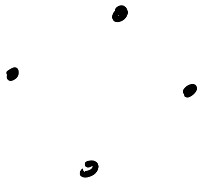
26/10/2021

Lecture - 10

$$\boxed{\begin{aligned} A\vec{x} &= \vec{b} \\ \vec{x} &\geq \vec{0}, \end{aligned}}$$

Either $\langle \vec{c}, \vec{x} \rangle$ is unbounded on F, or the optimal value of $\boxed{\langle \vec{c}, \vec{x} \rangle}$ is attained at a vertex of F. (LP is a finite problem.)

Brute-force idea to solve LPs:



vertices $\longleftrightarrow$ basic feasible solutions

(geometric) (computational)

$$A = \left[A_1 \mid \cdots \mid A_n \right]$$

$m \times n$ matrix of rank m.

$B = [B(1), \ldots, B(m)]$

$i_1, \ldots, i_m \in \{1, \ldots, n\}$,

$A_{B(1)} \ldots A_{B(m)}$ are linearly independent

$$\left[A_{B(1)} \middle| A_{B(2)} \middle| \cdots \middle| A_{B(m)} \right]$$

an invertible $m \times m$ matrix

$$A\vec{x} = \left[A_B \middle| A_N \right] \begin{bmatrix} \vec{x}_B \\ \vec{x}_N \end{bmatrix} = A_B \vec{x}_B + A_N \vec{x}_N = \vec{b}$$

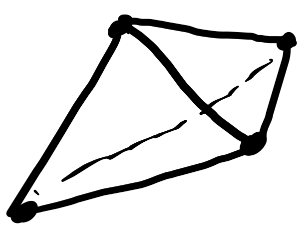
$$\boxed{A_B \vec{x}_B = \vec{b}}$$

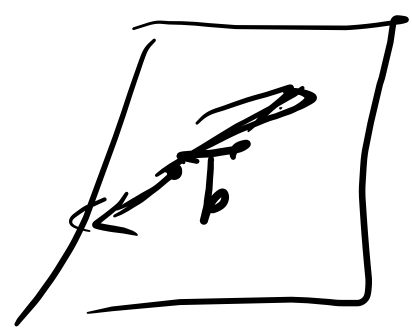
$$\vec{x}_B = A_B^{-1} \vec{b}.$$

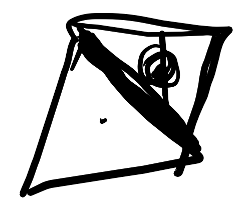
$$\langle \vec{c}, \vec{x} \rangle = \sum_{i=1}^{m} c_{B(i)} x_{B(i)}$$

The Simplex Algorithm:

Instead of brute-forcing our way through vertices of F, we want a systematic method to traverse these vertices in order to find the optimal <u>vertex</u>.

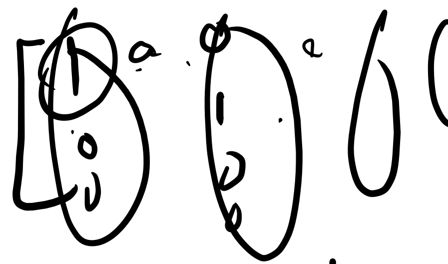


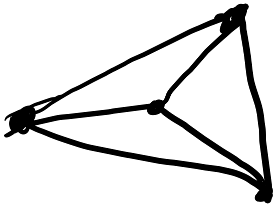


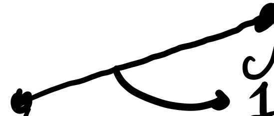


(Designed in 1947 by George Dantzig)

* The algorithm starts from an arbitrary vertex $\vec{v}$ of F and finds a cheaper neighboring vertex. If no neighboring vertex has cheaper cost, the current vertex can be proven to be globally optimal..



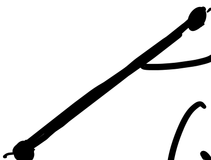




closed

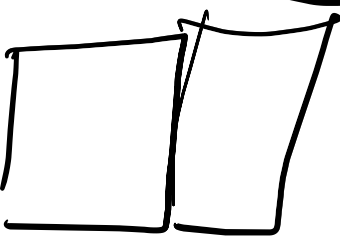
1-dimensional faces whose endpoints are the vertices

$P = \boxed{A\vec{x} \le \vec{b}}$



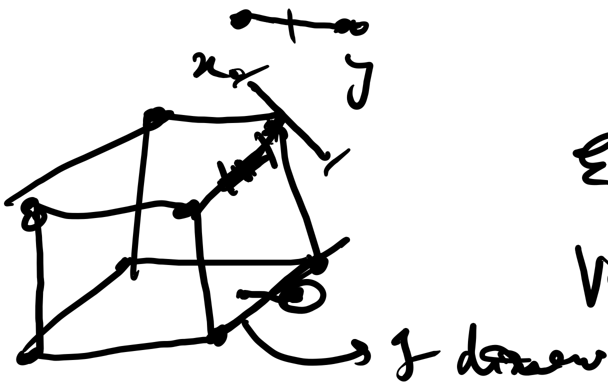
1-dimensional $\boxed{\text{face}}$ of F.

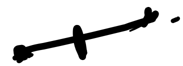
(vertex is given by n linearly independent constraints which are tight at the vertex $\vec{v}$.)



Face of K ; $\frac{x+y}{2} \in$ Face and $x, y \in K$.

then, $x, y \in$ Face





Edge – 1-dimensional face

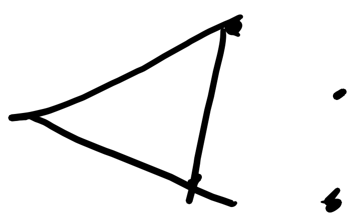
Vertex – 0 dimensional face

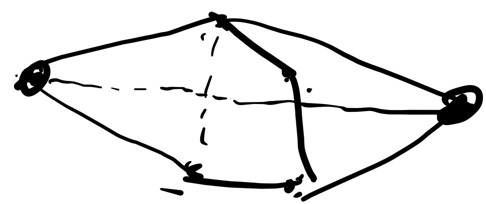
$\vec{v_1}, \vec{v_2}$ are neighboring vertices of K

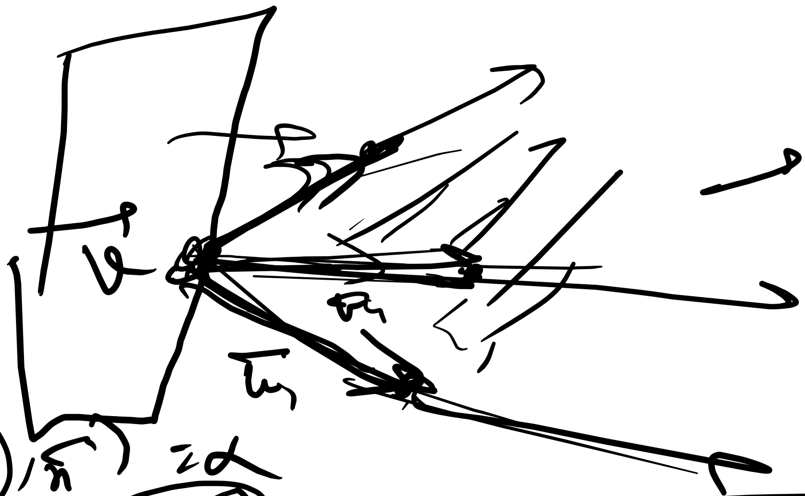
if $\{\lambda \vec{v_1} + (1-\lambda)\vec{v_2} : 0 \le \lambda \le 1\}$

= convex hull of $\vec{v_1}, \vec{v_2}$,

is a closed 1-D face of K.







contains all vectors

$\langle \vec{c}, \vec{n} \rangle = \alpha$

$$\langle \vec{c}, \vec{n} \rangle = \langle \vec{c}, \vec{v} \rangle \qquad \langle \vec{c}, \vec{v} \rangle \le \langle \vec{c}, \vec{v}_1 \rangle$$

$$\Rightarrow \quad 0 \le \langle \vec{c}, \vec{v}_1 - \vec{v} \rangle$$

$\langle \vec{c}, \vec{n} \rangle = \langle \vec{c}, \vec{v} \rangle$ is a supporting hyperplane for K.

Pivoting :

How to move from one bfs to a cheaper bfs ?

Let $\vec{x_0}$ be a bfs corresponding to the ordered basis $\mathcal{B} = \{B(1), \ldots, B(m)\}$

$$\left[A_1 \mid A_2 \mid \cdots \mid A_m \right]$$

$$B(1) \; \ldots \qquad B(m)$$

$x_{i0} \rightarrow$ $\underline{B(i)^{th} \text{ component of } \vec{x_0}}$, $1 \leq i \leq m$,

$$\begin{bmatrix} 0 \\ \vdots \\ x_{B_1}, \\ \vdots \\ \vdots \\ x_{B_m} \\ 0 \end{bmatrix} \qquad \begin{matrix} x_{B_1} \rightarrow x_{i0} \\ x_{B_m} \rightarrow x_{im} \end{matrix}$$

Because B is a basis, each nonbasic column j (A_j) as a linear combination of the basic columns for certain reals x_{ij} (j fixed),

$$\boxed{A_j = \sum_{i=1}^{m} x_{ij} A_{\underline{\underline{B(i)}}}}$$

[Note that j^{th} entry of $\vec{x_0}$ is $\underline{0}$.

$$\Rightarrow \theta A_j = \sum_{i=1}^{m} x_{ij} \, \theta A_{B(i)}$$

$$\left(\sum_{i=1}^{m} x_{ij} \, \theta A_{B(i)} \right) - \theta A_j = 0$$

$$\Rightarrow \sum_{i=1}^{m} x_{i0} A_{B(i)} - \sum_{i=1}^{m} x_{ij}\, \theta A_{B(i)} + \theta A_j = \vec{b}$$

$$A\vec{x} = (A_B | A_N) \begin{bmatrix} x_B \\ \hline x_N \end{bmatrix} = A_B x_B + A_N x_N$$

$$\Rightarrow \boxed{\sum_{i=1}^{m} \left(x_{i0} - \theta x_{ij} \right) A_{B(i)} + \theta A_j = \vec{b}}$$

$$\sum_{i \neq l} \left(x_{i0} - \theta_0 x_{ij} \right) A_{B'(i)} + \theta_0 \underbrace{A_j}_{B'(l)} = \vec{b}$$

Strategy to move to a neighboring bfs (vertex)

Define $\vec{x_0'}$ as follows:

(1) Its $\underline{\underline{j}}$ th component is θ (updated from θ)

(2) For each i, $1 \leq i \leq m$, its $B(i)^{th}$ component is $x_{i0} - x_{ij}\theta$. (choose appropriate θ to make one of the coordinate of $\vec{x_0}$ equal to 0, and rest feasible.)

(3) Other components are 0.

For (2), choose $\theta = \theta_0 = \min_{\{i : x_{ij} > 0\}} \frac{x_{i0}}{x_{ij}}$

Largest value of θ such that $\vec{x_0'}$ is feasible.

$i^* \in \arg\min_{\{i : x_{ij} > 0\}} \frac{x_{i0}}{x_{ij}}$

$x_{i0} \to x_{i0} - \theta x_{ij} \geq x_{i0} - \frac{x_{i0}}{x_{ij}} x_{ij}$

If $x_{ij} \leq 0$, $x_{i0} \to x_{i0} - \theta x_{ij} \geq 0$

If $x_{ij} > 0$, $x_{i0} \to x_{i0} - \theta x_{ij} \geq x_{i0} - \frac{x_{i0}}{x_{ij}} x_{ij} = 0$

Potential issues in the strategy:

* For all $1 \le i \le m$, $x_{i1} \le 0$. Then we may choose any $\theta_0 \geq 0$ and remain feasible.

* θ_0 may be 0, in which case $\overrightarrow{x_0'} = \overrightarrow{x_0}$ (but the ordered basis have changed). A pivot with $\theta_0 = 0$ is called <u>degenerate</u>.

Recall ~~of~~ def^n of degeneracy:

(A basic feasible solution is said to be <u>degenerate</u> if it has more than $n-m$ zeros).

[No updating of $\vec{x_0}$ happens.]

Theorem: given ordered basis $B(1), \dots, B(m)$ and the bfs x_0 corresponding to B.

(whose $B(i)^{th}$ component is x_{i0}, $1 \leq i \leq m$), let $j \notin B$.

Let $A_j = \sum_{i=1}^{m} x_{ij} A_{B(i)}$ (for appropriate x_{ij}'s)

Suppose $\{i : x_{ij} > 0\}$ is nonempty.

Let $\theta_0 := \min_{\{i : x_{ij} > 0\}} \frac{x_{i0}}{x_{ij}}$.

Suppose, $\theta_0 = \frac{x_{l0}}{x_{lj}}$ (with $x_{lj} > 0$).

Let $\mathcal{B}'(i) = \begin{cases} B(i), & \text{if } i \neq l \\ j, & \text{if } i = l \end{cases}$

$A_{B(l)}$ has been replaced by A_j.

Then $\mathcal{B}' = \{\mathcal{B}'(1), \dots, \mathcal{B}'(m)\}$ is a <u>basis</u> corresponding to the bfs $\vec{x_0}'$ whose $B'(i)^{th}$ component is $x_{i0} - \theta_0 x_{ij}$, if $i \neq l$, and whose j^{th} component is θ_0.

Proof :- Note that $\vec{x_0'}$ is feasible.

Need to show $A_{B'(1)}, \ldots, A_{B'(m)}$ are linearly independent.

For $i \neq l$, $A_{B(i)} = A_{B'(i)}$

$$A_j = \left(\sum_{i \neq l} x_{ij} A_{B(i)}\right) + A_{B(l)} x_{lj} \qquad x_{lj} > 0$$

$$A_j = A_{B'(l)} \Rightarrow \underline{\underline{A_{B(l)}}} = \frac{A_{B'(l)} - \sum_{i \neq l} x_{ij} A_{B(i)}}{x_{lj}}$$

Thus, Span $\langle A_{B'(1)}, \ldots, A_{B'(m)} \rangle$

contains the basis $\{A_{B(1)}, \ldots, A_{B(m)}\}$.

Hence, $\{A_{B'(1)}, \ldots, A_{B'(m)}\}$ is a basis. $\boxtimes$

<u>TABLEAUX</u> (graphic and pictorial representation of something)

<u>Notation</u>: $1 \leq i \leq m$, $e_i \to$ standard i^{th} vector in $\mathbb{R}^m$

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \to i^{th} \text{ coordinate}$$

$I_m = [e_1 | e_2 | \dots | e_m]$. $m \times m$ identity matrix.

$$\begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ \vdots & 0 & \vdots & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

If $\boxed{A_{B(1)} = e_1, \dots, A_{B(m)} = e_m}$, life would be much simpler in terms of pivoting.

$$A_j = \begin{bmatrix} \boxed{a_{1j}} \\ \vdots \\ a_{nj} \end{bmatrix}$$

$$\boxed{A_j = \sum_{i=1}^{m} a_{ij} A_{B(i)}}$$

e_i.

$(i,j)^{th}$ entry of A.

* We will force the basic columns to be I_m by executing elementary row operations.

$$
\begin{array}{c|cccc}
 & x_1 & x_2 & \cdots & x_n \\
\hline
b_1 & & & & \\
\vdots & A_1 & A_2 & \cdots & A_n \\
b_m & & & &
\end{array}
$$

Without changing solution space, we may put e_i on $A_{B(i)}$ by elementary row operations (on all $n+1$ columns).

Choose a nonbasic column j' and look at

$$\left[\theta_0 = \min_{\{i:\, a_{ij'} > 0\}} \frac{b_i'}{a_{ij'}}\right]$$

$a_{lj'}$

[Again by elementary rows operation.

(1) update column 0 so that it contains the basic entries of the new bfs.

$$\left[\begin{array}{ll} b_i' \to b_i' - \theta_0 a_{ij'} & \text{if } i \neq l \\ b_l' \to \theta_0, & \text{if } i = l \end{array}\right]$$

$A_B x = b$

$A_B^{-1} b$

(2) Put e_i in column $A_{B'(i)}$, for every i.

It suffices to put e_l onto A_j, since the remaining e's are already in required form.

Example. Consider the system

$$1 \cdot x_1 + 1 \cdot x_2 + 4x_6 = 6$$

$$-2x_1 + 2x_2 + x_3 - x_4 + x_6 = 14$$

$$x_1 - 2x_2 + x_4 + 2x_6 = -11$$

$$-x_1 - 3x_4 + x_5 - 5x_6 = 7, \quad x_j \geq 0.$$

	x_1	x_2	x_3	x_4	x_5	x_6
6	1	1	0	0	0	4
14	-2	2	1	-1	0	1
-11	1	-2	0	1	0	2
7	1	0	0	-3	1	-5

$$\mathcal{B} = \langle 2, 3, 4, 5 \rangle$$

$$
\begin{array}{c|cccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
\hline
6 & 1 & 1 & 0 & 0 & 0 & 4 \\
3 & -1 & 0 & 1 & 0 & 0 & 3 \\
1 & 3 & 0 & 0 & 1 & 0 & 10 \\
10 & 10 & 0 & 0 & 0 & 1 & 25
\end{array}
$$

$$
\begin{pmatrix} 0 \\ 6 \\ 3 \\ 1 \\ 10 \\ 0 \end{pmatrix}
$$

is a bfs for the LPS instance

Choose $j=1$,

$$\theta_0 = \min_{\{i : a_{i1} > 0\}} \left\{ \frac{6}{1}, \cancel{\frac{3}{4_1}}, \frac{1}{3}, \frac{10}{10} \right\}$$

$$= \frac{1}{3}, \quad \boxed{l = 3} \quad (x_{l0} - \theta_0 x_{lj} = 0)$$

Row operations to convert A_1 to e_3.
to get new bfs corresponding to $\{1, 2, 3, 5\}$

[$B(1) = B(3) = 4$
must be removed]

$$
\left[\begin{array}{c|cccccc}
17/3 & 0 & 1 & 0 & -1/3 & 0 & 2/3 \\
10/3 & 0 & 0 & 1 & 1/3 & 0 & 19/3 \\
1/3 & 1 & 0 & 0 & 1/3 & 0 & 10/3 \\
20/3 & 0 & 0 & 0 & -19/3 & 1 & -25/3
\end{array}\right]
$$

b/s correspon to ~~B(2,3), B(3,?)~~, (2, 3, [illegible] 1, 5]

107/3

$$\begin{bmatrix} 1/3 \\ 17/3 \\ 10/3 \\ 0 \\ 20/3 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1/3 \\ 17/3 \\ 10/3 \\ 0 \\ 20/3 \\ 0 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 17/3 \\ 10/3 \\ 1/3 \\ 20/3 \end{bmatrix}$$

* Dang 2) also does (1).

If $A_{B'(i)} = e_i$ for $1 \le i \le m$,
the 0th entry of column 0 must be
be the $B'(i)$th entry of the bfs
corresponding to B'.
(solution space is unchanged by elementary
row operations)

$$A\vec{x} = \vec{b}$$

Question: Which nonbasic column should we bring into the basis?

(In other words, which neighboring vertex of F should we travel to?)

The cost of a bfs $\vec{x_0}$ with ordered basis

$B(1), \ldots, B(m)$, i

$$\langle \vec{c}, \vec{x_0} \rangle = \sum_{i=1}^{m} c_{B(i)} x_{i0}$$

Qn. How will bringing j into the basis change the cost?

$$\theta_0 = \min_{\{i : x_{ij} > 0\}} \frac{x_{i0}}{x_{ij}}$$

(assuming this is nonempty).

Choose l such that $\boxed{\theta_0 = \frac{x_{l0}}{x_{lj}}}$.

$\boxed{\text{Column } j \text{ enters, Column } B(l) \text{ leaves}}$.

Updated cost

$$= \sum_{i \neq l} c_{B(i)} \left[\underline{\underline{x_{i0} - \theta_0}}\, x_{ij} \right] + c_j \underline{\theta_0}$$

$$\text{Old cost} = \sum_{i \neq l} c_{B(i)} x_{i0} + c_{B(l)} x_{l0}$$

$$= \sum_{i \neq l} c_{B(i)} x_{i0} + c_{B(l)} \theta_0 x_{lj}$$

Updated cost − old cost

$$= \theta_0 \left[c_j - \sum_{i \neq l} c_{B(i)} x_{ij} - c_{B(l)} x_{lj} \right]$$

$$= \theta_0 \left[c_j - \sum_{i=1}^{m} c_{B(i)} x_{ij} \right].$$

To decrease the cost, we want,

$$\left[c_j - \sum_{i=1}^{m} c_{B(i)} x_{ij} \right]$$ to be negative.

$$c_j - z_j$$

Let $\vec{z} \in \mathbb{R}^n$ s.t.

$$\boxed{z_j = \sum_{i=1}^{m} c_{B(i)}\, x_{ij}}$$

$$\vec{c'} = \vec{c} - \vec{z}$$

We may bring j onto the basis if

$c'_j < 0.$

Lemma. Let X be the tableau (without column 0) after s pivots, where A was the original matrix.

Let columns $B(1), \ldots, B(m)$ be the current ordered basis where column $B(i)$ of X is e_i.

Let B denote the $m \times m$ matrix whose i^{th} column is $A_{B(i)}$. Then $\boxed{X = B^{-1} A}$.

Furthermore, the 0^{th} column is exactly $B^{-1} b'$.

$$E_n \cdots E_1 A = I \implies A = E_n^{-1} \cdots E_1^{-1}$$
$$\implies = I \qquad = (E_n \cdots E_1)^{-1}$$

$$K_j \cdots K_1 \begin{bmatrix} A_1 & e_1 & \cdots & A_n \end{bmatrix} = X$$

Proof. If $K = K_j K_{j-1} \cdots K_1$

$$\boxed{K A_{B(i)} = e_i}$$

$$K = B^{-1}$$

$$K \begin{bmatrix} A_{B(1)} & \cdots & A_{B(m)} \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \cdots & e_m \end{bmatrix}$$

Theorem: If $\boxed{\vec{c} - \vec{z} \geq 0}$ at bfs $\vec{x_0}$, then $\vec{x_0}$ is optimal. (~~...~~ of