18/11/2021 Lecture - 15

Motivation for studying duality (in the context of linear programming):

(1) Finding lower bounds on the optimal value.

(2) Finding certificate for easy verification of a proposed/submitted optimal soln.

x x^2 x^9 x^8 x^n

$\left(x^{16} + \ldots + 4x^2 + x + 1 = 0 \right)$

Use your favourite method to find a soln.

Is the submitted set of roots of the above polynomial accurate?

Checking feasibility ~~is~~ of proposed optimal sol^n is easy $\boxed{A\vec{x} \le \vec{b}}$

Checking optimality is not so straightforward.

(3) Potentially having a simpler LP to solve and inferring the optimal sol^n to original LP from this simpler LP.

It's not the n in $\mathbb{R}^n$ that causes the simplex algorithm to be complicated, rather it is the number of constraints (m rn in the case of LPs $A\vec{x} \le \vec{b}$ $A_{m \times n}$, $\vec{x} \ge \vec{0}$)

$\boxed{A\vec{x} \le \vec{b}}$

PRIMAL

constraint i $\quad \langle \vec{a_i}, \vec{x} \rangle \geq b_i$

constraint i $\quad \boxed{\langle \vec{a_i}, \vec{x} \rangle = b_i}$

var j $\quad x_j \gtrless 0$

var j $\quad \boxed{x_j \geq 0}$

objective $\quad$ min $\langle \vec{c}, \vec{x} \rangle$

$\vec{x} \in \mathbb{R}^n$

$\boxed{m > n}$

DUAL

$\boxed{y_i \geq 0}$

$y_i \gtrless 0$

$\boxed{(A^T \vec{y})_j = c_j}$

$\boxed{(A^T \vec{y})_j \leq c_j}$

max. $\langle \vec{b}, \vec{y} \rangle$

$\vec{y} \in \mathbb{R}^m$

If PRIMAL has a large number of constraints compared to the number of variables, then DUAL has a small number of constraints compared to the number of variables.

Duality theorems:

Theorem. (Strong duality) If a linear program has an optimal soln (feasible and bounded), then so does its dual, (that is, DUAL is also feasible and bounded,) and their optimal costs are identical.

Theorem (Weak duality)

Let $\vec{x}$ be a feasible soln to PRIMAL, $\vec{y}$ be a feasible soln to DUAL, then

$$\langle \vec{c}, \vec{x} \rangle \geq \langle \vec{b}, \vec{y} \rangle.$$

Proof: PRIMAL

$$\min \langle \vec{c}, \vec{x} \rangle$$

s.t. $\boxed{(A)^i \vec{x} = b_i}$ $\quad (\langle \vec{a_i}, \vec{x} \rangle = b_i)$

$1 \leq i \leq h$

$(A)^i \vec{x} \geq b_i, \quad h+1 \leq i \leq m$

$x_j \geq 0, \quad 1 \leq j \leq l$

$x_j \gtrless 0, \quad l+1 \leq j \leq n.$

$\underline{\text{DUAL}}$:

$$\max \langle \vec{b}, \vec{y} \rangle$$

$$\boxed{y_i \gtrless 0} \quad , \quad 1 \le i \le h$$

$$y_i \geq 0 \quad , \quad \underline{\underline{h+1}} \le i \le \underline{m}$$

$$(A^T)^j \vec{y} \le c_j \quad , \quad 1 \le j \le l$$

$$(A^T)^j \vec{y} = c_j \quad , \quad l+1 \le j \le n .$$

$$\langle \vec{c}, \vec{x} \rangle \geq \sum_{j=1}^{l} (A^T \vec{y})^j x_j + \sum_{j=l+1}^{n} (A^T \vec{y})^j x_j$$

$$\left(\sum_{j=1}^{n} c_j x_j \right)$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j + \sum_{j=l+1}^{n}$$

$$= \sum_{i=1}^{m} y_i \left[\sum_{j=1}^{n} a_{ij} x_j \right]$$

$$= \sum_{i=1}^{n} y_i \left[\sum_{j=1}^{} a_{ij} x_j \right] + \sum_{i=n+1}^{m} y_i \left[\sum_{j=1}^{n} a_{ij} x_j \right]$$

$$\geq \sum_{i=1}^{m} b_i y_i = \langle \vec{b}, \vec{y} \rangle$$ ∎

<u>Strong duality theorem</u>

If PRIMAL ~~is feasible~~ has an optimal solution, then DUAL has an optimal sol^n, and the optimal value coincides.

Pf. PRIMAL $\min \langle \vec{c}, \vec{x} \rangle$

s.t. $A\vec{x} = \vec{b}$, $\vec{x} \geq \vec{0}$.

DUAL $\max \langle \vec{b}, \vec{y} \rangle$

s.t. $\vec{y} \geq \vec{0}$, $A^T \vec{y} \leq \vec{c}$

Let $\vec{v}^*$ be an optimal bfs for PRIMAL discovered by the Simplex Algorithm, and $B(1), \ldots, B(m)$ be the basis corresponding to $\vec{v}^*$.

$$\left[\quad A \quad \right]_{B(1) \cdots B(m)}, \quad B = \left[A_{B(1)} \; A_{B(2)} \; \cdots \; A_{B(m)} \right]_{m \times m}$$

$\vec{C}_B \longrightarrow$ vector in $\mathbb{R}^m$ whose i^{th} component is $c_{B(i)}$.

$$\langle \vec{C}, \vec{x} \rangle = \sum_{i=1}^{m} c_{B(i)} x_{B(i)} = \langle \vec{C}_B, \vec{x}_B \rangle$$

$\vec{c} = c - \vec{z}$, $\boxed{\vec{z}^T = c_B^T B^{-1} A}$

$B^{-1}\left(\; A \; \right)$

At termination, 0^{th} row of $\vec{c}^T$ is non-negative.

$$\vec{c} - \vec{z} \geq \vec{0} \Rightarrow \vec{c}^T \geq c_B^T B^{-1} A$$

$$A^T (B^{-1})^T \vec{c}_B$$

Let, $\vec{u}^* := (B^{-1})^T \vec{c}_B$

Thus, $\boxed{\vec{c} - A^T \vec{u}^* \geq \vec{0}}$ $\vec{u}^*$ is feasible for our DUAL.

$$\underline{\langle \vec{c}, \vec{v}^{*} \rangle} = c_B^T \cdot (\text{column } 0 \text{ of final tableau})$$

$B^{-1}\vec{b} \rightarrow \vec{x}_B$

optimal value of PRIMAL

$$= \vec{c}_B^{\,T} B^{-1} \vec{b}$$

$$= \vec{u}^{*T} \vec{b}$$

$$= \langle \vec{u}^{*}, \vec{b} \rangle$$

$\vec{b}$

From weak duality, $\langle \vec{y}, \vec{b} \rangle \leq \underline{\langle \vec{c}, \vec{v}^{*} \rangle}$

$\forall \vec{y}$ that's DUAL feasible sol.

Hence, $\vec{u}^{*}$ must be DUAL optimal. ∎

Corollary: If we run simplex on LPs, at termination, $\boxed{(B^{-1})^T \bar{c}_B}$ is dual optimal.

Corollary. Exactly one of these four cases happens.

(1) PRIMAL and DUAL are both infeasible.

(2) PRIMAL is unbounded, DUAL is infeasible.

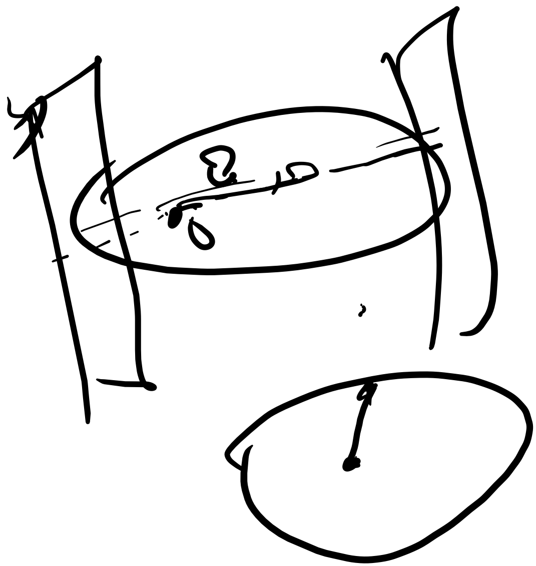
(3) PRIMAL is infeasible, DUAL is unbounded.

(4) PRIMAL and DUAL both have optimal points with equal optimal value.

If DUAL is easier to solve (by simplex) than PRIMAL, find optimal sol^n of dual (optimal sol^n) and corresponding basis. This will give you an optimal sol^n of the primal.

(Geometric proof of strong duality)

$$\left[h_K(\vec{u}) \right] = \left[\frac{1}{\rho_{K^\circ}(\vec{u})} \right]$$



Main assumption: $\vec{0} \in K$.

If $\vec{0} \in F$, we have already shown the result

$$\boxed{A\vec{x} \le \vec{b}} \quad \le$$

$$\boxed{A^T\vec{y} = \vec{c}, \quad \vec{y} \le \vec{0}.}$$

$$\langle \vec{b}, \vec{y} \rangle \le \langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A^T\vec{y} \rangle$$

$$= \langle \vec{x}, \vec{c} \rangle$$

Let PRIMAL be feasible and bounded. $\vec{x_0} \in F$. Translate F to $\boxed{F - \langle \vec{x_0} \rangle}$.

$$\boxed{b_i' = b_i - \langle \vec{x_0}, \vec{a_i} \rangle} \qquad \vec{b}' = \vec{b} - A\vec{x_0}$$

$$A\vec{x} \le \vec{b} \iff A\vec{x} - A\vec{x_0} \le \vec{b} - A\vec{x_0}$$

$$\iff A\vec{x}' \le \vec{b}'$$

$$\vec{x}' \in F - \langle \vec{x_0} \rangle$$

and $\vec{0} \in F - \langle \vec{x_0} \rangle$

$$\langle \vec{x}', \vec{x} \rangle = \langle \vec{c}, \vec{x} \rangle - \langle \vec{c}, \vec{x_0} \rangle$$

$$\text{(DUAL)} \quad \langle \vec{b}', \vec{y} \rangle \le \langle \vec{c}', \vec{x}' \rangle \quad \forall \vec{y} \text{ s.t. } \vec{y} \ge 0, \ \boxed{A^T\vec{y} = \vec{c}}$$

$$= \langle \vec{b}, \vec{y} \rangle - \sum_{i=1}^{m} \langle \vec{x_0}, \vec{a_i} \rangle y_i$$

$$= \langle \vec{b}, \vec{y} \rangle - \langle \vec{x_0}, A^T\vec{y} \rangle = \langle \vec{b}, \vec{y} \rangle - \langle \vec{x_0}, \vec{c} \rangle$$

$$\langle \vec{b}, \vec{y} \rangle - \langle \vec{c}, \vec{x}_0 \rangle \leq \langle \vec{c}, \vec{x} \rangle - \cancel{\langle \vec{c}, \vec{x} \rangle}$$

$$\Rightarrow \boxed{\langle \vec{b}, \vec{y} \rangle \leq \langle \vec{c}, \vec{x} \rangle}$$