25/11/2021 Lecture - 17

Economic interpretation of the dual.

(PRIMAL) $\min \langle \vec{c}, \vec{x} \rangle$

s.t. $A\vec{x} = \vec{b}$

$\vec{x} \geq \vec{0}$

$c^* \to$ optimal value

has a nondegenerate basic optimal soln,

$\vec{t} \in \mathbb{R}^n$,

(PRIMAL) $\min \langle \vec{c}, \vec{x} \rangle$

s.t. $A\vec{x} = \vec{b} + \vec{t}$

$\vec{x} \geq \vec{0}$.

$c^* + \langle \vec{u}^*, \vec{t} \rangle$

optimal

$$\langle \bar{u}^*, \bar{t} \rangle = \sum_{i=1}^{m} u_i t_i$$

$t_1 = 1, \quad t_2 = 0, \ldots, t_m = 0$

$$\langle \bar{u}^*, \bar{t} \rangle = \bar{u}_1^* t_1 = u_1^*$$

$u_i^* \longrightarrow$ marginal value of the i^{th} resource

$\langle \bar{c}, \bar{x}_1^* \rangle$

$\hookrightarrow$ primal optimal sol^n

$(c_1 + 1), c_2, \ldots, c_n$

Q: $\max \ \langle \vec{c}, \vec{x} \rangle$

$$A\vec{x} \le \vec{b}$$

$$\vec{x} \ge \vec{0}.$$

Introduce slack variables.

$$\begin{cases} A\vec{x} + I\vec{y} = \vec{b} \\ \vec{x} \ge \vec{0}, \ \vec{y} \ge \vec{0}, \end{cases} \qquad \boxed{\max. \ \langle \vec{c}', \vec{x}' \rangle} \qquad \vec{c}' = \begin{bmatrix} \vec{c} \\ \vec{0} \end{bmatrix}, \ \vec{x}' = \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}$$

If the above equivalent problem has a nondegenerate basic optimal sol'n, then the optimal value of

$$\max \ \langle \vec{c}, \vec{x} \rangle, \quad A\vec{x} \le \vec{b} + \vec{t}, \quad \vec{x} \ge 0.$$

is $\langle \vec{t}, \vec{u}^* \rangle$ is larger

Matrix games and the minimax theorem

von Neumann.

X → holds up one finger or two fingers

Y → does the same.

If X, Y make the same decision. Y wins Rs 100.

If X, Y make different decisions, X wins.

(1) Rs 100 if (s)he put up one finger.

(2) Rs. 200 if (s)he put up two fingers.

$$A = \begin{array}{cc} \begin{bmatrix} -100 & 200 \\ 100 & -100 \end{bmatrix} & \begin{array}{l} \text{one finger by } Y \\ \text{two fingers by } Y \end{array} \\ \begin{array}{cc} \text{one finger by } X & \text{two fingers by } X \end{array} & \end{array}$$

a_{ij} → how much ~~X~~ Y pays to X shows j^{th} fingers and Y shows i^{th} fingers

(payoff matrix)

If X does the same thing every time (or Y sees the pattern), Y will do the same and win every time.

Both players must have a <u>mixed</u> strategy

(Choice at every turn must be independent of previous turns)

<u>Strategy</u>: X puts up one finger with probability x_1

X " two finger with probability $x_2 = 1 - x_1$

(randomly).

If Y puts up one finger,

X will win $-100x_1 + 200x_2$

$= \boxed{200 - 300x_1}$ on average.

If Y puts up two fingers

X will win $100x_1 - 100x_2 = \boxed{200x_1 - 100}$

on average.

X reasons, Y will be able to guess x_1, x_2

after some games.

X wants to choose x_1 to maximize

$$\boxed{\min\{\underbrace{200-300x_1}, \underbrace{200x_1-100}\}}$$

Y reasons:

If I put up one finger with probability y_1, X will win $-100y_1+100y_2$ by repeatedly putting up one finger.

" " two fingers, X will win $200y_1-100y_2$ by putting up two fingers

$$\min \left(\max \{ -100y_1 + 200y_2 ,\ 200y_1 - 100y_2 \} \right)$$

General matrix game

A → payoff matrix

$m \times n$ matrix

~~The optimal values of $x_1, x_2, \ldots, y_1, \ldots$ are unchanged if one adds $\alpha \in \mathbb{R}$ to every entry of A.~~

Find $\alpha \in \mathbb{R}, \vec{x} \in \mathbb{R}^n$,

(P1)

max. α → $\min - \alpha$

s.t. $A\vec{x} \geq \alpha \vec{e}$ → $\begin{bmatrix} \vdots \end{bmatrix}$

$\langle \vec{e}, \vec{x} \rangle = 1$

$\vec{x} \geq \vec{0}$,

$\beta \geq 0$

$[\vec{x}, \alpha]$

$[A \;\; \vec{e}] \begin{bmatrix} \vec{x} \\ -\alpha \end{bmatrix} \geq \vec{0}$,

X's goal: Find $\beta \in \mathbb{R}, \vec{y} \in \mathbb{R}^n$

(P2)

min. β

s.t. $\vec{y}^T A \leq \beta \vec{e}^T$

$\vec{y} \geq \vec{0}$

$(\vec{y}, \beta)$

(P2) is in fact dual to (P1).

$\Rightarrow$ ~~Ø~~ = optimal value of (P1)

= optimal value of P2

Theorem. max min = min max

(von Neumann minimax theorem)

X and Y will have full information about each other's mixed strategies

Optimization: min. $f_0(\vec{x})$ → objective function

$\vec{x} \in \mathcal{D} \subseteq \mathbb{R}^n$

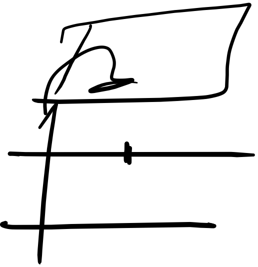
→ decision space

Our strategy to optimize f_0 depends on the nature of f_0 and the nature of $\mathcal{D}$.

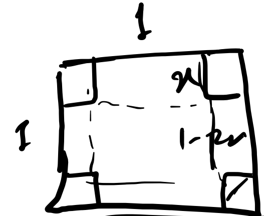
"In fact, the great watershed in optimization is not between linearity and non-linearity, but convexity and non-convexity" - Rockafellar, 1993

$\boxed{f_0 \longrightarrow \text{convex function}}$

$\mathcal{D} \to$ convex set in $\mathbb{R}^n$
(nonempty interior)



CONVEX OPTIMIZATION



$\boxed{x(1-2x)^2}$

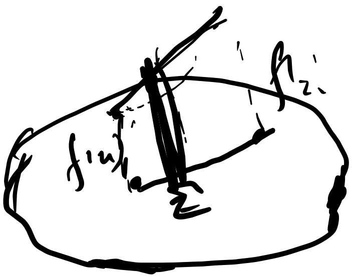
$0 \leq x \leq \frac{1}{2}$

$\mathcal{D} = [0, \frac{1}{2}]$

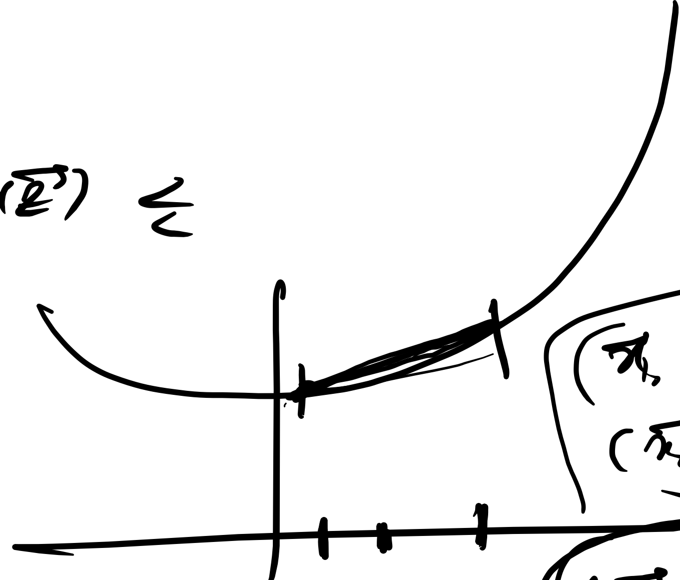
$f_0(x) = x(1-2x)^2$

Defn. $\Omega \subseteq \mathbb{R}^n$ be a convex domain.

$f : \Omega \longrightarrow \mathbb{R}^n$ is said to be a **convex function** if for any $\vec{x}_1, \vec{x}_2 \in \Omega$ and $\lambda \in [0,1]$ we have $f(\lambda \vec{x}_1 + (1-\lambda)\vec{x}_2) \leq \lambda f(\vec{x}_1) + (1-\lambda) f(\vec{x}_2)$



$f(\vec{z}) \le$



$(\vec{x}_1, f(\vec{x}_1))$

$(\vec{x}_2, f(\vec{x}_2))$

$\lambda \vec{x}_1 + (1-\lambda)\vec{x}_2$

$\lambda f(\vec{x}_1) + (1-\lambda) f_1$

Epigraph of f

$= \{ (\vec{x}, \xi) \in \mathbb{R}^n \times \mathbb{R} ;$

$f(\vec{x}) \le \xi \}$

↳ all points in $\mathbb{R}^n \times \mathbb{R}$ lying above the graph of f.

$\underline{f}$ is convex $\Leftrightarrow$ epigraph of f is an convex subset of $\mathbb{R}^n \times \mathbb{R}$.

Examples:

(1) Affine functions:

$f: \mathbb{R}^n \to \mathbb{R}$, $\quad \vec{u} \in \mathbb{R}^n$; $\alpha \in \mathbb{R}$,

$f(\vec{x}) = \langle \vec{u}, \vec{x} \rangle + \alpha$

(2) The supremum of (arbitrarily many) convex functions. (Exercise

$$f_1\left(\frac{\vec{x}+\vec{y}}{2}\right) \le \frac{f_1(\vec{x})+f_1(\vec{y})}{2}$$

$$f_2\left(\frac{\vec{x}+\vec{y}}{2}\right) \le \frac{f_2(\vec{x})+f_2(\vec{y})}{2}$$

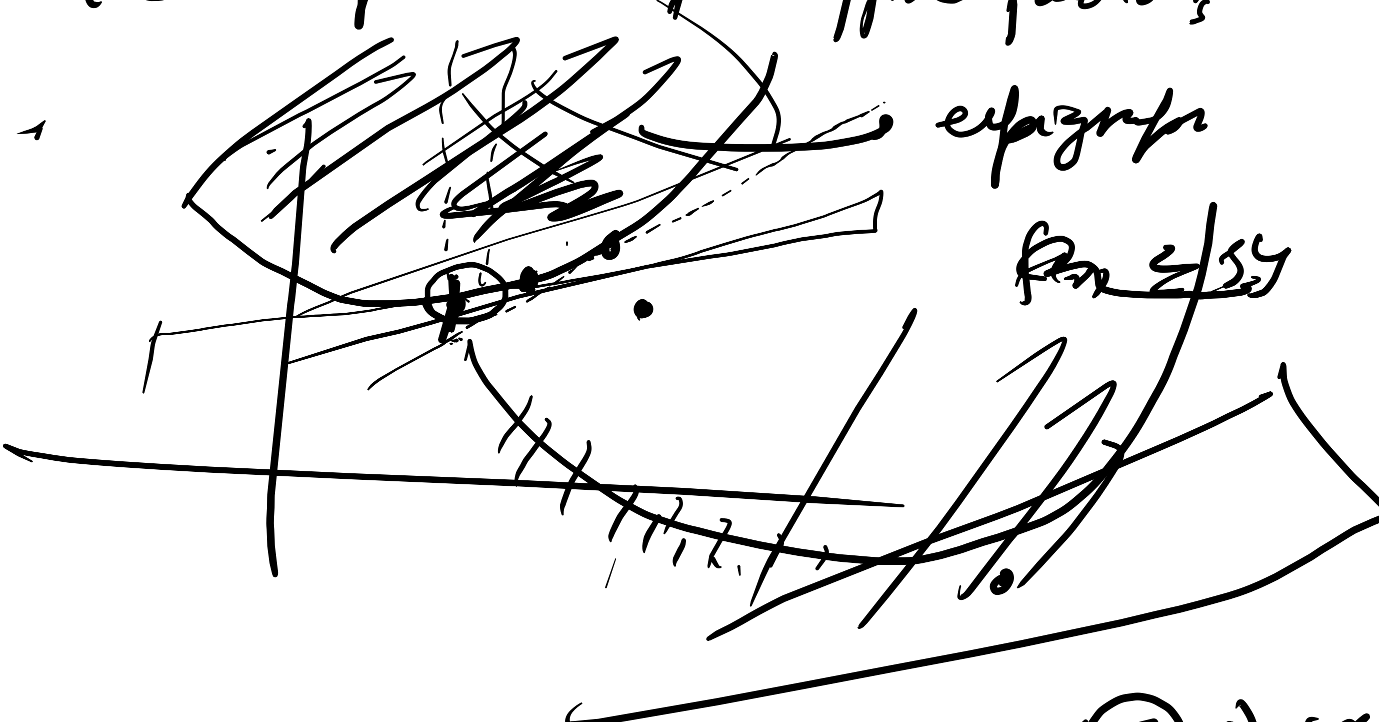
$\vec{x}, \vec{y} \in D$.

$\boxed{\max(f_1, f_2)}$

$\frac{k}{2}$

Theorem: Every convex function $f : \Omega \to \mathbb{R}^n$ is the supremum of affine functions

Proof:



$f(\vec{x}) \geq \langle \vec{u}, \vec{x} \rangle + \alpha$

$\forall \vec{x}$,