Harnack inequalities - from PDE to random graphs

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Plan of the lectures

Today

Lecture 1: Introduction to Harnack inequalities (formulation, history)

Lecture 2: Applications of Harnack inequalities, some ideas for proofs

Thursday

Lecture III: Applications to random graphs

Lecture IV: 'Fractal' graphs and the stability of the elliptic Harnack inequality

Brief aim of lectures. We will see how methods developed by researchers in PDE can be used to study Markov processes in discrete settings, such as random graphs.

Harmonic functions for a Markov process

Let $(X_t, t \in \mathbb{R}_+)$ be a Markov process on a metric space $(\mathcal{X}, d)$. $\mathbb{P}^x$ is the law of X started at x. Let $D \subset \mathcal{X}$ be a domain in $\mathcal{X}$. Define the exit time from D by

$$\tau_D = \inf\{t \geq 0 : X_t \notin D\}.$$

We say a function $h : \mathcal{X} \to \mathbb{R}$ is harmonic in D if $h(X_{t \wedge \tau_D}, t \geq 0)$ is a (local) martingale.

Equivalent definition (modulo some integrability/regularity questions). We can define the infinitesimal generator $\mathcal{L}_X$ by

$$\mathcal{L}_X f(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}^x f(X_t) - f(x)}{t}.$$

h is harmonic in D if

$$\mathcal{L}_X h(x) = 0, \quad x \in D.$$

We may also say X-harmonic.

Examples of infinitesimal generators

1. Let X be the continuous time Markov chain on the countable set $\mathbb{V}$, with

$$\mathbb{P}(X_{t+h} = y | X_t = x) = a(x,y)h + o(h), \quad y \neq x.$$

Then

$$\mathcal{L}_X f(x) = \sum_y a(x,y)(f(y) - f(x)).$$

2. Let W be Brownian motion on $\mathbb{R}^d$, and Δ be the Laplacian. Then

$$\mathcal{L}_W f(x) = \tfrac{1}{2}\Delta f(x) = \tfrac{1}{2}\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} f(x).$$

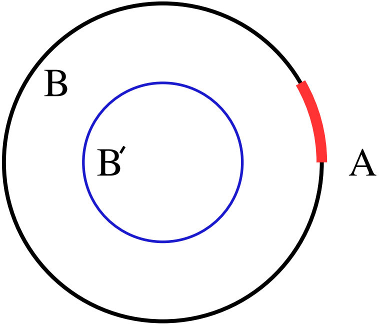
(W-harmonic is the same as the standard definition of harmonic.)

Elliptic Harnack inequality (EHI)

Definition. $(\mathcal{X}, d, X)$ satisfies the Elliptic Harnack inequality (EHI) if there exists $C_H < \infty$ such that whenever $h \geq 0$ is harmonic in $B = B(x, R)$ then writing $B' = B(x, R/2)$,

$$\sup_{B'} h \leq C_H \inf_{B'} h.$$

Meaning. A typical h is $h_A(x) = \mathbb{P}^x(X_{\tau_B} \in A)$ where $A \subset \partial B$. So EHI gives good 'mixing' properties of X - the probability of exiting B via A does not differ too much inside B'.



Brief History 1

Theorem. (Harnack 1887). Let $h : B(0,R) \to \mathbb{R}_+$ be harmonic (with respect to the Laplacian on $\mathbb{R}^d$). Let $r < R$. Then for any $x \in B(0,r)$,

$$\Big(\frac{R}{R+r}\Big)^{d-2} \frac{R-r}{R+r} \le \frac{h(x)}{h(0)} \le \Big(\frac{R}{R-r}\Big)^{d-2} \frac{R+r}{R-r}.$$

Proof. Easy from the Poisson formula; if $r < s < R$ then

$$h(x) = \frac{s^2 - |x|^2}{\omega_d s} \int_{\partial B(0,\, s)} \frac{h(y)}{|x-y|^d} \sigma(dy).$$

Remark. The EHI as stated on the previous slide follows immediately.

Divergence form PDE in $\mathbb{R}^d$

Let $a(x) = (a_{ij}(x), 1 \leq i,j \leq d)$ be a symmetric matrix, which is uniformly elliptic: there exists $A \geq 1$ such that for all $x \in \mathbb{R}^d$

$$A^{-1}|\xi|^2 \leq \sum_{i,j} \xi_i a_{ij}(x)\xi_j \leq A|\xi|^2.$$

Define

$$\mathcal{L}_a f(x) = \nabla(a.\nabla f) = \sum_i \frac{\partial}{\partial x_i}\Big(\sum_j a_{ij}(x) \frac{\partial}{\partial x_j} f(x) \Big).$$

Elliptic divergence form PDE in a domain $D \subset \mathbb{R}^d$:

$$\begin{aligned} \mathcal{L}_a f(x) &= 0, \quad x \in D, \\ f(x) &= g(x), \quad x \in \partial D. \end{aligned}$$

Brief History 2

A major open problem in the late 1950s: go beyond the classical Schauder estimates (1930s) to obtain regularity for solutions of divergence form PDEs. Solved independently by de Giorgi, Nash and Moser.

Moser 1961: proved EHI for solutions to divergence form PDE.

Moser 1964: proved the stronger parabolic Harnack inequality (PHI) for solutions to heat equation $\partial_t u = \mathcal{L}_a u$.

Bombieri-Giusti 1972: EHI in manifold context.

Fabes-Stroock 1986: Proved PHI using ideas of Nash, made two-way connection between PHI and Gaussian heat kernel bounds.

Li-Yau 1986: PHI using gradients for manifolds.

Grigoryan 1992, Saloff-Coste 1992: gave characterization of PHI via conditions which are (a) 'stable' (b) often easy to check.

1990s: Extensions to metric spaces (Sturm), graphs (Delmotte).

Applications of EHI: Hölder continuity

Let h be harmonic in a domain D, assume EHI holds. Set

$$\operatorname{Osc}(h, B) = \sup_B h - \inf_B h.$$

Suppose $B = B(x, R) \subset D$. Choose a, b so that $u = a + bh$ satisfies $\sup_B u = 1$, $\inf_B u = 0$, $u(x) \geq \frac{1}{2}$. Then the EHI gives

$$\tfrac{1}{2} \leq \sup_{B'} u \leq C_H \inf_{B'} u, \qquad \text{i.e. } \inf_{B'} u \geq 1/(2C_H).$$

So writing $\delta = 1/(2C_H)$,

$$\operatorname{Osc}(u, B') \leq (1 - \delta) = (1 - \delta)\operatorname{Osc}(u, B).$$

By linearity this holds for h, and iterating

$$\operatorname{Osc}(h, B(x, 2^{-n}R)) \leq (1 - \delta)^n \operatorname{Osc}(h, B(x, R)),$$

which gives Hölder continuity of h.

Applications of EHI: Liouville property

Definition. X satisfies the strong Liouville property (SLP) if whenever $h \geq 0$ is harmonic on the whole space $\mathcal{X}$ then h is constant.

Theorem. If X satisfies the EHI then the SLP holds.

Proof. Suppose $h \geq 0$ is non-constant and harmonic. Replacing h by $h - \inf h$ we can assume $\inf h = 0$. Choose $x \in \mathcal{X}$ with $h(x) > 0$. Let $y \in \mathcal{X}$. Then by the EHI in the ball $B(y, 4d(x, y))$,

$$h(y) \geq \inf_{B(y,2d(x,y))} h \geq C_H^{-1} \sup_{B(y,2d(x,y))} h \geq C_H^{-1} h(x),$$

so $\inf h > 0$, a contradiction.

SLP holds for $\mathbb{R}^d$, and this is connected with the fact that there is only one way for BM to 'go to infinity'.

Weighted graphs

The PDE methods of Moser etc are very general, and can be applied to processes on manifolds, metric spaces and graphs. I will mainly discuss *continuous time simple random walks on weighted graphs*.

Weighted graphs. Let $(\mathbb{V}, E)$ be a connected graph (finite or infinite) and $a : E \to (0, \infty)$ be edge weights. We call $\mathbb{G}_a = (\mathbb{V}, E, a)$ a weighted graph.
The natural weights are $a(e) = 1$ for all $e = 1$.
We extend a to a function $a : \mathbb{V} \times \mathbb{V} \to [0, \infty)$ by setting $a_{xy} = 0$ if $\{x, y\}$ is not an edge. We set $a_x = \sum_y a_{xy}$. Note that $a_{xy} = a_{yx}$.
Assume:
(1) No multiple edges or self-loops -i.e. $\{x, x\}$ is not an edge.
(2) $\mathbb{G}$ is locally finite: for each x the set $\{y : y \sim x\} = \{y : \{x, y\} \in E\}$ is finite.
We write $d(x, y)$ for the shortest path metric on $\mathbb{G}$, and define balls by

$$B(x, r) = B_{\mathbb{G}}(x, r) = \{y : d(x, y) \leq r\}.$$

Random walk on a weighted graph

The continuous time simple random walk (CTSRW) $X = (X_t, t \in \mathbb{R}_+)$ on $\mathbb{G}_a$ makes jumps from x to $y \sim x$ at rate a_{xy}/a_x:

$$\mathbb{P}^{\cdot}(X_{t+h} = y | X_t = x) = h \frac{a_{xy}}{a_x} + o(h).$$

Properties and notation

(1) X is reversible or symmetric with respect to a:

$$a_x \mathbb{P}^x(X_t = y) = a_y \mathbb{P}^y(X_t = x).$$

(2) Define the heat kernel to be the transition density of X with respect to a:

$$p_t(x, y) = \frac{\mathbb{P}^x(X_t = y)}{a_y}.$$

Then $p_t(x, y) = p_t(y, x)$.

This symmetry is very important: the methods I will discuss do not work well, or at all, for non-symmetric processes.

Overall goal

We have a weighted graph $\mathbb{G}_a$. Using various kinds of information about the geometry of $\mathbb{G}_a$ we want to prove results about the long term behaviour of the CTSRW X on $\mathbb{G}_a$.

We will be particularly interested in obtaining estimates on the heat kernel $p_t(x, y)$, as these enable us to 'read off' many properties of the process. For example, X is transient iff

$$\int_1^\infty p_t(x, x) dt < \infty.$$

Parabolic Harnack inequality

Let $T = R^2$ and $Q = B(x, R) \times [0, T]$ be a space-time cylinder. Set

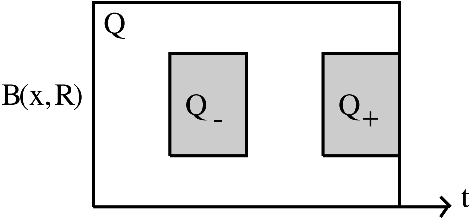
$$Q_- = B(x, \tfrac{1}{2}R) \times [\tfrac{1}{4}T, \tfrac{1}{2}T], \quad Q_+ = B(x, \tfrac{1}{2}R) \times [\tfrac{3}{4}T, T].$$

The PHI states that if $u = u(x, t)$ is a non-negative solution of

$$\frac{\partial u}{\partial t} = \mathcal{L}u, \quad (x, t) \in Q, \tag{1}$$

(we say that u is caloric) then

$$\sup_{Q_-} u \le C_P \inf_{Q_+} u.$$



If h is harmonic in B, then $u(x, t) = h(x)$ is caloric, so PHI $\Rightarrow$ EHI.

Characterization of PHI

Theorem (Grigoryan, Saloff-Coste, Sturm, Delmotte)

Given suitable local regularity, the following are equivalent:

(a) PHI holds,

(b) Gaussian heat kernel bounds (GB) hold,

(c) $\mathcal{X}$ satisfies VD= 'volume doubling' plus PI= Poincaré inequality.

Remarks. (0) See the next few slides for the definition of GB, VD, PI.

(1) This theorem has versions for manifolds, metric spaces and graphs.

(2) For weighted graphs local regularity is 'controlled weights', which we will assume from now on: there exists $p_0 > 0$ such that

$$\frac{a_{xy}}{a_x} \geq p_0 > 0 \quad \text{for all } y \sim x.$$

Gaussian heat kernel bounds (GB)

We say that (GB) hold if there exist c, C such that whenever $x, y \in \mathbb{V}$ and $t \geq d(x,y) \vee 1$ then

$$p_t(x,y) \leq \frac{C}{\mu_a(B(x, ct^{1/2}))} \exp(-c\frac{d(x,y)^2}{t}),$$
$$p_t(x,y) \geq \frac{c}{\mu_a(B(x, Ct^{1/2}))} \exp(-C\frac{d(x,y)^2}{t}).$$

Notes. (1) $\mu_a(A) = \sum_{x \in A} a_x$, i.e. we use $a.$ to define a measure on $\mathbb{V}$.
(2) If $\mathbb{G} = \mathbb{Z}^d$ and a are natural weights, then $\mu_a(B(x,r)) \asymp r^d$ and we obtain the familiar $ct^{-d/2} \exp(-Cd(x,y)^2/t)$.
(3) If $t \leq d(x,y)$ then we no longer have Gaussian bounds; the dominant term is the tail of the Poisson distribution.

VD and PI

Volume doubling. There exists C_V such that

$$\mu_a(B(x,2r)) \le C_V \mu_a(B(x,r)) \text{ for } x \in \mathcal{X}, r > 0.$$

Remarks. (1) VD holds for $\mathbb{Z}^d$, as $\mu_a(B(x,r)) \asymp r^d$.
(2) VD implies polynomial volume growth, i.e. there exists $\alpha < \infty$ such that $a(B(x,r)) \le c_x r^\alpha$. So this condition excludes graphs with exponential growth such as the binary tree.

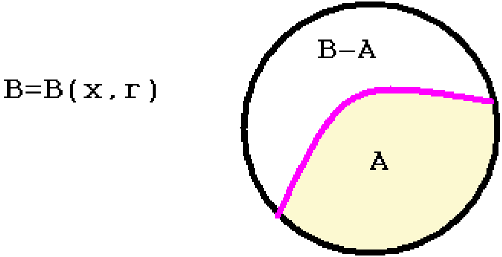
Poincaré inequality. There exist C_P such that if $f : 2B = B(x_0, 2R) \to \mathbb{R}$ and $\overline{f}_B$ is the average of f on B,

$$\sum_{y \in B(x_0,R)} (f(y) - \overline{f}_B)^2 a_y \le C_P R^2 \frac{1}{2} \sum_{x,y \in 2B} a_{xy}(f(x) - f(y))^2 = C_P R^2 \mathcal{E}_{2B}(f,f).$$

Poincaré inequality (PI)

The PI follows from an isoperimetric inequality. Suppose for all $B = B(x,R)$ and subsets $A \subset B$ with $\mu_a(A) \le \frac{1}{2}\mu_a(B)$ we have

$$\mathcal{F}_a(A, B-A) = \sum_{x\in A}\sum_{y\in B-A} a_{xy} \ge \frac{c\mu_a(A)}{R}.$$



Poincaré inequality from isoperimetric inequality

Let $f = 1_A$. Then

$$\sum_x |f(x) - \bar{f}_B| a_x \asymp \mu_a(A), \quad \sum_{x,y} a_{xy} |f(x) - f(y)| = \mathcal{F}_a(A, B - A).$$

So the isoperimetric inequality $\mu_a(A)/R \leq c\mathcal{F}(A, B - A)$ implies a '1-1' PI for this f:

$$\sum_x |f(x) - \bar{f}_B|^1 a_x \leq cR \sum_{x,y} a_{xy} |f(x) - f(y)|^1.$$

One can then extend to general f, and the '2-2' PI follows by an appropriate use of Cauchy-Schwarz.

A graph for which PI fails

$$\sum_{B(x,R)} |f(x) - \bar{f}_B|^2 a_x \le cR^2 \sum_{x,y \in B(x,R)} a_{xy} |f(x) - f(y)|^2 \qquad \text{(PI)}$$

Consider two copies of $\mathbb{Z}^3$ connected at their origins 0_1 and 0_2. Let $f = 1$ on one copy, and $f = -1$ on the other. Let $B = B(0_1, R)$. Then $\bar{f}_B \simeq 0$ and

$$\sum_B |f(x) - \bar{f}_B|^2 a_x \asymp R^3, \quad \sum_{x,y \in B} a_{xy} |f(x) - f(y)|^2 = (f(0) - f(0'))^2 = 1.$$

Remark. The EHI also fails for this graph. Call the two copies $\mathbb{V}_1$ and $\mathbb{V}_2$. As SRW on $\mathbb{Z}^3$ is transient, ultimately X will remain in one copy. Let

$$h(x) = \mathbb{P}^x(X_t \in \mathbb{V}_1 \text{ for all large } t).$$

Then $0 < h < 1$, h is harmonic and non-constant, so the SLP fails. As EHI implies SLP, EHI must also fail. Also easy to see directly by looking at h in $B(0, R)$.

Recall the PHI theorem

Theorem

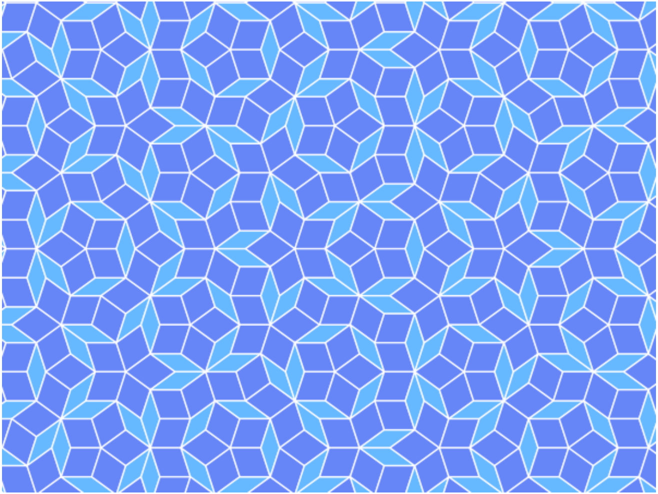
Given suitable local regularity, the following are equivalent:

(a) PHI holds,

(b) Gaussian heat kernel bounds (GB) hold,

(c) $\mathcal{X}$ satisfies VD= 'volume doubling' plus PI= Poincaré inequality.

Why is this theorem useful?



Random walk on graph given by Penrose tiling

The Penrose tiling gives a non-periodic bounded degree graph $\mathbb{G}_{Pen}$ embedded in $\mathbb{R}^2$.
We expect that the CTSRW X^{Pen} on this graph (with natural weights) will have similar long term behaviour to the CTSRW on $\mathbb{Z}^2$.
I do not know how to prove this by probabilistic methods.

On the other hand, for the graph $\mathbb{G}_{Pen}$:
(1) One has $|B_{\mathbb{G}_{Pen}}(x,r)| \asymp r^2$, which implies (VD).
(2) The isoperimetric inequality for $\mathbb{G}_{Pen}$ follows from the isoperimetric inequality for $\mathbb{R}^2$ or $\mathbb{Z}^2$ fairly easily. (If there were a 'bad set' for $\mathbb{G}_{Pen}$ then one could construct a similar 'bad set' for $\mathbb{Z}^2$.)

So, the conditions (VD) + (PI) hold for $\mathbb{G}_{Pen}$, and the implication (c) $\Rightarrow$ (b) then gives (GB).

Perturbations of graphs: example

We have (GB) for the CTSRW on $(\mathbb{Z}^d, E_d)$ (with natural weights a_{Nat}). Suppose we have $E' \subset E_d$ and look at the CTSRW associated with the weights a with

$$a(e) = 1, \quad e \in E - E',$$
$$a(e) = 2, \quad e \in E'.$$

Then $a \asymp a_{Nat}$, and it is straightforward to verify that (VD) and (PI) hold for $(\mathbb{Z}^d, E_d, a)$. Hence the CTSRW on this graph satisfies (GB).

Perturbations of graphs II

Recall the 'PHI' Theorem gives that PHI $\Leftrightarrow$ GB $\Leftrightarrow$ (VD)+(PI).

Theorem. *Let* $(\mathbb{G}, a)$ *be a weighted graph and suppose that the CTSRW on this graph satisfies (GB). Let* $a'(e), e \in E$ *be weights such that* $a' \asymp a$*, i.e.*

$$c_1 a(e) \le a'(e) \le c_2 a(e) \text{ for all } e \in E.$$

Then the CTSRW on $(\mathbb{G}, a')$ *satisfies (GB).*

Proof. (1) Use (GB) $\Rightarrow$ (VD) + (PI) to deduce that $(\mathbb{G}, a)$ satisfies (VD) + (PI) .

(2) If the conditions (VD) + (PI) hold for $(\mathbb{G}, a)$ then they hold for $(\mathbb{G}, a')$. (These conditions are stable under perturbation of weights).

(3) (VD) + (PI) $\Rightarrow$ (GB) to deduce that $(\mathbb{G}, a')$ satisfies (GB).

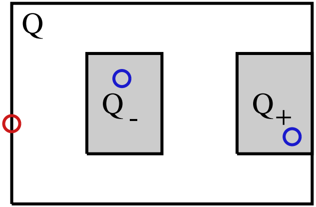
Proofs

Recall the 'PHI' Theorem gives that PHI $\Leftrightarrow$ GB $\Leftrightarrow$ (VD)+(PI).

There are several proofs of this result, all of which use ideas of de Giorgi, Moser, Nash.
The easier arguments are for the equivalence of PHI, GB, and that these imply (VD)+(PI). The proof that the 'low level' conditions (VD)+(PI) imply PHI or GB is harder.

From GB to PHI

Recall PHI says for $u \geq 0$ caloric, $T = R^2$ one has $\sup_{Q-} u \leq C \inf_{Q+} u$.



Suppose that we have

$$u(y,t) = \sum_{z \in B(x,R)} v(z) p_t(z,y). \qquad (1)$$

(GB) implies that

$$c_1 T^{-\alpha/2} \leq p_t(z,y) \leq c_2 T^{-\alpha/2} \text{ if } z,y \in B(0,R) \text{ and } t \in [\tfrac{1}{4}T, T].$$

Feeding this estimate into (1) gives PHI for u.

Note. For a full proof, one also needs to consider 'mass coming in from the side of the cylinder'.

PHI to GB

The whole argument takes several steps, to get the GB bounds (upper and lower) in different 'regimes'.

Start of the argument: let $B' = B(x_0, R/2)$, fix $x \in \mathbb{V}$ and set $u(y,t) = p_t(x,y)$. Then

$$p_{T/2}(x,y) \le \sup_{Q_-} u \le C_H \inf_{Q_+} u \le C_H p_T(x,y') \quad \text{for } y, y' \in B'.$$

Multiply by the weight $a_{y'}$ and sum over $y' \in B$ to obtain

$$p_{T/2}(x,y) \sum_{y' \in B'} a_{y'} \le C_H \sum_{y' \in B'} p_T(x,y') a_{y'} \le C_H,$$

which gives

$$p_{T/2}(x,y) \le \frac{C_H}{\mu_a(B(x_0, R/2))}.$$

Moser's proof

I will now sketch some of the ideas in Moser's proof of the EHI; the proof of the PHI uses the same methods but is more complicated.

I will do it in the context of a divergence form operator on $\mathbb{R}^d$, but explain how the methods generalize.

Moser's proof 1

Setup and notation. $B(r) = B(0,r) \subset \mathbb{R}^d$; we will always have $1 \le r \le 2$.
$\mathcal{L}_a = \nabla a.\nabla$ divergence form operator acting on functions $f : B(2) \to \mathbb{R}$.
$h \ge 0$ a solution of $\mathcal{L}_a h = 0$ in $B(2)$, with $h = h_0$ on $\partial B(2)$.
We write $\int f$ for $\int f dx$.

Our aim is to prove the EHI: $\sup_{B(1)} h \le C \inf_{B(1)} h$.

Lemma M1. (Sobolev inequality). There exists $\kappa > 1$ such that if $f : B(r) \to \mathbb{R}$ then

$$\int_{B(r)} |f|^{2\kappa} \le c_1 \int_{B(r)} \left(|\nabla f|^2 + |f|^2 \right).$$

Proof. Follows from the PI. (Saloff-Coste).

Moser's proof 2

Lemma M2. Let $g = h^p$ with $p \in (-\infty, -\frac{1}{2}) \cup (\frac{1}{2}, \infty)$. If $1 < s < r < 2$ then

$$\int_{B(s)} |\nabla g|^2 \leq c|r-s|^{-2} \int_{B(r)} g^2.$$

Proof. Choose η with $\eta = 1$ on $B(s)$, $\eta = 0$ outside $B(r)$ and $|\nabla \eta| \leq (r-s)$.

$$\int_{B(s)} |\nabla g|^2 \leq \int_{B(r)} \eta^2 |\nabla g|^2 \leq c \int_{B(r)} |\nabla \eta|^2 |g|^2 \leq \frac{c}{|r-s|^{-2}} \int_{B(r)} |g|^2$$

The step $\leq$ uses integration by parts, $\mathcal{L}_a h = 0$, and Cauchy-Schwarz.

Important remark. Lemmas M1 and M2 hold very generally, e.g. for the CTSRW on a graph, if we replace $\int_D |\nabla f|^2$ by the energy form

$$\sum_{x,y \in D} a_{xy} (f(x) - f(y))^2.$$

Moser's proof 3

Set $p_n = \kappa^n$, $r_n = 1 + 2^{-n}$, $B_n = B(r_n), f = h$ or $f = h^{-1}$.

$$\int_{B_{n+1}} |f|^{2p_{n+1}} = \int_{B_{n+1}} |f|^{2\kappa p_n} \leq c_1 \int_{B_{n+1}} \left(|\nabla f^{p_n}|^2 + |f^{p_n}|^2\right) \qquad \text{by M1}$$

$$\leq (c_1 + 1)|r_n - r_{n+1}|^{-2} \int_{B_n} |f^{p_n}|^2 \leq c_2 2^{2n} \int_{B_n} |f^{p_n}|^2. \qquad \text{by M2}$$

So

$$||f||_{B_{n+1},2p_{n+1}} = \left(\int_{B_{n+1}} |f|^{2p_{n+1}} \right)^{1/2p_{n+1}} \leq (c_2 2^{2n})^{1/2p_{n+1}} ||f||_{B_n,2p_n}.$$

Since $\prod_n (c_2 2^{2n})^{1/2p_{n+1}} = C < \infty$ we deduce by iterating

$$\sup_{B(1)} f = ||f||_{B(1),\infty} \leq C||f||_{B(2),2} = C \int_{B(2)} f^2.$$

Hence

$$\sup_{B(1)} h \leq c \int_{B(1)} h^2, \quad \sup_{B(1)} h^{-1} = \frac{1}{\inf_{B(1)} h} \leq c \int_{B(1)} h^{-2}.$$

Moser's proof 4

It remains to prove

$$\int_{B(1)} h^2 \leq \frac{c}{\int_{B(1)} h^{-2}}.$$

For this one needs to use the Poincaré inequality again.
(Moser's original proof used the John-Nirenberg inequality, but Bombieri-Giusti found an easier way.)

Remark. I have given Moser's proof in its original context of divergence form elliptic PDEs, as it's the easiest context in which to explain the ideas. However, the two key Lemmas M1, M2, also work for graphs, and one can use the same basic proof strategy. One looks at balls $B(x, t_n R)$ with $t_n = 1 + 2^{-n}$. To continue the iteration when $2^{-n}R \leq 1$, and so one can no longer use Lemma M2, one uses the local regularity of the graph to show that if $x \sim y$ then $|h(x) - h(y)| \leq ch(x)$.