Harnack inequalities - from PDE to random graphs
Lecture 3: Random walks and percolation

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Introduction

In the first two lectures we met Harnack inequalities and saw how they relate to Gaussian heat kernel bounds, and how these can be proved using geometric information on the graph - such as bounds on the volumes (i.e. sizes) of balls, and an isoperimetric inequality.

Today we will see how these ideas work in two less familiar situations:

Random walk in random environment

Random walks on 'fractal graphs'

Random walks with random transition probabilities

Take the base graph to be $\mathbb{Z}^d$, with $d \geq 2$.
There are two common models:

(1) **Random walk in random environment (RWRE).** For each $x \in \mathbb{Z}^d$ choose the transition probabilities from x to its neighbours i.i.d. at random from some fixed distribution. The RW $(X_n, n \in \mathbb{Z}_+)$ then moves according to these probabilities.
This model is hard: papers by Kalikow (1981), Sznitman (2000+) and many others.
One cause of difficulties: the random walk is not reversible, and it is not possible to compute its stationary measure.

I will not talk about this model.

Random walks with random transition probabilities II

(2) **Random conductance model (RCM).** Choose random weights (conductances) $a_{xy}(\omega)$ for the edges $\{x, y\}$ for $\mathbb{Z}^d$; these are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Look at the CTSRW X with generator

$$\mathcal{L}_{a(\omega)} f(x) = \frac{1}{a_x(\omega)} \sum_{y \sim x} a_{xy}(\omega)(f(y) - f(x)).$$

We call $a(\omega)$ the *environment* given by $\omega \in \Omega$, and write P^x_ω for the law of the CTSRW $X = (X_t, t \in \mathbb{R}_+)$ with generator $\mathcal{L}_{a(\omega)}$.
The process X jumps from x to $y \sim x$ at rate a_{xy}/a_x; if $a_{xy} = 0$ then X cannot move along the edge $\{x, y\}$.

The RCM model is much easier than RWRE because the process has known stationary measure μ_a with $\mu_a(\{x\}) = a_x$.

Examples. (1) (Percolation). $(a_e, e \in E_d)$ i.i.d. Bernoulli(p) r.v.
(2) (iid RCM) $(a_e, e \in E_d)$ i.i.d with some distribution on $[0, \infty)$.
(3) $(a_e, e \in E_d)$ stationary and ergodic

Percolation on $\mathbb{Z}^d$

This was introduced by Broadbent and Hammersley (1957).

Fix $p \in [0, 1]$. For each edge $e = \{x, y\}$ keep the edge with probability p, delete it with probability $1 - p$, independently of all the others.
Let $\mathcal{O} = \mathcal{O}(\omega)$ be the set of edges which are kept, which are called open edges. The connected components of the random graph $(\mathbb{Z}^d, \mathcal{O}(\omega))$ are called (open) clusters.

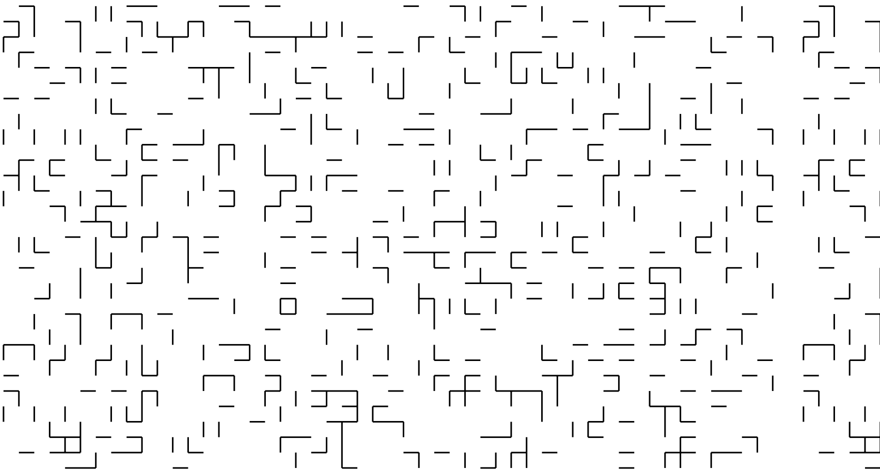
We are interested in the infinite clusters.

There exists $p_c = p_c(d) \in (0, 1)$ such that, with probability 1:

- if $p < p_c$ all clusters are finite (subcritical regime),
- if $p > p_c$ then there exists a unique infinite cluster, C_∞ (supercritical regime),

If $p = p_c$ (critical regime) it is conjectured that all clusters are finite, but only proved in some cases ($d = 2$, $d \geq 11$ or so).

Percolation



Random walk on percolation clusters

In terms of the weights a defined earlier,

$$a_e(\omega) = \begin{cases} 1 & \text{if } e \text{ is open, i.e. } e \in \mathcal{O}(\omega) , \\ 0 & \text{if } e \text{ is closed i.e. } e \notin \mathcal{O}(\omega) \end{cases}$$

So X will not jump across a closed bond, and if $X_t = x$ then X is equally likely to jump across any of the open bonds e which have x as one end. X cannot (a.s.) move between open clusters.

If $p < p_c$ then the long range behaviour of X is not interesting – it will remain in the (typically small) finite cluster it started in.

So we restrict to the case $p > p_c$, and would like to have:
(a) Gaussian bounds/PHI,
(b) Quenched functional CLT (QFCLT): the rescaled random walk $X_t^{(n)} = n^{-1/2} X_{nt}$ converges to Brownian motion.

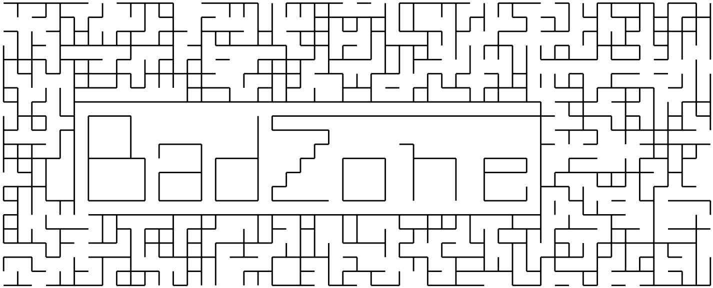
The PHI Theorem – graph case

Theorem *(T. Delmotte, 1999). Let $\mathbb{G}$ be a (locally finite) graph with local regularity (controlled weights). The following are equivalent:*
(a) $\mathbb{G}$ satisfies (VD) and (PI),
(b) Solutions of the heat equation on $\mathbb{G}$ satisfy a PHI,
(c) The heat kernel $p_t(x, y)$ satisfies (GB).

We would like to use this result for percolation, i.e. for the random subgraph $(\mathcal{C}_\infty, \mathcal{O}|_{\mathcal{C}_\infty})$ of $\mathbb{Z}^d$. This graph does satisfy 'controlled weights'.

$p > p_c$: random walk on $\mathcal{C}_\infty$

However, neither VD nor PI hold for $\mathcal{C}_\infty$. The reason is that if we look far enough we can find arbitrarily large ‘bad regions’:



BUT: Big bad regions are a long way away

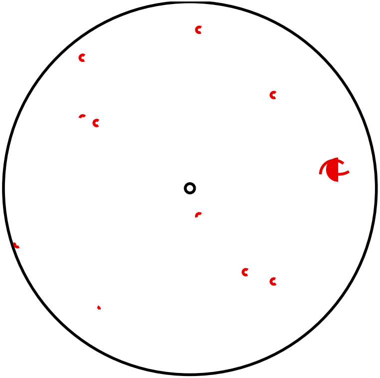
Suppose we are looking for a specific bad configuration of volume r. This has probability of order e^{-cr}.

So to find it in $B(0,R)$ we need $R^d e^{-cr} \approx 1$, or $r \approx \log R$.

Hence one expects the biggest 'bad region' in $B(0,R)$ to be of size $O(\log R)$.

Will these cause 'log corrections' in (GB)?

No. The time to leave a bad region is about $(\log R)^2$, and this is much less than the time to leave $B(0,R)$, which is R^2.



Isoperimetric inequalities for percolation

Fix suitable (non random) constants $C_1 C_2, C_3$. Call a ball $B(x,r) \subset \mathcal{C}_\infty$ *good* if both volume and PI are 'about right' for $B(x,r)$:

$$C_1 r^d \le |B(x,r)| \le C_2 r^d,$$

PI holds (with constant C_3) for $B(x,r)$.

Theorem *(Benjamini-Mossel, Mathieu-Remy, MB.) If $p > p_c$ then*

$$\mathbb{P}(B(x,R) \text{ is good }) \ge 1 - e^{-R^\delta}.$$

Proof. It is enough to look at connected sets A, and there are $e^{c_1 n}$ connected sets A with $0 \in A$ and $|A| = n$. If $p > 1 - \varepsilon$ then for each A the isoperimetric inequality fails with probability $e^{-b(\varepsilon)n}$, where $b(\varepsilon) > c_1$.
For general $p \in (p_c, 1)$ one then uses a renormalization argument.

Natural guess: if $B(x_0, R)$ is 'good', then $p_t(x,y)$ should satisfy (GB) when

$$t \approx R^2, \quad x, y \in B(x_0, R/2).$$

Very good balls

This is the right general idea, but ‘good’ is not enough. The proofs of (GB) all use iterative methods or differential inequalities, which rely on the space being regular over a range of length scales.

To control $p_t(x,y)$ as above one needs to have $B(z,r)$ ‘good’ for all $z \in B(x_0,R)$, and $R^{1-\varepsilon} < r \leq R$.

Corollary. *If* $p > p_c$ *and if* $\theta \in (0,1)$ *then*

$$\mathbb{P}(\text{ every ball } B(y,r) \subset B(x,R) \text{ with } R^\theta \leq r \leq R \text{ is good}) \geq 1 - e^{-cR^{\theta\delta}} \quad (*)$$

Call a ball satisfying the condition in (*) *very good*: the current proofs need ‘very good’ not just ‘good’.

Gaussian bounds for $\mathcal{C}_\infty$

Theorem *(MB, 2004) Let $p > p_c$. For each $x \in \mathbb{Z}^d$ there exist r.v. $T_x(\omega) \ge 1$ with*

$$\mathbb{P}(T_x \ge n, x \in \mathcal{C}_\infty) \le c \exp(-n^\varepsilon) \tag{3}$$

and (non-random) constants $c_i = c_i(d,p)$ such that the transition density of X satisfies, for $x, y \in \mathcal{C}_\infty(\omega)$, $t \ge \max(T_x(\omega), c|x-y|)$:

$$\frac{c_1}{t^{d/2}} e^{-c_2|x-y|^2/t} \le p_t^\omega(x,y) \le \frac{c_3}{t^{d/2}} e^{-c_4|x-y|^2/t}, \tag{GB}$$

1. The randomness of the environment is taken care of by the $T_x(\omega)$, which depend on the percolation configuration near x. These r.v. will usually be small, but will be large for points in big bad regions.
2. Good control of the tails of the r.v. T_x is essential for applications.
3. The proof used 'Nash' rather than 'Moser'.

Nash's idea (PDE setting)

- The key hard step in Nash's 1958 paper was to prove that if $M_x(t) = \sum_y |x - y| p_t(x, y) dy$ then

$$c_1 t^{1/2} \leq M_x(t) \leq c_2 t^{1/2}. \quad (1)$$

- He considered the entropy $Q_x(t) = -\sum_y p_t(x, y) \log p_t(x, y) dy$, and found an ingenious, but not very transparent argument using three inequalities between M_x and Q_x:

$$Q_x(t) \geq c + \tfrac{1}{2} d \log t, \quad (2)$$

$$M_x(t) \geq c e^{Q_x(t)/d}, \quad (3)$$

$$Q_x'(t) \geq c M_x'(t)^2. \quad (4)$$

Lemma *(Nash (1958).) If functions Q, M satisfy* (**??**) – (**??**)*, and* $M(0) = 0$*, then M satisfies (1).*

Proof. First year calculus.

Nash-Bass method

- Nash just obtained Hölder continuity for $u(x,t)$, but Richard Bass showed how the upper bound on M leads to (GB).
- This technique also works for graphs. It is useful for percolation clusters, because if we fix a base point x then 'distant bad regions' have little effect on $M_x(t)$ and $Q_x(t)$.
- One has to prove the three inequalities (**??**) – (**??**):
- $Q_x(t) \geq c + \frac{1}{2} d \log t$ follows from an upper bound on $p_t^\omega(x,x)$ proved by Mathieu and Remy, which comes from VD+PI for 'very good' balls.
- $M_x(t) \geq c e^{Q_x(t)/d}$ just uses $|B(x,r)| \leq c r^d$.
- $Q_x'(t) \geq c M_x'(t)^2$ holds in general.

Chaining to obtain Gaussian upper bound

We have $M_x(t) = \mathbb{E}^x|X_t - x| \le ct^{1/2}$, and it follows that

$$\mathbb{P}^x(\tau(x,r) < c_1 r^2) \le \tfrac{1}{2}, \text{ where } \tau(x,r) = \inf\{t \ge 0 : |X_t - x| > r\}.$$

Look at the sequence of times $\sigma_1 = \tau(x,r)$,

$$\sigma_2 = \inf\{s \ge 0 : |X_{\sigma_1+s} - X_{\sigma_1}| > r\}, \text{ etc.}$$

Then

$$S = \sum_{i=1}^{n} \sigma_i \ge \sum_{i=1}^{n} c_1 r^2 1_{(\sigma_i \ge c_1 r^2)} \ge c_1 r^2 \texttt{Bin}(n, \tfrac{1}{2}),$$

so that

$$P(|X_{c_1\delta n r^2} - x| \ge nr) \le P(S < c_1 \delta n r^2) \le e^{-cn}.$$

Let $R = nr$, $T = c_1\delta n r^2 = c_1 \delta n R^2$; then

$$P(|X_T - x| > R) \le P(S < T) \le e^{-c' R^2/T}.$$

The full Gaussian upper bound for X follows.

Gaussian lower bound

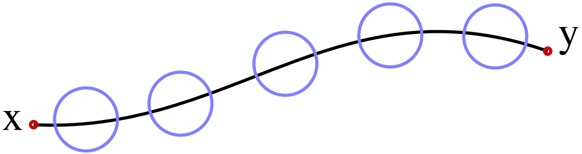
This follows by a very general argument of Fabes-Stroock.

Step 1. Use the PI to show that the function $\log p_t(x, \cdot)$ cannot oscillate too much on the ball $B(x, t^{1/2})$. This gives the 'near diagonal lower bound'

$$p_t(x, y) \geq ct^{-d/2} \quad \text{if } d(x, y) \leq ct^{1/2}.$$

Step 2. Use 'chaining' to obtain the general Gaussian lower bound.
If $d(x, y) = R$ and we want to bound $p_T(x, y)$, choose $n \simeq R^2/T$ so that if $r = R/n$, $t = T/n$ then $r \simeq t^{1/2}$. Choose a sequence of balls $B(z_i, r)$ linking x, y; using the CK equations for $p_t(., .)$ one gets

$$p_T(x, y) \geq ct^{-d/2} e^{-cn} \geq cT^{-d/2} e^{-cR^2/T}.$$



Functional CLT

Theorem *(Sidoravicius and Sznitman, ($d \geq 4$), Berger and Biskup, Mathieu and Piatnitski). Let $p > p_c$. For a set of ω with probability one, a FCLT holds for X, i.e. the rescaled SRW*

$$X_t^{(n)} = n^{-1/2} X_{nt}$$

converges to (a constant multiple of) Brownian motion.

For $d \geq 3$ the proofs use the Gaussian upper bounds.

The CLT is *quenched*, i.e. it holds for a set of environments $a_e(\omega)$ which has probability 1.

Proof of CLT

The basic strategy of the proof is to perturb $\mathcal{C}_\infty \subset \mathbb{Z}^d$ into a graph which is harmonic. So we 'move' $x \in \mathcal{C}_\infty$ to $\varphi(x) = \varphi_\omega(x) \in \mathbb{R}^d$, where φ satisfies

$$\mathcal{L}_a \varphi(x) = \sum_{y \sim x} a_{xy}(\varphi(y) - \varphi(x)) = 0, \quad x \in \mathcal{C}_\infty.$$

Set $\chi_\omega(x) = x - \varphi_\omega(x)$; χ_ω is called the ***corrector***. We have

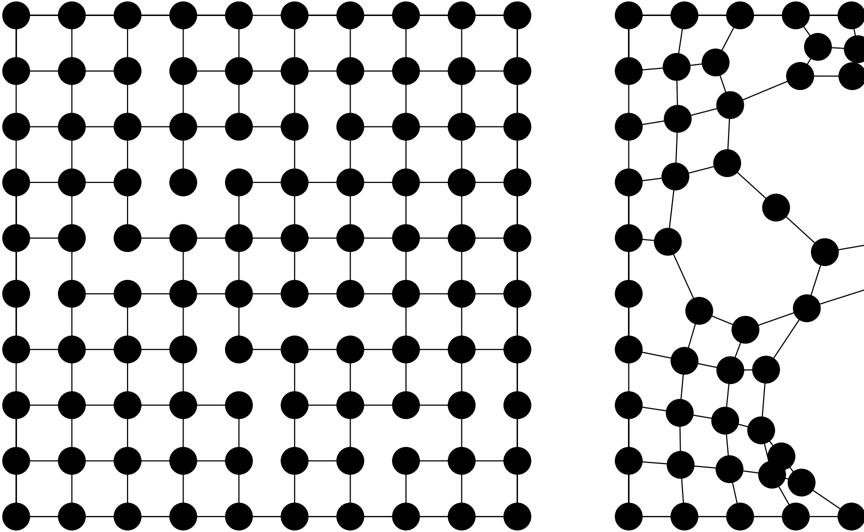
$$X_t = \varphi_\omega(X_t) + \chi_\omega(X_t).$$

$M_t = \varphi_\omega(X_t)$ is a martingale with stationary ergodic increments; a CLT for martingales implies that the rescaled processes $M^{(n)}$ converge to σW, where W is Brownian motion.

The hard part of the argument is to control the corrector: we want

$$\limsup_n \sup_{0 \le s \le nt} \left| \frac{\chi_\omega(X_s)}{\sqrt{n}} \right| \to 0.$$

The function φ in a fixed box



Control of the corrector

Theorem *(Biskup and Prescott). Suppose (GB) hold, and χ has polynomial growth, and is sublinear on average, i.e.*

$$\lim_n n^{-d} \sum_{|x| \le n, x \in \mathcal{C}_\infty} 1_{\{|\chi(x)| \ge \varepsilon n\}} = 0. \qquad (SoA)$$

Then

$$\lim_n \max_{|x| \le n, x \in \mathcal{C}_\infty} n^{-1} |\chi(x)| = 0.$$

Idea of proof. Since $\varphi(M_t)$ is a martingale, and $\varphi(x) = x - \chi(x)$,

$$0 = E^x_\omega(\varphi(X_t) - \varphi(x)) = E^x_\omega(X_t - x) - E^x_\omega(\chi(X_t) - \chi(x)).$$

So using (GB)

$$|\chi(x)| \le E^x_\omega |X_t - x| + |E^x_\omega \chi(X_t)| \le ct^{1/2} + |E^x_\omega \chi(X_t)|.$$

If $t = \varepsilon^2 n^2$ then the final term is small using (SoA) plus (GB).

PHI and Local Limit Theorem

As in other cases, once one has (GB) the PHI follows quite easily. Set

$$p_t^{(n,\omega)}(0,x) = n^{d/2} p_{nt}^{\omega}(\lfloor 0 \rfloor, \lfloor n^{1/2} x \rfloor),$$
$$k_t(x,y) = (2\pi\sigma^2)^{-d/2} \exp(-|x|^2/2\sigma^2 t).$$

The quenched CLT implies that in any small ball $U = B(x_0, \varepsilon)$

$$\int_U p_t^{(n,\omega)}(0,y)dy \to \int_U k_t(0,y)dy.$$

The PHI gives Hölder continuity of $p_t^{(n,\omega)}(x,y)$, and so one can replace the convergence of integrals by pointwise convergence:

Theorem *(MB and Hambly). A local limit theorem also holds:*

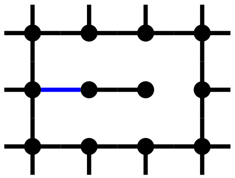
$$p_t^{(n,\omega)}(0,x) \to k_t(x,y).$$

So if one has both (GB) and a CLT one gets very nice pointwise limits.

General RCM

Theorem *(Mathieu, Biskup-Prescott, Andres-MB-Deuschel-Hambly)*. Let a_e be independent and identically distributed with $\mathbb{P}(a_e > 0) > p_c$. Let $\mathcal{C}_\infty$ be the infinite cluster associated with the percolation process $\{e : a_e > 0\}$. Then a QFCLT holds for X started in $\mathcal{C}_\infty$.

Remark. Berger, Biskup, Hoffmann and Kozma showed that GB do **not** hold in general. The reason is that 'traps' are possible:



The blue bond represents an edge with $0 < a(e) \ll 1$, and the black bonds edges with $a_e = O(1)$.
However, (GB) do hold for the SRW on a modified cluster $\mathcal{C}'_\infty$.

Beyond i.i.d.: the stationary ergodic case

It is also interesting to consider a_e with long range correlations.

So now assume just that a_e are stationary and ergodic.

One cannot expect (GB) to hold in general, so one needs to find methods for proving the QFCLT which do not rely on these.

Theorem (Biskup). *Let $d = 2$, and suppose*

$$\mathbb{E}\mu_e < \infty, \quad \mathbb{E}\mu_e^{-1} < \infty.$$

Then a QFCLT holds with limit σW, with $\sigma > 0$.

The proof uses that the fact (special to $d = 2$) that sublinearity on average for the corrector χ implies pointwise sublinearity.

Theorem *(Andres, Deuschel, Slowik 2014–2015).* Let $d \geq 2$. Let (a_e) be stationary and ergodic with $\mathbb{P}(a_e > 0) = 1$, and assume there exist p, q with

$$p^{-1} + q^{-1} < 2/d$$

such that

$$\mathbb{E}\, a_e^p < \infty, \quad \mathbb{E}\, a_e^{-1/q} < \infty.$$

Then the QFCLT holds.

The proof of this Theorem uses Moser's ideas, rather than those of Nash.

A counterexample (MB, Burdzy, Timar) shows that the QFCLT may fail if $\mathbb{E} a_e = \infty$.

Conjecture. The QFCLT holds if $\mathbb{E}\, a_e < \infty$.