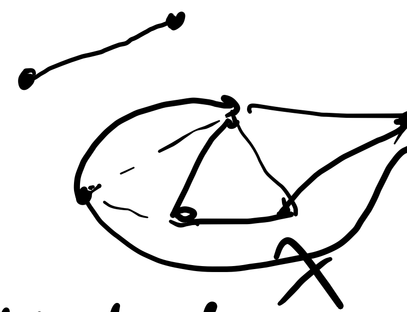
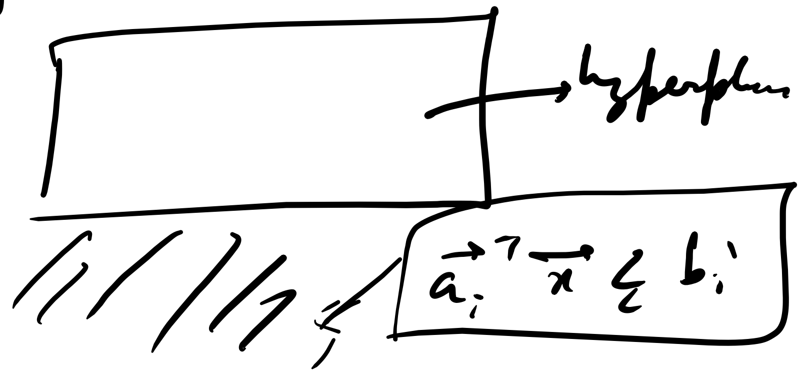
02/11/2021

Lecture - 12



Def^n: Let $P = \{\vec{x} \in \mathbb{R}^n : A\vec{x} \leq \vec{b}\}$ be a polyhedron in $\mathbb{R}^n$. Two vertices of P, $\vec{v_1}, \vec{v_2}$ are said to be *neighboring vertices* if they are the endpoints of a closed 1-dimensional face of P.

In other words, $\vec{v_1}$ and $\vec{v_2}$ are neighbours if there are n linearly independent constraints amongst $A\vec{x} \leq \vec{b}$ which are tight at $\vec{v_1}$, and n l.i constraints amongst $A\vec{x} \le \vec{b}$ which are tight at $\vec{v_2}$, such that precisely $(n-1)$ of those tight constraints are shared by $\vec{v_1}, \vec{v_2}$.



Defn. $A\vec{x} = \vec{b}$, $\vec{x} \ge \vec{0}$.

Two basic feasible solutions (corresponding to bases B_1, B_2) are said to be adjacent/neighbouring if $\#(B_1 \cap B_2) = n-1$.

Lemma -

$$\begin{bmatrix} I_n \\ \hline A \end{bmatrix} \qquad [\,A\,] = [A_1 | A_2 | \dots | A_n]$$

$(n+m) \times n$ $\qquad$ $m \times n$

Let $\{j_1, \dots, j_m\}$ be distinct members of $[n] = \{1, 2, \dots, n\}$

$A_{j_1}, \dots, A_{j_m}$ is a basis of $\mathbb{R}^m$

if and only if

the m rows of A + the $n-m$ rows $\{e_j^T : j \notin \{j_1, \dots, j_m\}\}$ of I_n are linearly independent.

Proof :- Equivalently,

$A_{j_1}, \ldots, A_{j_m}$ are linearly independent

$$\Updownarrow$$

m rows of A + ($n-m$ rows $\{e_j^T : j \notin J\}$ of I_n) are linearly independent.

Equivalently,

$A_{j_1}, \ldots, A_{j_m}$ are linearly dependent

$$\Updownarrow$$

[m rows of A + $n-m$ rows $\{e_j^T ; j \notin J\}$ of I_n are linearly dependent.]

($\Rightarrow$) $A_J = [A_{j_1} | \cdots | A_{j_m}]$

~~If the~~ since $A_{j_1}, \ldots, A_{j_m}$ are linearly dependent, A_J is <u>not</u> an invertible matrix,

$[a_{j_1} \quad \cdot \quad a_{j_m}] \begin{bmatrix} A_J \end{bmatrix} = 0$, $[a_{j_1} \ldots a_{j_m}]$
$\neq [0 \cdots 0]$

$A = [A_J | A_{J^2}]$

$$\begin{bmatrix} a_1 & \cdots & a_{n-m} \end{bmatrix} A = \begin{bmatrix} 0 & \vec{a}_J^{\,T} A_{J^c} \end{bmatrix}$$

$$\left[\begin{array}{c|c} 0 & I_{n-m} \\ \hline A_J & A_{J^c} \end{array}\right] \Rightarrow \left[\begin{array}{c|c} 0 & I_{n-m} \\ \hline 0 & \end{array}\right]$$

$$A$$

$$\begin{bmatrix} a_J & a_{J^c} \end{bmatrix}$$

$$\begin{bmatrix} I_{n-m} \\ \hline A_{J^c} \end{bmatrix}$$

rank $n-m$

number of rows $n \geq n-m$

$$\left[\begin{array}{c|c} 0 & I_{n-m} \\ \hline A_J & A_{J^c} \end{array}\right]$$

$[a_{J^c}]$

I_n

$$\left[\begin{array}{c|c} 0 & I_{n-m} \\ \hline \begin{matrix} 0 & I_n & \cdots \\ 0 & \cdots & \\ 0 & \cdots & \end{matrix} & X A_{J^c} \end{array}\right]$$

$\bar{a}_{J^c}$

$$\left[\begin{array}{c} I_{n-m} \\ \hline X A_{J^c} \end{array}\right] \therefore \vec{\sigma}$$

$[\vec{a}_{J^c}^T \mid \vec{a}^T]$ $[\vec{a}_J^T \mid \vec{c}_J^T]$

$$\begin{bmatrix} 0 & I_{n-m} \\ A_J & A_{J^c} \end{bmatrix}$$

$$\left[\, 0 \quad 1 \quad \vec{a}_{J^c}^T + \vec{a}_J^T A_{J^c} \,\right] \quad \mathbb{R}^{n \times m}$$

$\vec{a}_{J^c}^T \quad - \vec{a}_J \cdot A_{J^c} \quad [\quad]$

$1 \times m$ $m \times n$ row vector.

$1 \times m$ $\vec{a} \neq \vec{0}$

($\Leftarrow$)

rows of $\left[\begin{array}{c|c} 0 & I_{n-m} \\ \hline A_J & A_{J^c} \end{array}\right]^T$ are

linearly dependent.

$\overline{\langle \vec{a}^T}\left[\begin{array}{c|c} 0 & I_{n-m} \\ \hline A_J & A_{J^c} \end{array}\right] = \vec{0}$

$[\vec{a}_J \cdot A_J \mid + \vec{a}_{J^c} + \vec{a}_I A_{J^c} = \vec{0}$

Just looking at $j_1 \dots j_m$ with

$$\underline{\underline{\begin{bmatrix} a_{j_1} & \cdots & a_{j_m} \end{bmatrix}}} \begin{bmatrix} A_{j_1} \,|\, \cdots \,|\, A_{j_m} \end{bmatrix} = \vec{0}$$

(Not all of $a_{j_1}, \dots, a_{j_m}$ can be equal to 0).

$\vec{a}_{J^c} A = \vec{0}$ full rank..

$\vec{a} \neq 0$

and $\vec{a}_{J^c} = 0$

$\Rightarrow \vec{a}_J \neq 0$

<u>Corollary 1</u>: A vector $\vec{v} \in F$ (of LPS) is a vertex if and only if $\vec{v}$ is a basic feasible soln.

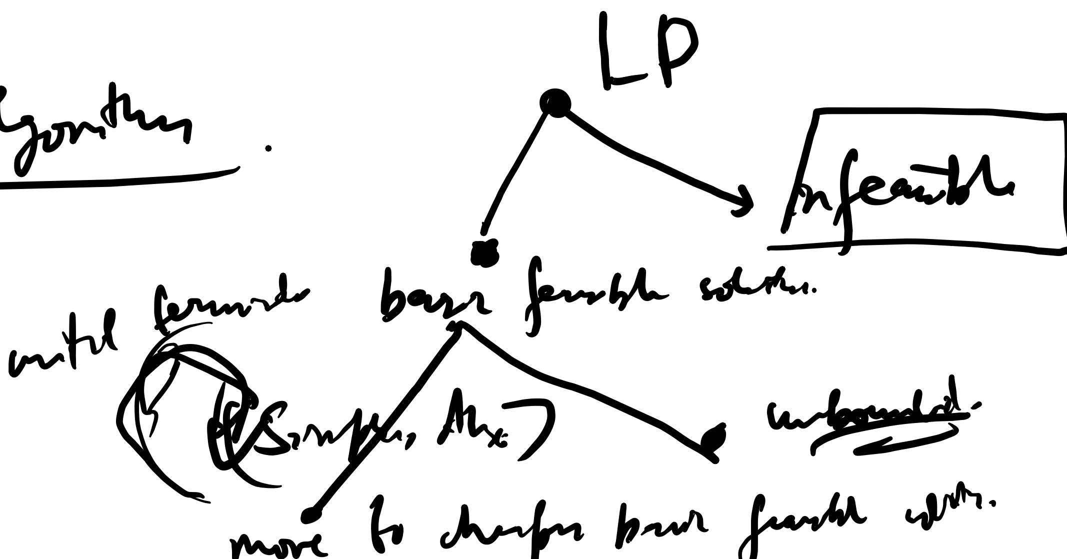
Proof:

$$\left[\begin{array}{c} I_n \\ \hline A \end{array}\right] \left[\begin{array}{c} \vec{x} \end{array}\right] \geq \left[\begin{array}{c} \vec{0} \\ \vdots \\ \hline \vec{b} \end{array}\right]$$

[$\vec{v} \rightarrow n$ linearly independent constraints which are tight at $\vec{v}$ are containing the constraints $\boxed{A\vec{x} \geq \vec{b}}$].

Corollary 2: $\vec{v_1}, \vec{v_2} \in F$ (of LPS) are neighbouring vertices if and only if they correspond to adjacent basic feasible solutions.

Simplex Algorithm.



Pivoting (Strategy of moving to a neighboring basic feasible solution of cheaper cost).

$$\begin{array}{c|ccc} & \boxed{c_1} & \cdots & c_n \\ & j_1 & \cdots & j_m \\ \hline b & & A & \end{array}$$

(1) Find a column j whose row 0 entry is strictly negative. (If there is no such column, we are done.)

$$\left(c_j - \sum c_{j(i)} x_{ij}\right) < 0$$

(2) Compute $\boxed{\theta_0 = \min_{\{i : x_{ij} > 0\}} \frac{x_{i0}}{x_{ij}}}$ (j chosen in step 1)

assume nondeg.

(3) Find an l such that $\theta_0 = \dfrac{x_{l0}}{x_{lj}}$

(and $x_{lj} > 0$).

Pivot on the $(l, j)^{th}$ entry.

$\boxed{\text{Column } j \text{ enters, column } B(l) \text{ leaves}}$

__Theorem.__ If $\bar{c}_j < 0$ and $x_{ij} \leq 0$ for all $1 \leq i \leq m$, then the LPS instance is unbounded.

Proof. Let $\theta \geq 0$.

$\vec{x_0} \rightarrow$ current bfs.

$$\vec{w^\theta} := \vec{x_0} + \begin{bmatrix} \alpha_1^\theta \\ \vdots \\ \alpha_n^\theta \end{bmatrix}$$

where,

$$\alpha_k^\theta = \begin{cases} -\theta a_{ij} & \text{if } k = B(i) \\ \theta & \text{if } k = j \\ 0 & \text{if } k \neq j \text{ and } k \text{ is nonbasic} \end{cases}$$

Clearly, $\vec{w^\theta} \geq 0$ (as $x_{ij} \leq 0$ for all i and all $\theta \geq 0$)

$$A\vec{w}_\theta = A\vec{x}_0 + \theta\left[A_j - \sum_{i=1}^{m} x_{ij} A_{B(i)}\right]$$

$$= \vec{b}$$

$$\left(X = [A_{j_1} | A_{j_m}]^{-1} A \right)$$

$\vec{w}_\theta \in F$.

$$\langle \vec{c}, \vec{w}^\theta \rangle = \vec{c}^T \vec{x}_0 + \theta\left[c_j - \sum_{i=1}^{m} x_{ij} c_{B(i)}\right]$$

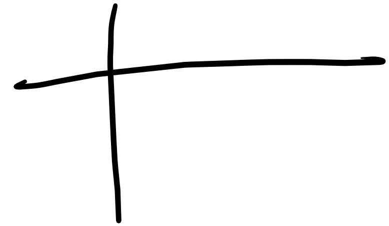
$$= \vec{c}^T \vec{x}_0 + \theta\alpha$$

$$(\alpha < 0)$$

So, $\langle \vec{c}, \vec{w}^\theta \rangle$ can be made arbitrary small by choosing θ sufficiently large. Thus LPS is unbounded. ∎

~~Toooch~~

Need a recipe for choice of:



(1) $j \rightarrow$ entering column.

(2) $l \rightarrow$ exiting column

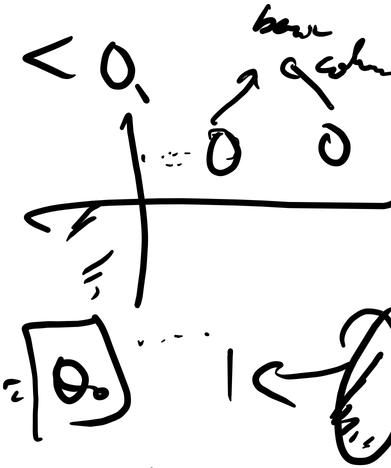
The issue of cycling:

If $x_{r_i} = 0$ for some $1 \le i \le m$, ~~then~~ when $x_{ij} > 0$, we may be stuck at a vertex/bfs.

Bland's anticycling rule for tie-breaking:

* A method to select the entering and exiting columns.

- Choose 'j' (nonbasic column) minimal such that $\bar{c}_j < 0$.



- Choose 'l' for which $\underline{B(l)}$ is as small as possible from among all l satisfying $x_{lj} > 0$ and $\frac{x_{l0}}{x_{lj}} = \boxed{\theta_0}$

(Choose l such that $\underline{B(l)}$ is minimum, not l itself.)

Theorem: The simplex algorithm terminates if Bland's rule is used to choose the pivot entry (entering column, exiting column).

Proof. (By contradiction.)

Assume using Bland's rule, we fall into a cycle.

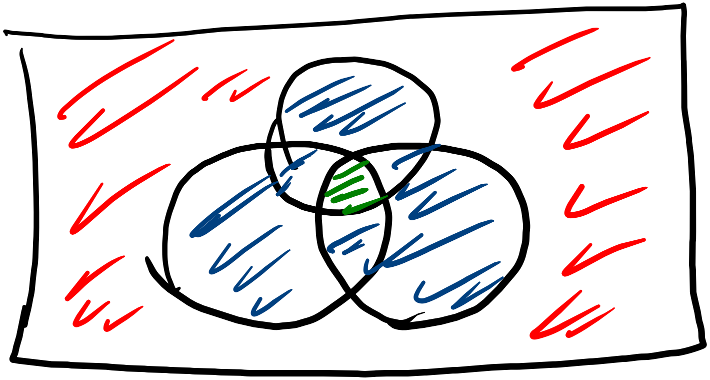
$$\begin{tikzcd} B_1 \arrow[r] & \cdots \arrow[r] & B_q \arrow[ll, bend left] \end{tikzcd}$$

($B_i \to$ ordered basis, we are cycling through bfs corresponding to these ordered basis

Outline: Partition $[n] = \{1, 2, \ldots, n\}$

$\{B_1(1), B_1(2), \ldots, B_1(m)\}$

$\{B_q(1), \ldots, B_q(m)\}$



$J_- = \{ j : j \notin \bigcup_{i=1}^{q} B_i \}$ → columns corresponding to $j \in J_-$ never enters in this cycle.

(red)

$J_+ = \{ j : j \in \bigcap_{i=1}^{q} B_i \}$ → columns corresponding to $j' \in J_+$ never exit in this cycle.

$J_0 = \{ j : j \in B_{i_1},\ j \notin B_{i_2}$ for some $i_1, i_2 \in [q] \}$

"fickle" indices

Define, $l := \max\{j : j \in J_0\}$

(maximum of fickle indices.)

$l \in J_0 \Rightarrow \exists$ basis $\mathcal{B}$ in the cycle such that l exits. Say k enters.

$$\mathcal{B} \longrightarrow \mathcal{B} \cup \{k\} \setminus \{l\}$$

(Clearly, $k < l$ and $k \in J_0$)

k is also "fickle".

Partition indices again (into 5 parts)

$$\left.\begin{array}{l}\{k\} \\ N \setminus \{k\}\end{array}\right\}$$ non-basic indices

$N \rightsquigarrow$ non-basic cols

$$\left.\begin{array}{l}\{l\} \\ B \cap J_0 \setminus \{l\} \\ B \setminus J_0\end{array}\right\}$$ basic indices.

i) .. The

A: $\left[A_1 \mid A_2 \mid \dots \mid A_n \right]$

$\hookrightarrow$ partition of $[n]$

$[n] \rightsquigarrow$ basic indices (corresponding to basic columns)

nonbasic indices.

Start with $\vec{x}$, get $\vec{x}'$ by pivoting.

$$\vec{x}_{\underline{\underline{N\setminus\{k\}}}} = 0 \longrightarrow \vec{x}'_{\underline{\underline{N\setminus\{k\}}}} = 0$$

$$x_k = 0 \longrightarrow x_k = \theta_0$$

$$\vec{x}_B \longrightarrow \vec{x}'_B = \vec{x}_B - \theta_0 X_k$$

To get from $\vec{x}$ to $\vec{x}'$, we move in the direction

$A \to [\;]$

$X = B^{-1}A$

$$\vec{d} = \begin{bmatrix} \vec{0} \\ 1 \\ -x_k \end{bmatrix} \begin{array}{l} \leftarrow N \setminus \{k\} \\ \leftarrow \{k\} \\ \leftarrow B \end{array}$$

$$\boxed{\langle \vec{c}, \vec{d} \rangle < 0}$$

$$\vec{x} \xrightarrow{\vec{d}} \vec{x}'$$

$$\vec{x} + \theta_0 \vec{d} = \vec{x}'$$

$$\langle \vec{c}, \vec{x}' \rangle = \langle \vec{c}, \vec{x} \rangle + \theta_0 \underline{\underline{\langle \vec{c}, \vec{d} \rangle}} < \langle \vec{c}, \vec{x} \rangle$$

$$\theta_0 > 0$$

Claim I: $\vec{d} \in$ nullspace of X. ($X\vec{d} = \vec{0}$).

Proof.

$$X\vec{d} = X_{N\setminus\{k\}}\vec{d}_{N\setminus\{k\}} + X_k d_k + X_B \vec{d}_B$$

$$= \vec{0} + X_k + (-X_k) = \vec{0}$$

At some iteration, l will reenter the basis (say at basis $\hat{B}$). $l \notin \hat{B}$ and $l \in S(\hat{B}^{-1})$.

Claim 2 $\left[\langle \vec{\bar{c}}, \vec{d} \rangle = \langle \vec{c}, \vec{d} \rangle\right]$

Proof 1 $\quad \vec{\bar{c}} = \vec{c} - \vec{c}_B^{\,T} X$

$$\langle \vec{\bar{c}}, \vec{d} \rangle = \langle \vec{c}, \vec{d} \rangle - \langle \vec{c}_B, \underbrace{X\vec{d}}_{0} \rangle$$

$$= \langle \vec{c}, \vec{d} \rangle$$

We will show that $\langle \vec{c}, \vec{d} \rangle > 0$.

	$\bar{c}$ (wrt $\hat{\mathcal{B}}$)	$\bar{d}$ (wrt $\mathcal{B}$)
$\{k\}$	≥ 0	1
$N \setminus \{k\}$	?	0
$\{l\}$	< 0	< 0
$B \cap J_0 \setminus \{k\}$	≥ 0	≥ 0
$B \setminus J_0$	$= 0$	?

① $\bar{c}_{J_f} = 0$ (J_f is in every basis)

$\bar{c}_l < 0$ (l enters basis at $\hat{\mathcal{B}}$)

Fickle indices:

$$\bar{c}_{\mathcal{I}_0 \cap \hat{\mathcal{B}}} = 0$$ (for all indices in $\hat{\mathcal{B}}$)

$$\bar{c}_{\mathcal{I}_0 \setminus \underline{\underline{\hat{\mathcal{B}} \cup \{\ell\}}}} \geq 0$$ (because of Bland's rule)

$[\ \bar{c}_{\ell} < 0\ ]$

Together, $\underline{\underline{\bar{c}_{\mathcal{I}_0 \setminus \{\ell\}} \geq 0}}$ (fickle indices outside of 'ℓ').

$d_k = 1 > 0$

$d_{N\setminus\{k\}} = 0$

$$\begin{array}{c} \begin{bmatrix} 0 \\ 1 \\ -X_k \end{bmatrix} \end{array} \begin{array}{l} \to N\setminus\{k\} \\ \to k \\ \leftarrow B \end{array}$$

$\boxed{d_l = -X_{rk} < 0}$ (for some $r = 1, \dots, m$)

($B(r)$)

$l \to r^{th}$ basic index w.r.t. B.

$\boxed{l = B(r)}$.

$$\theta_0 = \min\left\{ \frac{b_i}{X_{ik}} \;;\; X_{ik} > 0 \right\}$$

$\Rightarrow X_{rk} > 0$)

$d_{B \cap I_0 \setminus \{l\}}$

If $\boxed{d_j < 0}$, then $j \in B$.

$j \to s^{th}$ basic index

$\Rightarrow d_j = -X_{sk} < 0$

$\Rightarrow X_{sk} > 0$

$j \in B \cap I_0 \setminus \underline{\{l\}} \Rightarrow j < l$

$\Rightarrow \underline{B(s) < B(r)}$

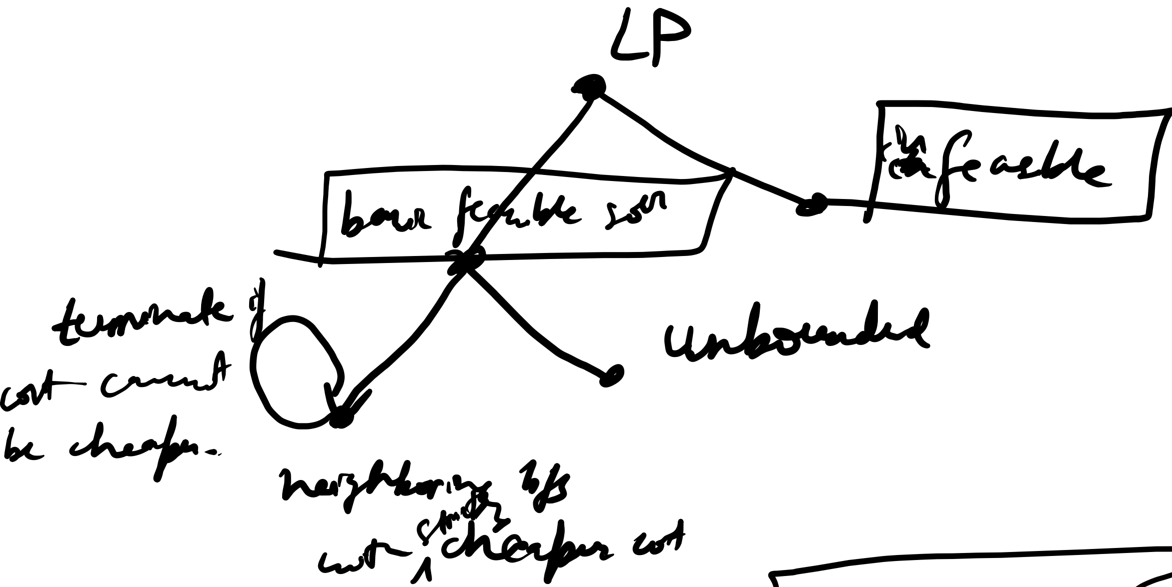
and $x_{sk} > 0$

Simplex + Bland's rule would have chosen $\tilde{s}$ instead (contradicting our choice of r).

$$\langle \vec{\bar{c}}, \vec{d} \rangle > 0$$

$$\langle \vec{c}, \vec{d} \rangle \leq 0.$$

Contradiction!!!



Getting started:

$\vec{x} \geq \vec{0}$, $A\vec{x} = \vec{b}$

(1) If $A\vec{x} = \vec{b}$ is infeasible, declare infeasible and terminate.

(2) Add m ($\leq \operatorname{rank}(A)$) artificial variables

$$\begin{bmatrix} x_1' & x_2' & \cdots & x_m' \end{bmatrix}$$

$$\left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & A \end{array}\right] \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_m' \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}$$

$$\boxed{\text{minimize } \sum_{i=1}^{m} x_i'}$$

and $x_i' \geq 0$ for $1 \leq i \leq m$,

$x_j \geq 0$ for $1 \leq j \leq n$.

(Without loss of generality, we may assume $\vec{b} \geq 0$, by modifying A.)

Lemma: The new problem has optimum value 0 if and only if the original problem is feasible.

~~Proof ($\Leftarrow$)~~

New LP is Feasible and has a bfs.

$$[\,b_1, \ldots, b_m, 0, \ldots, 0\,]$$

$$x_1 \ldots \vec{x}\;\vec{y} \qquad \begin{bmatrix} b_1 \\ \vdots \\ b_m \\ 0 \\ \vdots \end{bmatrix}$$

If optimal value of new LP is > 0, report infeasibility and halt.

(3)

~~If the original LP is~~

If $\bar{x}$ is an optimal bfs of new LP, then $(x_1^*, \ldots, x_n^*)$ is a bfs of original LP,

Worst-Case Performance

$$\max \sum_{j=1}^{n} 10^{n-j} x_j$$

subject to

$$2 \sum_{j=1}^{i-1} 10^{i-j} x_j + x_i \leq 100^{i-1}, \quad 1 \leq i \leq n$$

$$x_j \geq 0$$