})}{\mathrm{Cap}(Q_{j,k}|8Q_{j,k})}, \qquad (*)$$

The measure $\mu = \nu_{k_0}$ where $8^{-k_0}R \asymp 1$.
The inequality (*) ensures 'good behaviour' of the functions $\Psi_\mu(x, R)$.

Remainder of proof

Once we have the measure $\mu = \nu_{k_0}$ we can use straightforward extensions of known methods to complete the argument:

(2) The space $(\mathbb{V}, E, d_G, \mathcal{E}_a, \mu)$ satisfies VD by the VK construction. It satisfies conditions similar to those assumed by Bass, so one can use his methods to prove PI$(\Psi_\mu(.))$ and CS$(\Psi_\mu(.))$.

(3) The proof that if $(\mathbb{V}, E, d_G, \mathcal{E}_a, \mu)$ satisfies VD, PI$(\Psi_\mu(.))$ and CS$(\Psi_\mu(.))$ then EHI holds is a straightforward extension of the methods of Grigor'yan, Hu (2014), which in turn are extensions of the Moser, de Giorgi arguments.

Characterization of EHI

Our Theorem gives a stable characterization of the EHI, but the conditions are hard to check.

Given a ball $B = B(x,R)$, and balls $B_i = B(z_1, R/4)$ write

$$C_{\text{eff}}(B_1, B_2; B) = \inf\{\mathcal{E}_B(f,f) : f = 1 \text{ on } B_1 \text{ and } f = 0 \text{ on } B_2\}.$$

Let $\mathcal{A}(x,R)$ be the set of pairs of balls $(B(z_1,R/8), B(z_2,R/8))$ with $d(z_1,z_2) \geq R/3$, $z_i \in B(x,R/2)$.
We say the dumbbell condition (DC) holds for $(\mathcal{X}, d, m, \mathcal{E})$ if

$$\frac{\sup_{(B_1,B_2)\in\mathcal{A}(x,R)} C_{\text{eff}}(B_1,B_2;B)}{\inf_{(B_1,B_2)\in\mathcal{A}(x,R)} C_{\text{eff}}(B_1,B_2;B)} \leq C.$$

Conjecture. EHI is equivalent to MD and DC.