16/12/2021 Lecture-23

$f_0 \longrightarrow$ objective function

$\boxed{\mathcal{D}} \longrightarrow$ decision space

min. f_0 on $\mathcal{D}$.

Structure of $\mathcal{D}$, and nature of f_0 affects our choice of algorithm for optimization.

$$\left[\min_{\vec{x} \in \mathbb{R}^n} \|A\vec{x} - \vec{b}\|_2 \quad , \quad \min \|\vec{x}\|_2 \text{ given } A\vec{x} = \vec{b}\right)$$

$\mathcal{D}$ — closed convex set

Examples of closed convex sets

(1) Affine spaces (translates of linear subspace of $\mathbb{R}^n$)

(2) Hyperplanes $\quad H_{\vec{a}, b} = \{ \vec{x} \in \mathbb{R}^n : \langle \vec{a}, \vec{x} \rangle = b \}$

$\vec{a}$ normal vector (non-zero vector)

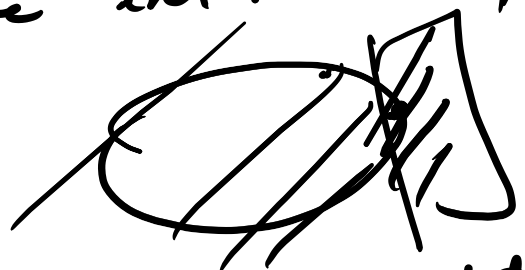
b offset from origin

(3) $\boxed{\text{closed half spaces}}$

$$H^-_{\vec{a}, b} = \{ \vec{x} \in \mathbb{R}^n : \langle \vec{a}, \vec{x} \rangle \le b \}.$$

Two main representation theorems for closed convex sets in $\mathbb{R}^n$

Theorem 1: Each nonempty closed convex set in $\mathbb{R}^n$ is the intersection of its supporting halfspaces.



Theorem 2. (Extremal representation) Each nonempty closed convex set C, in $\mathbb{R}^n$ is of the form $C = \bar{C} \oplus V$, where V is a linear subspace of $\mathbb{R}^n$, and $\bar{C}$ is a

line-free closed convex set in a subspace complementary to V.

+

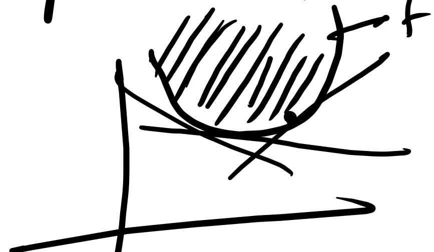
Each line-free closed convex set $C \subseteq \mathbb{R}^n$ is the convex hull of its extreme points and extreme rays.

Minkowski's theorem:

Each compact convex set $K \subseteq \mathbb{R}^n$ is the convex hull of its extreme points.

Supporting hyperplane theorem:

Let $K \subseteq \mathbb{R}^n$ be nonempty, convex and closed. Then through each boundary point of K, there is a support plane of K.



Separating hyperplane theorem:

If A, B are nonempty convex sets in $\mathbb{R}^n$ s.t. $\boxed{A \cap B = \phi}$

then A, B can be separated by a hyperplane.

($A - B$ is convex. Separation of $\vec{0}$ and $A - B$).

$\vec{0} \notin A - B$)

Polyhedron. Determined by (intersection of) finitely many halfspaces and hyperplanes.

Polytope. A compact polyhedron.
finitely many extreme points.

LINEAR PROGRAMMING

min. $\langle \vec{c}, \vec{x} \rangle$

subject to $A\vec{x} \leq \vec{b}$ (general form)

subject to $\boxed{\begin{array}{l} A\vec{x} = \vec{b} \\ \vec{x} \geq \vec{0} \end{array}}$ standard form)

$m \times n$

$m \leq n$. $m = \text{rank } A$.

Lemma: Suppose $P = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} \le \vec{b}\}$ is a polyhedron. Then $\boxed{\vec{v}}$ is (non) an extreme point of P iff there are n linearly independent constraints among the constraints $A\vec{x} \le \vec{b}$, that are tight at $\vec{v}$, that is, satisfy equality at $\vec{v}$.

$\vec{v}$

$$\begin{bmatrix} I_n \\ A \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix} \begin{matrix} = \\ \ge \\ \le \end{matrix} \begin{bmatrix} \vec{0} \\ \vec{b} \end{bmatrix}$$

$m \leq n$.

n linearly independent rows of $\left[\begin{array}{c} I_n \\ \hline A \end{array}\right]$

$$
\left[\begin{array}{c|c}
I_{n-m} & 0 \\
\hline
A_{j_1} \cdots A_{j_{n-m}} & \\
\end{array}\right]
$$

$j_1, \ldots, j_{n-m}$

$x_{j_1} = 0 \quad \ldots \quad x_{j_{n-m}} = 0$

$m \times m$

ii) invertible or full-rank

iff

i) invertible

$\{j_1, \ldots, j_{n-m}\}^c$ in $\{1, \ldots, n\}$

Motivates us to define basic sol'ns.

$$\left[A_1 \; \ldots \; \middle| A_n \right]$$

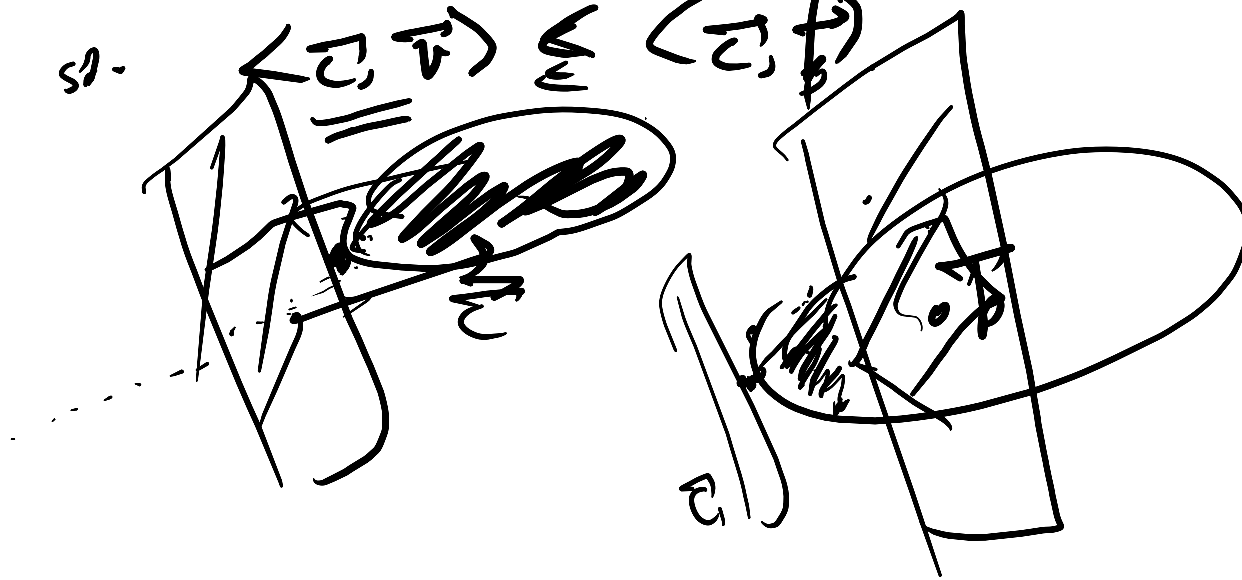
$j_1, \ldots, j_m$

$A_{j_1} \ldots A_{j_m}$ are linearly independ't

$\{j_1, \ldots, j_m\}$ is a basis for $\mathbb{R}^m$.

$[A_{j_1} \ldots | A_{j_m}]^{-1} \vec{b}$ and 0's elsewhere bfs's are vertices of the feasible set.

Theorem: In LPS, let $\vec{p} \in F$. Either LPS is unbounded or there is a vertex $\vec{v}$ of F s.t. $\langle \vec{c}, \vec{v} \rangle \leq \langle \vec{c}, \vec{p} \rangle$



Theorem: In LPs, if $F \neq \phi$, and $\langle \vec{c}, \vec{x} \rangle \geq B$ for all $\vec{x} \in F$, then there is an optimal vertex.

(LP is a finite problem)

SIMPLEX ALGORITHM: Systematic way of traversing vertices of F:

Neighbouring vertex: shares $n-2$ linear (in)equality constraints that are tight.

corresponding basis differs by exactly one column.

Tableaux:

$$\begin{array}{c|ccc} & 1, \ldots & & x_n \\ \hline b_1 & A_1 & \vdots & A_n \\ \vdots & & & \\ b_m & & & \end{array}$$

Pivoting: process of moving to neighbouring bfs (with lower cost)

How to choose a pivot?

$A_{j_1} \to e_1$

$A_{j_m} \to e_m$

(1)

$$\begin{array}{c|cccc}
0 & c_1 & c_2 & \cdots & c_n \\
\hline
b_1 & 1 & 0 & & \\
\vdots & 0 & 1 & & \\
 & 0 & 1 & & \\
b_m & 1 & & &
\end{array}$$

B^{-1}

(2) If j is basic, $\bar{c}_j$ is 0

$$\text{row}(0) = \text{row}(0) - c_{B(i)}\, \text{row}(i)$$

Find column j whose row 0 entry $\bar{c}_j$ is strictly negative. (If no such j exists, then the current basis corresponds to an optimal sol)

(3) j^{th} entry

$$c_j - \sum_{i=1}^{m} x_{ij}\, c_{B_i} = \bar{c}_j$$

$$\begin{bmatrix} \vdots & \begin{bmatrix} x_{1j} \\ \vdots \\ x_{mj} \end{bmatrix} & \end{bmatrix}$$

$$\boxed{\theta_0 = \min_{\{i : x_{ij} > 0\}} \frac{x_{i0}}{x_{ij}}}$$

j enters B

$1 \le i \le m$

$$\boxed{x_{ij} \le 0, \quad \bar{c}_j < 0}$$

LPS is unbounded

$$\theta_0 = \frac{x_{l0}}{x_{lj}}$$

$$\boxed{B(l) \text{ exits}}$$

Bland's rule :-

$\boxed{0}\ \boxed{0}\ \boxed{\bar{c}_j} \cdots < 0 \ \bigcirc$

c_j minimal s.t. $\bar{c}_j < 0$.

Choose l s.t. $\boxed{B(l)}$ is smallest satisfying

$\theta_0 = \dfrac{x_{l0}}{x_{lj}}$.

DUALITY :-

PRIMAL

$\geq$ constraint

no sign constraint

$\geq$ sign constraint

DUAL

$\geq$ sign constraint

$(A^T y)_j = c$

$A^T y \leq \vec{c}$.

Weak duality: DUAL optimal value

$\leq$ PRIMAL optimal value.

Strong duality: (DUAL optimal value

$=$ PRIMAL optimal value.)

Dual simplex algorithm.

Reducing number of vertices can ~~fo~~ speed up simplex.

$A_{m \times n}$ $\quad m >> n$

$\boxed{n << m}$

$\boxed{\text{DUAL}}$ of DUAL is PRIMAL