23/11/2021 Lecture - 16

PRIMAL

$\boxed{B} \rightarrow$ optimal bfs

$B = [A_{B(1)} \cdots A_{B(m)}]$

DUAL

$\vec{u}^* = (B^{-1})^T c_B$

dual optimal

(1) find a lower bound for LPP.

(2) verification certificate for alleged optimal soln.

(3) solve LPP ~~within~~ faster. $x_1, \ldots, x_n$ $\quad m < n$

$A_{m \times n}$

$a_{11}x_1 \cdots + a_{1n}x_n \leq b_1$

$\vdots$

$a_{m1}x_1 \cdots + a_{mn}x_n \leq b_m$

$m \gg n$

look at the dual LPP.

How to verify $\vec{v^*}$ is in fact an optimal solution to LPP?

COMPLEMENTARY SLACKNESS:

Theorem: Let P be a linear program in general form.

$$P: \quad \min \langle \vec{c}, \vec{x} \rangle$$

$$\text{s.t.} \quad (A)^i \vec{x} = b_i, \quad 1 \le i \le h,$$

$$(A)^i \vec{x} \geq b_i, \quad h+1 \le i \le m$$

$$x_j \geq 0, \quad 1 \le j \le l,$$

$$x_j \gtrless 0, \quad l+1 \le j \le n,$$

D:

$$\max \ \langle \vec{b}, \vec{y} \rangle$$

$$y_i \gtrless 0, \quad 1 \le i \le h$$

$$y_i \geq 0, \quad h+1 \le i \le m$$

$$A_j^T \vec{y} \le c_j, \quad 1 \le j \le l$$

$$A_j^T \vec{y} = c_j, \quad l+1 \le j \le n.$$

Let $\vec{w} \in \mathbb{R}^n$ be PRIMAL feasible, and $\vec{u} \in \mathbb{R}^m$ be DUAL feasible. Then $\vec{w}$ is primal optimal and $\vec{u}$ is DUAL optimal if and only if

$$\left(\left(\sum_{j=1}^{n} a_{ij} w_j\right) - b_i\right) u_i = 0 \quad \text{for } i = 1, \dots, m$$

$$\downarrow$$

$$\left[\left((A)^i \vec{w} - b_i\right) u_i = 0\right]$$

and

$$\left(\left(\sum_{i=1}^{m} a_{ij} u_i\right) - c_j\right) w_j = 0 \quad \text{for } j = 1, \dots, n.$$

$$\downarrow$$

$$\left[w_j \left(c_j - A_j^T \vec{u}\right) = 0\right].$$

(If a dual variable is non-zero, then the corresponding primal constraint must be tight. If a primal variable is non-zero then the corresponding dual constraint must be tight).

Proof:

$$(\cancel{A})((A)^i \vec{w} - b_i) u_i \geq 0, \quad i = 1 \dots m$$

$$w_j (c_j - A_j^T \vec{u}) \geq 0, \quad j = 1 \dots n$$

$$t := \sum_{i=1}^{m} \boxed{((A)^i \vec{w} - b_i) u_i} + \sum_{j=1}^{n} \boxed{w_j (c_j - A_j^T \vec{u})}$$

$$= -\sum_{i=1}^{m} b_i u_i + \sum_{j=1}^{n} w_j c_j$$

$$+ \boxed{\sum_{i=1}^{m} u_i (A)^i \vec{w}} \qquad \sum_{j=1}^{n} w_j (A_j^T \vec{u})$$

$$\langle \vec{u}, (A\vec{w}) \rangle \qquad \langle \vec{w}, A^T \vec{u} \rangle$$

$$\left[\, t = \langle \vec{c}, \vec{w} \rangle - \langle \vec{b}, \vec{u} \rangle \,\right]$$

~~So~~ $\vec{w}$ is primal optimal and $\bar{u}$ is dual optimal ~~if and only if~~ $\Leftrightarrow t = 0$ by strong duality.

$$\Longleftrightarrow \quad \left(\boxed{A_i^T \vec{w} - b_i}\right) u_i = 0, \quad i = 1, \dots, m$$

$$w_j \left(c_j - A_j^T \vec{u} \right) = 0, \quad j = 1, \dots, n$$

$\blacksquare$

What is complementary slackness?

The two conditions in the previous theorem look simple with slack variables.

$$x_{n+i} = (A)^i \vec{x} - b_i, \quad i = 1, \dots, m$$

$$y_{m+j} = c_j - A_j^T \cdot \vec{y}, \quad j = 1, \dots, n.$$

$$\begin{array}{c|c|c|c|c|c}
x_1 & \cdots & x_n & x_{n+1} & \cdots & x_{n+m} \\
\hline
y_{m+1} & \cdots & y_{m+n} & y_1 & \cdots & y_m
\end{array}$$

The two conditions imply that amongst the m+n matching pairs, at least one variable must be zero

Theorem. (Certificate for optimality)

A feasible soln $\vec{w}$ of LP is optimal if and only if there are numbers $u_1, \dots, u_m$ such that

(1) $\boxed{\sum_{i=1}^{m} a_{ij} u_i = c_j}$ whenever $w_j > 0$

(2) $u_i = 0$ whenever $\boxed{\sum_{j=1}^{n} a_{ij} w_j > b_i}$.

(3) [$\vec{u}$ is feasible for dual of LP]

Proof: If $\vec{w}$ is optimal, then DUAL has an optimal soln $\vec{u}$. This $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$

$u_1, \ldots, u_m$ — the entries of $\vec{u}$ satisfy the given conditions.

If $u_1, \ldots, u_m$ satisfy (1), (2), (3), then by complementary slackness, $\vec{w}$ is primal optimal.

∎

Corollary. If $\vec{w}$ is a $\boxed{\text{nondegenerate}}$ bfs, then the system of equations (1), (2) has a unique solⁿ.

Proof. $\sum_{i=1}^{m} a_{ij} u_i = c_j$, whenever $w_j > 0$.

$\vec{w}$ ~~const~~ is a bfs corresponding $\mathcal{B} = [B(1), \ldots, B(m)]$

$$\boxed{\sum_{i=1}^{m} a_{B(i) B(j)} u_i = c_{B(j)}}$$

$B = [A_{B(1)}, \ldots A_{B(m)}]$

$B^T \vec{u} = \vec{C_B}$

$$\boxed{\vec{u} = (B^{-1})^T \vec{c_B}}$$

Example: Consider the following claim:

$$\vec{x}^* = \begin{bmatrix} 4 \\ -1 \\ 2 \\ 0 \\ -3 \end{bmatrix}$$ is an optimal sol^n to the LPP given below;

$$\min \; \boxed{-4}\, x_1 + 2x_2 - 11x_3 + 4x_4 + 0 \cdot x_5$$

s.t. $2x_1 - x_2 + 3x_3 - 2x_4 + x_5 = 6$ (tight at $\vec{x}^*$)

$x_1 + 4x_2 - x_3 - x_4 - 3x_5 \geq 4$ (tight at $\vec{x}^*$)

$4x_1 + 6x_2 - x_3 - 5x_4 + x_5 \geq -8$ (not tight at $\vec{x}^*$)

$x_1 \geq 0$

$x_2 \leq 0$

$x_3, x_4 \geq 0$

$x_5 \leq 0$

DUAL

$$\max \quad 6y_1 + 4y_2 - 8y_3$$

s.t.

$$2y_1 + y_2 + 4y_3 \leq -4$$

$$-y_1 + 4y_2 + 6y_3 = 2$$

$$3y_1 - y_2 - y_3 \leq -11$$

$$-2y_1 - y_2 - 5y_3 \leq 4$$

$$y_1 - 3y_2 + y_3 \leq 0$$

$y_1 \gtrless 0$

$y_2 \geq 0$

$y_3 \geq 0$

$$\begin{bmatrix} \leq \\ = \\ \geq \end{bmatrix}$$

If $\vec{x}^*$ is primal optimal, CST asserts that

if $\vec{y}^*$ is dual optimal,

$$2y_1^* + y_2^* + 4y_3^* = -4 \quad (\text{since } x_1^* > 0)$$

$$3y_1^* - y_2^* - y_3^* = -11 \quad (\text{since } x_3^* > 0)$$

$$y_3^* = 0$$

$$\vec{y}^* = \qquad \begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1^* \\ y_2^* \\ y_3^* \end{bmatrix} = \begin{bmatrix} -4 \\ -11 \\ 0 \end{bmatrix}$$

$$\vec{y}^* = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$ must be dual optimal.

But, $\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$ is not even <u>dual feasible</u>.

Hence, $\vec{x}^*$ is not primal <u>optimal</u>.

APPLICATIONS OF DUALITY

Theorems of the Alternative

Farkas' Lemma (I): Exactly one of the following alternatives holds.

(1) $A\vec{x} \leq \vec{b}$ has a sol^n, $\vec{x} \in \mathbb{R}^n$

(2) There is a vector $\vec{y} \in \mathbb{R}^m$ such that
$\vec{y} \geq \vec{0}$, $A^T\vec{y} = \vec{0}$, $\langle \vec{b}, \vec{y} \rangle < 0$.

Proof: PRIMAL max $\langle \vec{0}, \vec{x} \rangle$ $= 0$
s.t. $A\vec{x} \leq \vec{b}$, $\vec{x} \geq \vec{0}$.

DUAL: $\min \langle \vec{b}, \vec{y} \rangle$

$$\boxed{\begin{aligned} A^T\vec{y} &= \vec{0} \\ \vec{y} &\geq \vec{0} \end{aligned}}$$

$\{\varepsilon, 2\varepsilon\}$

(1) $\Rightarrow$ (2):

If PRIMAL is feasible and hence has 0 as its optimal cost. By strong duality, DUAL has optimal cost 0, and hence

$\langle \vec{b}, \vec{y} \rangle \geq 0$, whenever $A^T\vec{y} = \vec{0}$ and $\vec{y} \geq \vec{0}$.

$(2)^c \Rightarrow (1)$

DUAL is clearly not unbounded.

$\langle \vec{b}, \vec{z} \rangle \geq 0$

Then DUAL is feasible and bounded.

$\Rightarrow$ PRIMAL is feasible and bounded.

$\Rightarrow$ $A\vec{x} \leq \vec{b}$ is feasible.

$\Rightarrow$

Farkas' Lemma (2): Exactly one of the two alternatives hold.

(1) $[A\vec{x} = \vec{b},\ \vec{x} \geq \vec{0}]$ has a sol^n;

(2) $\exists\ \vec{y} \in \mathbb{R}^n$ s.t. $A^T\vec{y} \leq 0$ and $\langle \vec{b}, \vec{y} \rangle > 0$ | ($\vec{y}$ → certificate of infeasibility)

Proof: Left as an exercise.

Geometric interpretation of Farkas' lemma ?

$A\vec{x} = \vec{b}$, $\vec{x} \geq 0$

$\vec{b} = x_1 A_1 + \dots + x_n A_n$

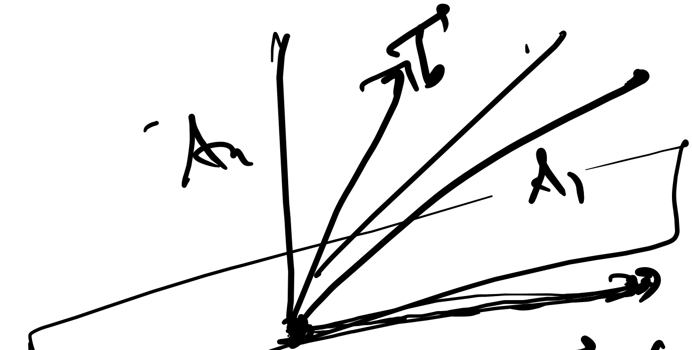
1st column of A

nth column of A

$x_1 \geq 0 \dots, x_n \geq 0$

$A\vec{x} = \vec{b}$, $\vec{x} \geq 0$ is feasible

$\iff$ $\vec{b} \in$ convex cone generated by $A_1, \dots, A_n$



~~Alte~~ Alternative, $\vec{b} \notin$ convex cone generated by $A_1, \ldots, A_n$.

$\Longleftrightarrow \exists$ a hyperplane passing through the origin that strictly separates $\vec{b}$ and the columns ~~vectors~~ of A. Let $\vec{y}$ be normal to the hyperplane

$$\boxed{\langle \vec{y}, A_i \rangle \leq 0,} \quad i = \ldots, n$$

$$\langle \vec{y}, \vec{b} \rangle > 0$$

$$\boxed{A^T \vec{y} \leq \vec{0}}$$

$$\langle \vec{b}, \vec{y} \rangle > 0$$

~~Fardas~~ WEAK DUALITY was easy to show.

PRIMAL

$\min \langle \vec{c}, \vec{x} \rangle$

$A\vec{x} = \vec{b}$

$\vec{x} \geq 0, \; \alpha^T$

DUAL

$\max \langle \vec{b}, \vec{y} \rangle$

s.t. $A^T \vec{y} \leq \vec{c}$

$\vec{y} \gtrless \vec{0},$

$\beta^T.$

Pf: $\vec{x}$ is feasible, $A\vec{x} = \vec{b}$

$\vec{y}$ is feasible in DUAL $A^T\vec{y} \le \vec{c}$,

$$\langle \vec{y}, \vec{b} \rangle = \langle \vec{y}, A\vec{x} \rangle = \langle A^T\vec{y}, \vec{x} \rangle$$
$$\le \langle \vec{c}, \vec{x} \rangle$$

$\Rightarrow \langle \vec{y}, \vec{b} \rangle \le \alpha^*$, $\forall\ \vec{y} \in$ DUAL feasible region

$\Rightarrow \beta^* \le \alpha^*$

STRONG DUALITY revisited:

If either PRIMAL or DUAL is feasible, then $\alpha^* = \beta^*$.

Proof. Wlog, PRIMAL is feasible.

If PRIMAL is unbounded, $\alpha^* = -\infty = \beta^*$ (by weak duality), we are done.

Let $\vec{x}^*$ be an optimal sol^n for PRIMAL.

$$\langle \vec{c}, \vec{x}^* \rangle = \alpha^*.$$

There are ~~no~~ NO vectors $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$, $\vec{x} \geq \vec{0}$, and $\langle \vec{c}, \vec{x} \rangle = \alpha^* - \varepsilon$ ($\varepsilon > 0$)

$$\begin{bmatrix} A \\ -\vec{c}^{T} \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{b} \\ -\alpha^* + \varepsilon \end{bmatrix}$$

$\vec{x} \geq \vec{0}$ is infeasible.

By Farkas lemma 2, $\exists \; (\vec{u}, z) \in \mathbb{R}^{m+1}$

such that $\begin{bmatrix} A \\ -\vec{c}^{T} \end{bmatrix}^{T} \begin{bmatrix} \vec{u} \\ z \end{bmatrix} \leq \vec{0}$. $\begin{bmatrix} \vec{b} \\ -\alpha^* + \varepsilon \end{bmatrix}^{T} \begin{bmatrix} \vec{u} \\ z \end{bmatrix} > 0$.

$\iff \left[A^{T}\vec{u} \leq z\vec{c}^{*} \right.$; $\langle \vec{b}, \vec{u} \rangle > z(\alpha^* - \varepsilon)$

$\vec{u} \in \mathbb{R}^{m}$, $z \in \mathbb{R}$.

Claim: ~~We~~ $\langle \vec{b}, \vec{u} \rangle \le z\alpha^*$.

Proof. (If not, then $\exists$, $\underline{\langle \vec{b}, \vec{x} \rangle > z\alpha^*}$, $\begin{bmatrix} \vec{x} \\ z \end{bmatrix}^T$

$$\left(\begin{bmatrix} A \\ -\vec{c}^T \end{bmatrix}^T \begin{bmatrix} \vec{x} \\ z \end{bmatrix} \le \vec{0}, \quad \begin{bmatrix} \vec{b} \\ -\alpha^* \end{bmatrix}^T \begin{bmatrix} \vec{u} \\ z \end{bmatrix} > 0. \right)$$

then, $\begin{bmatrix} A \\ -\vec{c}^T \end{bmatrix}^T \vec{x} = \begin{bmatrix} \vec{b} \\ -\alpha^* \end{bmatrix}$ is not feasible

$\vec{x} \ge \vec{0}$

$A\vec{x} = \vec{b}$, $\langle \vec{c}, \vec{x} \rangle = \alpha^*$

$\vec{x} \ge \vec{0}$.

Contradiction!!

$$\boxed{A^T \vec{u} \leq z \vec{c}}$$

$$z(\alpha^* - \epsilon) < \langle \vec{b}, \vec{u} \rangle \leq z \alpha^*,$$

$$\Rightarrow \quad -z\epsilon < 0$$

$$\Rightarrow \quad z\epsilon > 0$$

$$\Rightarrow \quad \boxed{z > 0}$$

We may assume $z = 1$, wlog.

For some $\vec{u} \in \mathbb{R}^m$, $\boxed{A^T \vec{u} \leq \vec{c}}$

and $\alpha^* - \varepsilon < \langle \vec{b}, \vec{u} \rangle$

In other words there is a DUAL feasible vector $\vec{u}$ s.t. $\alpha^* - \varepsilon < \langle \vec{b}, \vec{u} \rangle \leq \beta^*$

$\Rightarrow \quad \alpha^* - \varepsilon \leq \beta^* \quad , \forall \varepsilon > 0$

$\Rightarrow \quad \alpha^* \leq \beta^*$

$\Rightarrow \quad \boxed{\alpha^* = \beta^*}$. ∎

Economic interpretation of primal is clear.

ECONOMIC INTERPRETATION OF THE DUAL

$T \sim 2\text{kg}$ $\quad \frac{T}{\delta}$ $\quad f(b_j)$

Dimension analysis

$$A\underline{x} \le \underline{b}$$

$$a_{11}x_1 + \dots + a_{1n}x_n \le b_1$$

$$\vdots$$

$$a_{m1}x_1 + \dots + a_{mn}x_n \le b_m$$

$x_j \longrightarrow$ quantity of product P_j

$b_i \to$ quantity of resource R_i

$a_{ij} \to$ units of resource R_i per unit of product P_j.

$c_j \to$ profit per unit of product P_j.

units of $c_j \to$ rupees per unit of product P_j

units of $a_{ij} \to \frac{\text{amount of resource } R_i \text{ per unit}}{\text{of product } P_j}$.

$A\vec{x} \le \vec{b}$, $A^T\vec{y} \ge \vec{c}$ $\vec{y}^T A$

$\sum a_{ij} y_i \ge \vec{c}$

unit of y_i → value per unit of resource i.
(dual variable)

<u>Theorem</u> – Consider

PRIMAL (P) : $\min \langle \vec{c}, \vec{x} \rangle$ (in LPS)

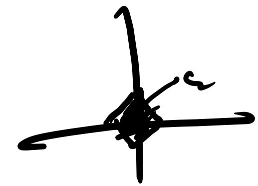
s.t. $A\vec{x} = \vec{b}$

$\vec{x} \geq \vec{0}$

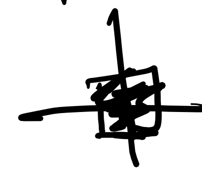
Suppose that P has at least one <u>nondegenerate</u> basic optimal solⁿ, then there is a unique $\vec{u}^* \in \mathbb{R}^m$ which is ~~the dual~~ optimal in the dual of LP.

Furthermore, there is an $\varepsilon > 0$ with the following property..

Suppose $\vec{t} \in \mathbb{R}^m$ and $|t_i| \le \varepsilon$ $\forall i = 1, \ldots, m$



Then

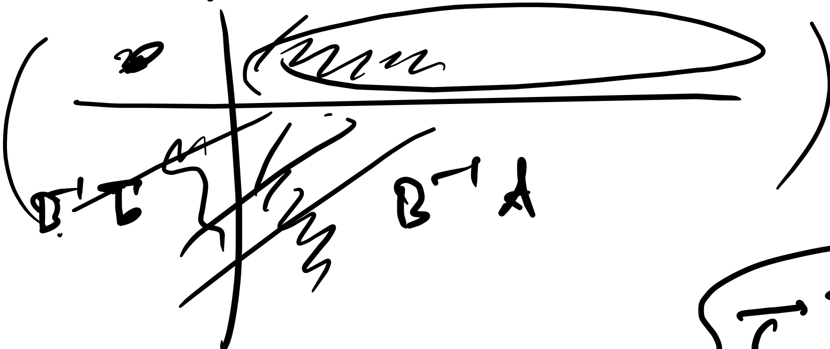


$$(P') \quad \min \langle \vec{c}, \vec{x} \rangle \quad A\vec{x} = \vec{b} + \vec{t} \quad \vec{x} \ge \vec{0}.$$

has an optimal sol^n and its value is $c^* + \langle \vec{u}^*, \vec{t} \rangle$ where c^* is the optimal cost of P and $\vec{u}^*$ is dual optimal.

Proof. $B = B(1) \dots B(m)$ corresponding to the optimal nondegenerate bfs $\vec{x}$.

$B = [A_{B(1)} \quad A_{B(m)}]$



cost c^* of $\vec{x}$ = $\vec{c}_B^{\,T} B^{-1}\vec{b}$

row 0 = $\vec{c}^{\,T} - \vec{c}_B^{\,T} B^{-1} A$

$B^{-1}\vec{b} > \vec{0}$, as $\vec{b}$ is nondegenerate.

$\boxed{\delta} \to$ smallest component of $B^{-1}\vec{b}$.

$f : \mathbb{R}^n \to \mathbb{R}^n$

$x \to B^{-1}x$ is continuous.

$\exists\, \varepsilon > 0$ s.t. when $\|\vec{t}\|_\infty \leq \varepsilon$, we have

$$\|B^{-1}t\|_\infty \leq \delta.$$

$$\Rightarrow \quad B^{-1}(\vec{b}+\vec{\varepsilon}) = B^{-1}\vec{b} + B^{-1}\vec{\varepsilon} \geq \vec{0}.$$

B yields a bfs for P'.

Tableau corresponding to $\mathcal{B}$ for P' is the same everywhere (except column 0) as the tableau corresponding to $\mathcal{B}$ for P.

In particular, 0^{th} row (except $(0,0)^{th}$ entry) has no negative entries. Hence $\mathcal{B}$ is an optimal basis for P'.

$$C' = C_B^T B^{-1} (\vec{b} + \vec{t}) \quad \text{(optimal cost for } P'\text{)}$$

$$\boxed{\vec{u}^{*} = (B^{-1})^T C_B.}$$

$$= C_B^T \ldots \vec{u}^{*T} (\vec{b} + \vec{t})$$

$$C' = \langle \vec{u}^*, \vec{b} + \vec{t} \rangle$$

$$= \langle \vec{u}^*, \vec{b} \rangle + \langle \vec{u}^*, \vec{t} \rangle$$

$$= C^* + \langle \vec{u}^*, \vec{t} \rangle$$

vector of dual optimal variables

$$\underline{u_1^* t_1} + u_2^* t_2 \ldots$$

$u_i^* \rightarrow$ <u>marginal value</u> of the ith resource

if the ith resource is increased by ε sufficiently small, the profit will increase by $\boxed{u_i^* \varepsilon}$