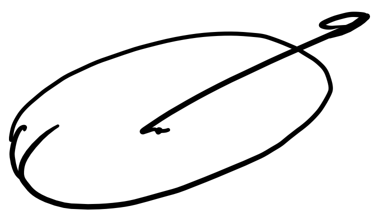
16/11/2021

Lecture - 14

Duality:

Earlier, we had seen two representation theorems for convex bodies



Duality translates results on boundary points of closed convex sets into results on support planes (or normal vectors) associated with a "dual" convex set.

Example : K_1, K_2

$\cap H_i$ $\qquad \cap H_i'$

$$K_1 \cap K_2 = \left(\cap H_i\right) \cap \left(\cap H_i'\right)$$

For intersection of convex sets the supporting hyperplane description easily carries forward.

$$\boxed{\text{conv}(K_1 \cup K_2)} = \text{conv}\{x_1, \ldots, x_k, \ldots, x_{\ldots}, \ldots, x_{m}\}$$

$K_1 = \text{conv}\{x_1, \ldots, x_k\}$, $K_2 = \text{conv}\{x_{\ldots}, \ldots, x_m\}$

Defn: Let K be a closed convex set in $\mathbb{R}^n$. The **polar body** (or **dual body**) of K, denoted by K^*, is defined as follows.

$$K^* = \{\vec{y} \in \mathbb{R}^n \mid \langle \vec{x}, \vec{y} \rangle \leq 1 \text{ for every } \vec{x} \in K\}$$

$\in \mathbb{R}^n$.

Properties of polar bodies

(1) K^* is non-empty (contains $\vec{0}$)

(2) If K $\subseteq \mathbb{R}^n$ is closed, convex and contains $\vec{0}$, then

$(K^*)^* \subseteq K$

Pf: $\boxed{K \subset (K^*)^*}$ $\quad \vec{x} \in K$

$\vec{y} \in K^*$ $\quad$ ~~$\vec{x} \in (K^*)^*$~~

$\boxed{\langle \vec{x}, \vec{y} \rangle \leq 1}$, $\forall \vec{x} \in K$, $\underline{\vec{y}} \in K^*$

$\langle \vec{y}, \vec{x} \rangle \leq 1$

$(K^*)^*$ is closed, convex

and $0 \in (K^*)^*$

closed: L^* is closed for any convex set L in $\mathbb{R}^n$

$\vec{y}_n \in L^*$ s.t. $\vec{y}_n \to \vec{y}$ $\quad \langle \vec{x}, \vec{y}_n \rangle \leq 1$ $\forall \vec{x} \in L$

$\Rightarrow \langle \vec{x}, \vec{y} \rangle \leq 1$, $\forall \vec{x} \in L$

convex: $\vec{y}_1, \vec{y}_2 \in L^*$,

$\langle \vec{x}, \vec{y}_1 \rangle \le 1 \qquad \forall \vec{x} \in L.$

$\langle \vec{x}, \vec{y}_2 \rangle \le 1$

$\langle \vec{x}, \lambda \vec{y}_1 + (1-\lambda)\vec{y}_2 \rangle \le 1, \ \forall \vec{x} \in L$

$\lambda \in [0,1]$

Let $\vec{z} \in \mathbb{R}^n \setminus K$

There is a hyperplane $H_{\vec{a},b}$

$(\vec{a} \neq \vec{0}, b \in \mathbb{R})$

with $K \subset H^-_{\vec{a},b}$ $(\langle \vec{a}, \vec{x} \rangle \le b)$ and $\boxed{\langle \vec{z}, \vec{a} \rangle > b}$

Here, $b > 0$ since $\vec{0} \in K$.

(using strong separation theorem for a closed convex set in $\mathbb{R}^n$ and a point outside of it)

For all $\vec{y} \in K$, we have $\langle \vec{y}, \vec{a}/b \rangle \leq 1$

so that $\frac{1}{b}\vec{a} \in K^* \to L$

$$\langle \vec{z}, \vec{a}/b \rangle > 1 \Rightarrow \vec{z} \notin L^* = (K^*)^*$$

We have shown that $(K^*)^* \subseteq K$.

(3) Polarity reverses containment: $\boxed{K \subset L}$ (closed, convex sets in $\mathbb{R}^n$)

$$\Rightarrow L^* \subset K^*.$$

(4) K_1, K_2 closed, convex sets in $\mathbb{R}^n$.

$$\boxed{(K_1 \cap K_2)^* = \operatorname{conv}(K_1^* \cup K_2^*)}$$

Proof: $K_1 \cap K_2 \subset K_i$, $i=1,2$

$$\Rightarrow \quad K_i^* \subset (K_1 \cap K_2)^*$$

$$\Rightarrow \quad \operatorname{conv}(K_1^* \cup K_2^*) \subset (K_1 \cap K_2)^*$$

$$K_i \subset \operatorname{conv}(K_1 \cup K_2), \quad i=1,2$$

$$\Rightarrow (\operatorname{conv}(K_1 \cup K_2))^* \subset K_i^*, \quad i=1,2$$

$$\Rightarrow \left(\operatorname{conv}(K_1 \cup K_2)\right)^* \subset K_1^* \cap K_2^*$$

$$\Rightarrow \left(K_1^* \cap K_2^*\right)^* \subset \operatorname{conv}(K_1 \cup K_2)$$

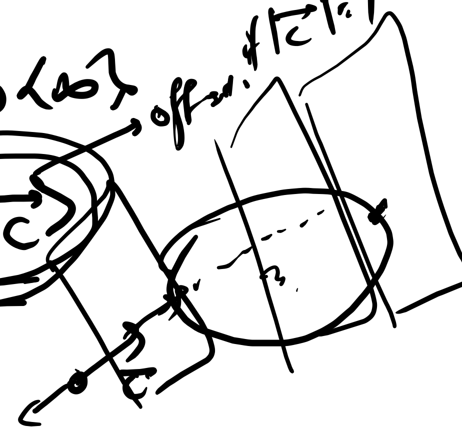
$$\Rightarrow \left(K_1 \cap K_2\right)^* \subset \operatorname{conv}(K_1^* \cup K_2^*)$$

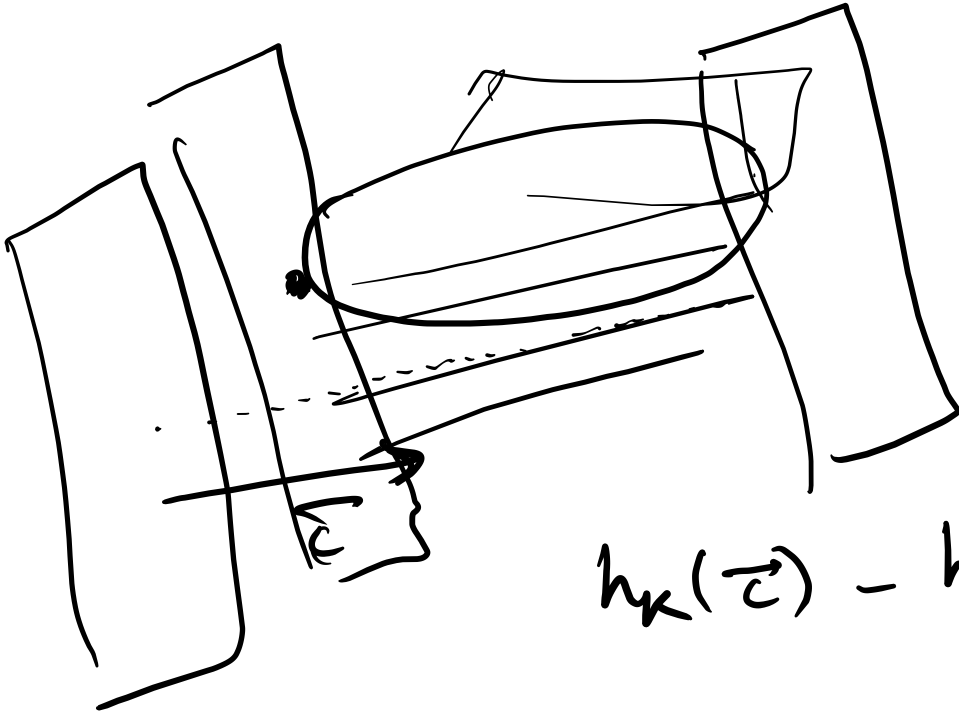
__Defn__: support function of K.

$$h_K : \mathbb{R}^n \setminus \{\vec{0}\} \longrightarrow \mathbb{R} \cup \{\infty\}$$

$$\underline{\underline{h_K(\vec{c})}} = \sup_{\vec{z} \in K} \langle \vec{z}, \vec{c} \rangle$$

offset, if $|\vec{c}| = 1$



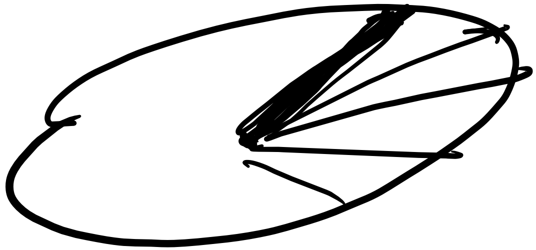


$h_K(\vec{c}) - h_K(-\vec{c})$

for $\vec{c}$ a unit vector

= width of K along the direction of $\vec{c}$.

Radial function of K (containing $\vec{0}$)

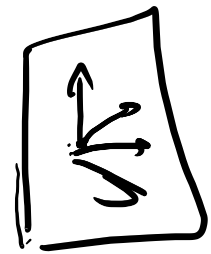


$$\rho_K(\vec{b}) = \sup\{\lambda \geq 0 : \lambda\vec{b} \in K\}$$

Lemma: Let $K \subseteq \mathbb{R}^n$ closed, convex and containing $\vec{0}$. For every non-zero vector $\vec{u} \in \mathbb{R}^n$,

$$h_K(\vec{u}) = \frac{1}{\rho_{K^*}(\vec{u})}, \text{ and } h_{K^*}(\vec{u}) = \frac{1}{\rho_K(\vec{u})}$$

with the convention, $\frac{1}{\infty} = 0$, and $\frac{1}{0} = \infty$.



Proof

$$\rho_{K^*}(\vec{u}) = \sup_{\lambda \geq 0} \langle \lambda ; \lambda \vec{u} \in K^* \rangle$$

$$\langle \lambda \vec{u}, \vec{x} \rangle \leq 1 \quad \forall \vec{x} \in K$$

$$\Rightarrow \langle \vec{x}, \vec{u} \rangle \leq \frac{1}{\lambda}, \forall \vec{x} \in K$$

$$h_K(\vec{u})$$

If $\rho_{K^*}(\vec{u}) = 0$

$\varepsilon \vec{u} \notin K^*$, $\forall \varepsilon > 0$

$\exists$ some $\vec{x}_\varepsilon \in K$ s.t. $\langle \varepsilon \vec{u}, \vec{x}_\varepsilon \rangle > 1$.

$$\langle \vec{u}, \vec{x}_\varepsilon \rangle > \frac{1}{\varepsilon}$$

$$h_K(\vec{u}) > \frac{1}{\varepsilon}$$

In geometric terms, LP asks for finding value of support function of specific convex bodies (<u>polyhedra</u>) in a particular direction (determined by the cost vector).

Computing the polar body of a polyhedron

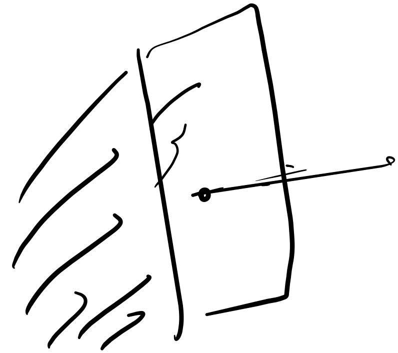
$\vec{0} \in H^-_{\vec{a}, b}$

so that

$b \geq 0$

$$\left(H^-_{\vec{a}, b}\right)^* \qquad \{\langle \vec{a}, \vec{x} \rangle \leq b\}$$

$\boxed{b > 0}$, $b = 0$



Case 1. $b > 0$

$$\left(H^-_{\vec{a}, b}\right)^* = \left[0, \frac{1}{b}\vec{a}\right] \qquad \left(H^-_{\vec{a}, 0}\right)^* = \{\lambda \vec{a} : \lambda \geq 0\}$$

$\left(\left[0, \frac{1}{b}\vec{a}\right]\right)^*$ where $\boxed{b>0}$

$\left\{\vec{x} \in \mathbb{R}^n : \langle \vec{x}, \lambda\vec{a} \rangle \leq 1 \right.$
for $0 \leq \lambda \leq \frac{1}{b}\left.\right\}$.

$= \left\{\vec{x} \in \mathbb{R}^n : \langle \vec{x}, \vec{a} \rangle \leq b \right\}$

$= \quad \overline{H}_{\vec{a}, b}$

Case II : $b = 0$,

$$\left([0,\infty)\vec{a}\right)^*$$

$$= \{\vec{x} \in \mathbb{R}^n : \langle \vec{x}, \lambda\vec{a}\rangle \le 1 \quad \forall\ 0 \le \lambda\}$$

$$= \{\vec{x} \in \mathbb{R}^n : \langle \vec{x}, \vec{a}\rangle \le 0\}$$

$$= \bar{H}_{\vec{a},0}$$

✓

Proposition: Let K be a convex polyhedron in $\mathbb{R}^n$ containing $\vec{0}$ (not necessarily in the interior). Let

$$K = \{ A \vec{x} \le \vec{b} \}$$

A: $m \times n$ matrix; $\vec{x} \in \mathbb{R}^n$; $\vec{b} \in \mathbb{R}^m$

$$\boxed{\langle \vec{a_i}, \vec{x} \rangle \le b_i}$$

$\vec{a_i}$: ith row vector of A.

Then:

$$K^* = \left\{ \sum_{i=1}^{m} \mu_i \vec{a_i} \; ; \; \mu_i \ge 0 \text{ for all } i = 1, \dots, m \text{ and } \boxed{\sum_{i=1}^{m} \mu_i b_i} \le 1 \right\}.$$

Proof: $\boxed{(K_1 \cap \dots \cap K_m)^\circ} = \mathrm{conv}(K_1^\circ \cup K_2^\circ \dots \cup K_m^\circ)$

$K_i = H_{\vec{a}_i, b_i}$ $\quad K_i^\circ = \left[\vec{0}, \frac{1}{b_i}\vec{a}_i\right]$

$$K^\circ = \mathrm{conv}\left(\left\{\vec{0}, \frac{1}{b_1}\vec{a}_1, \dots, \frac{1}{b_m}\vec{a}_m\right\}\right)$$

$$\ni \underbrace{\frac{\lambda_1}{b_1}}_{\mu_1}\vec{a}_1 + \dots + \underbrace{\frac{\lambda_m}{b_m}}_{\mu_m}\vec{a}_m + \lambda_{m+1}\vec{0}$$

$$\boxed{\lambda_1 + \dots + \lambda_{m+1} = 1}$$

$\lambda_1 + \dots + \lambda_m = \mu_1 b_1 + \dots$ then ≤ 1.

Consider an LPP $\left(\min \langle \vec{c}, \vec{x} \rangle \text{ s.t. } \boxed{A\vec{x} \le \vec{b}}\right)$ such that

$\vec{0} \in F$ (so that $\vec{b} \geq \vec{0}$)

$$\nu_{\min} = \boxed{h_K(-\vec{c})} \quad \text{(optimal value)}$$

$$= \boxed{\frac{1}{\nu_{\min}}} ? \boxed{\frac{1}{\boxed{h_K(-\vec{c})}} = \boxed{\rho_{K^*}(-\vec{c})}}$$

$$= \sup_{\rho \geq 0} \langle \rho : -\rho \vec{c} \in K^* \rangle$$

The polar set of $\vec{c}$ is ($K = \vec{P}$)

$$K^{*} = \{ \sum_{i=1}^{m} z_i \vec{a}_i \; ; \; \vec{z}_i \geq 0, \langle \vec{b}, \vec{z} \rangle \leq 1 \}.$$

We want the largest non-negative number ρ s.t.

$$\boxed{-\rho \vec{c} = \sum_{i=1}^{m} z_i \vec{a}_i}, \quad \vec{z}_i \geq 0, \quad \boxed{\langle \vec{b}, \vec{z} \rangle \leq 1}$$

(If LPP is bounded) we may pick $\rho > 0$

s.t. $-\rho \vec{c} \in K^{*}$. Define, $y_i = -\frac{z_i}{\rho} \leq 0$.

$$\boxed{\vec{c} = \sum_{i=1}^{m} y_i \vec{a}_i}, \quad \vec{y} \leq \vec{0}$$

$$\langle \vec{b}, \vec{y} \rangle \geq -\frac{1}{\rho}$$

$$\left(\vec{b} \geq \vec{0},\ \vec{y} \leq \vec{0} \right.$$

$$\left. \Rightarrow \langle \vec{b}, \vec{y} \rangle \leq \vec{0} \right)$$

$$\boxed{-\frac{1}{\rho} \leq \langle \vec{b}, \vec{y} \rangle \leq 0}$$

Maximizing ρ is equivalent to maximizing $\langle \vec{b}, \vec{y} \rangle$.

Maximize $\langle \vec{b}, \vec{y} \rangle$

subject to $A^T \vec{y} = \vec{c}$, $\vec{y} \leq \vec{0}$

The computational viewpoint

Example: $\min.\ x_1 + 2x_2 - 7x_3$

$$\begin{pmatrix} 1 \\ 2 \\ -7 \end{pmatrix}$$

s.t. $3x_1 - x_2 + 5x_3 = 6$

$2x_1 + 0x_2 - x_3 = 12$

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$

F

$\vec{w} \in F$. $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$.

$$f = y_1 \underbrace{(3w_1 - w_2 + 5w_3)}_{6} + y_2 \underbrace{(2w_1 - w_3)}_{12}$$

$$= \boxed{6y_1 + 12y_2}$$ has no dependence on $\vec{w}$

$$= \underline{(3y_1 + 2y_2)} w_1 + (-y_1) w_2$$
$$+ \underline{(5y_1 - y_2)} w_3$$

$$\leq w_1 + 2w_2 - 7w_3$$
$$= \langle \vec{c}, \vec{w} \rangle$$

If the coefficients of y's satisfied

$$\left[\begin{aligned} 3y_1 + 2y_2 &\le 1 \\ -y_1 &\le 2 \\ 5y_1 - y_2 &\le -7 \end{aligned}\right] \; - \; F'$$

(No sign-constraints on y_1, y_2)

then, f is a lower bound for $\langle \vec{c}, \vec{x} \rangle$.

Maximizing $f = \boxed{6y_1 + 12y_2}$ on F' would lead to an even better lower bound for LPP.

Remark on why talk about these lower bounds when simplex method already gives optimal solⁿ (greatest lower bound)?

Sanity check / verification of obtained solution.

Example: min. $3x_1 - x_2 - 4x_3$

sub. to. $\begin{bmatrix} 5x_1 - 6x_2 + 7x_3 = 9 \\ 6x_1 - 9x_2 - 2x_3 \geq -7 \end{bmatrix}$

$x_1 \geq 0$, $\boxed{x_2 \geq 0,\ x_3 \gtrless 0.}$

Let $\vec{w} \in F$, $w \geq 0$,

$$f = y_1(5w_1 - 6w_2 + 7w_3) + y_2(6w_1 - 9w_2 - 2w_3)$$

If $y_2 \geq 0$, ~~y_1~~

$$\boxed{f \geq 9y_1 - 7y_2}$$

$$f = (5y_1 + 6y_2)w_1 + (-6y_1 - 9y_2)w_2 + (7y_1 - 2y_2)w_3$$

$$\leq 3w_1 - w_2 - 4w_3$$

$$\left[\begin{array}{l} 5y_1 + 6y_2 \leq 3 \\ -6y_1 - 9y_2 \leq -1 \\ 7y_1 - 2y_2 \leq -4 \end{array}\right.$$

$$y_2 \geq 0 \quad \Bigg]$$

Then, $\boxed{9y_1 - 7y_2}$ maximizing gives a lower bound

for $\boxed{3x_1 - x_2 - 4x_3}$.

The new problem is the DUAL of the original PRIMAL problem.

	PRIMAL	DUAL
constraint i	$\langle \vec{a_i}, \vec{x} \rangle = b_i$	$y_i \gtrless 0$
constraint i	$\langle \vec{a_i}, \vec{x} \rangle \geq b_i$	$y_i \geq 0$
var j :	$x_j \gtrless 0$	$(A^T \vec{y})_j = c_j$
var j :	$x_j \geq 0$	$(A^T \vec{y})_j \leq c_j$
obj. fn..	$\min \langle \vec{c}, \vec{x} \rangle$	$\max \langle \vec{b}, \vec{y} \rangle$

Theorem: DUAL of the DUAL is the PRIMAL.

Proof. Exercise.

Duality theorems:

Theorem (Weak duality theorem). Let $\vec{x}$ be a feasible soln to PRIMAL and $\vec{y}$ be a feasible soln to DUAL. Then $\langle \vec{c}, \vec{x} \rangle \geq \boxed{\langle \vec{b}, \vec{y} \rangle}$.

Proof: $F = \langle A\vec{x} \le \vec{b} \rangle$.

Since $\vec{x}$ is feasible, $A\vec{x} \le \vec{b}$.

Since $\vec{y}$ is feasible, $\vec{y} \le \vec{0}$, $A^T\vec{y} = \vec{c}$.

$$\langle \vec{b}, \vec{y} \rangle \le \langle A\vec{x}, \vec{y} \rangle \quad (\vec{y} \le \vec{0})$$

$$= \langle \vec{x}, A^T\vec{y} \rangle$$

$$= \langle \vec{x}, \vec{c} \rangle$$

$$\left(\langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A^T\vec{y} \rangle \right)$$

$$\boxed{\vec{y}^T A\vec{x}} = \vec{x}^T A^T\vec{y}$$

Remark:

$\boxed{\alpha^*}$ → optimal PRIMAL sol^n

β^* → optimal DUAL sol^n

$\boxed{\alpha^* \geq \beta^*}$ (Weak duality)

(1) If $\alpha^* = -\infty$, then DUAL is infeasible → PRIMAL is unbounded.

(2) If $\beta^* = \infty$, then PRIMAL is infeasible → DUAL is unbounded

Theorem (Strong Duality)

If PRIMAL is feasible and bounded, then DUAL is feasible and bounded, with $\alpha^* = \beta^*$.