14/10/2021

Lecture - 8

Standard form of LP:

$$\min \ \langle \vec{c}, \vec{x} \rangle = \vec{c}^T \vec{x}$$

subject to $\boxed{A\vec{x} = \vec{b}}$ (equality constraints)

$\boxed{\vec{x} \geq \vec{0}}$ (sign constraints)

* We can assume A is full-rank

$\boxed{A} \to m \times n$ matrix

n = dimension of $\vec{x}$.

m = number of equality constraints

$m \leq n$ and $m = \text{rank}(A)$.

$A_1, \ldots, A_n \longrightarrow$ columns of A.

$$\begin{array}{cc} j_1 & j_m \\ \left[\begin{array}{cc} A_{j_1} & A_{j_m} \end{array}\right] \end{array}$$

If $A_{j_1}, \ldots, A_{j_m}$ are linearly independent,
$(J = \langle j_1, \ldots, j_m \rangle)$

$A_J = [A_{j_1} | \ldots | A_{j_m}] \longrightarrow$ $m \times m$ invertible matrix

$A\vec{x} = \vec{b}$

$\vec{x} = [x_J \| x_{J^c}]^T$

$$x_J = A_J^{-1}\vec{b}$$

$$x_{J^c} = \vec{0},$$

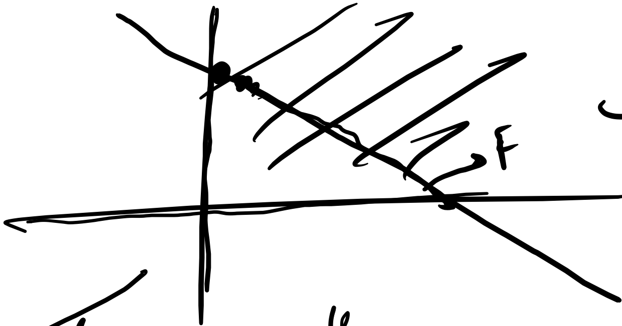
$$A\vec{x} = \left([A_J \mid A_{J^c}] \begin{bmatrix} x_J \\ \hline x_{J^c} \end{bmatrix} \right)$$

$$= A_J x_J + A_{J^c} x_{J^c}$$

$$= \vec{b} + \vec{0} = \vec{b}$$

$\vec{x} = [x_J \mid \vec{0}]$ is a <u>basic sol^n</u>.

Iff $\boxed{x_J \geq 0}$, $\vec{x}$ is a <u>basic feasible sol^n</u>.



$\boxed{x_1 \geq 0}$

$\boxed{x_2 \geq 0}$

Goal: If the feasible region of LPS is nonempty, it must have a basic feasible soln. → vertex.

$\boxed{(x_1 + t a_1, \dots, x_n + t a_n)}$

$t \in \mathbb{R}$,

Theorem. In LPS, suppose that $\vec{w} \in \mathbb{R}^n$. Then $\vec{w}$ is a vertex (or extreme point) of F if and only if $\vec{w}$ is a basic feasible solution of the LPS instance.

Proof. ($\Rightarrow$)

$\vec{w}$ is a vertex of F.

$$I = \{ i : w_i > 0 \}$$

Claim. $\{A_j : j \in I\}$ are linearly independent in $\mathbb{R}^m$

$$\sum_{j \in I} z_j A_j = 0 \qquad A\vec{z} = 0$$

$\vec{w}$ $\qquad \vec{z} \qquad A\vec{z} = 0$

$$A(\vec{w} + \theta \vec{z}) = \vec{b} = A\vec{w}$$

$\{A_j : j \in I\}$ extends to a basis (J) of $\mathbb{R}^m$:

using columns of A.

$w_j = 0$ if $j \notin I$

$\Rightarrow$ $w_j = 0$ if $j \notin J$ ($I \subseteq J$).

$[A_{j_1} \ldots A_{j_m}]$ is invertible.

$\vec{w} = [w_J][w_{J^c}]$ and $\boxed{\vec{w} \in F}$.

$\to 0$

($\Leftarrow$) Let $\vec{w}$ be a bfs of

$A\vec{x} = \vec{b}$, $\vec{x} \geq \vec{0}$, (LPS).

where A has rank m.

Choose $J \subseteq \{1, 2, \ldots, n\}$ corresponding to $\vec{w}$. ($\#(J) = m$ and for $j \notin J$, $w_j = 0$)

[Furthermore $\vec{w}$ is the unique vector in F satisfying $w_j = 0$ if $j \notin J$.] ✓

Let $\boxed{\vec{w} = \dfrac{\vec{w_1} + \vec{w_2}}{2}}$ for $\underline{\underline{\vec{w_1}, \vec{w_2} \in F}}$,

$\forall j \in \{1, \dots, n\}$,

$w_{1,j} \geq 0$, $w_{2,j} \geq 0$

$w_j = \dfrac{w_{1,j} + w_{2,j}}{2}$

$\Rightarrow w_{1,j} = 0$, $w_{2,j} = 0$ $\forall j \in J^c$.

By a previous remark $\vec{w} = \vec{w_1} = \vec{w_2}$.

<u>Corollary</u>: There are finitely many vertices of LPS.

<u>Proof</u> - Each vertex of F is a bfs.

bfs correspond to subsets of size m in $\{1, \ldots, n\}$

$$\begin{array}{c} j_1 < \cdots < j_m \\ \left[A_1 \,\middle|\, \cdots \,\middle|\, A_n \right] \end{array}$$

There ~~at~~ are at most $\binom{n}{m}$ ~~choose~~ collections of columns of A which form a basis of $\mathbb{R}^m$.

The number of bfs's is at most $\binom{n}{m}$.

Defn: A basic feasible soln is <u>degenerate</u> if it has strictly more than $n-m$ zeros.

$$\left[A_1 | \ldots \quad | A_n \right]$$

* Degeneracy does not mean redundancy.

Theorem: If two different bases (from the columns of A) correspond to a single bfs $\vec{v}$, then $\vec{v}$ is degenerate.

Pf → $J_1, J_2 \subseteq \{1, \ldots, n\}$, $\#(J_1) = \#(J_2) = m$.

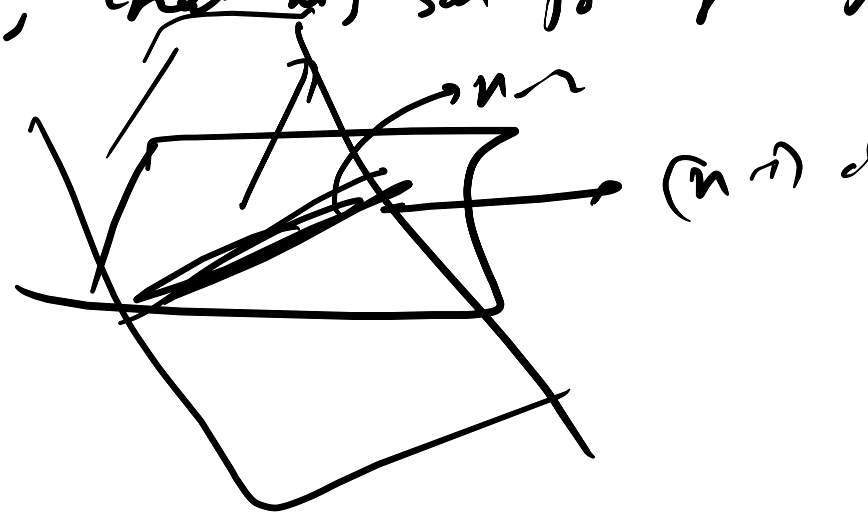
$v_j = 0$ for $j \notin J_1$ $(j \in J_1^c)$, $v_j = 0$, for $j \notin J_2$. $(j \in J_2^c)$

$v_j = 0$, for $j \notin J_1 \cap J_2$. $\left(j \in J_1^c \cup J_2^c = (J_1 \cap J_2)^c \right)$

Hence, $v_j = 0$ for strictly more than $n - m$ indices j.

Exercise: If $\vec{v}$ is degenerate, is it necessarily true that there are two different bases (comprised of columns of A) corresponding to $\vec{v}$? Prove or provide counterexample.

Lemma: Suppose $P = \{\vec{x} \in \mathbb{R}^n \mid \underline{\underline{A}}\vec{x} \leq \vec{b}\}$ is a polyhedron. Then $\vec{v}$ is an extreme point of P if and only if there are ~~two~~ $\boxed{n}$ linearly independent constraints (amongst $A\vec{x} \leq \vec{b}$) that are **tight** at $\vec{v}$, that is, satisfy equality at $\vec{v}$.



Proof: ($\Rightarrow$)

Let $\vec{v}$ be an extreme point of P.

$B\vec{x} \le \vec{b'}$ be those constraints amongst $A\vec{x} \le \vec{b}$ (B consists of a subset of rows of A) that are tight at $\vec{v}$.

$\boxed{B\vec{v} = \vec{b'}}$ $\boxed{C\vec{v} < \vec{b''}}$.

Claim: rank$(B) = n$

Proof of Claim: $\boxed{\text{rank}(B) \le n-1}$ $\Rightarrow \vec{w}$

$A\vec{x} = \vec{b}$

Choose $\vec{w} \in$ nullspace of B.
nonzero vector $\vec{v'} = \vec{v} + \gamma\vec{w}$.

satisfies $B\vec{v}' = \vec{b}'$. for any $\gamma \in \mathbb{R}$

For a sufficiently small positive γ_0

both $\vec{v} + \gamma_0 \vec{w}$ and $\vec{v} - \gamma_0 \vec{w}$ satisfy the

remaining constraints.

$$\vec{v} = \frac{1}{2}(\vec{v} + \gamma_0 \vec{w}) + \frac{1}{2}(\vec{v} - \gamma_0 \vec{w})$$

$c_1 x_1 \dots + c_n x_n < \alpha_i$

contradicting the fact that $\vec{v}$ is extreme.

Φ

($\Leftarrow$) $\boxed{B\vec{x} \le \vec{b'}}$ are tight at $\vec{v}$,

and rank $(B) = n$.

$$\vec{v} = \frac{\vec{v_1} + \vec{v_2}}{2} \quad \text{where } \vec{v_1}, \vec{v_2} \in P.$$

$$\vec{b'} = B\vec{v} = \frac{1}{2} B\vec{v_1} + \frac{1}{2} B\vec{v_2} \le \frac{1}{2}\vec{b'} + \frac{1}{2}\vec{b'} = \vec{b'}$$

$$B\vec{v} = B\vec{v_1} = B\vec{v_2}$$

$$\Rightarrow \quad \vec{v} = \vec{v_1} = \vec{v_2}$$

$\Rightarrow$ $\vec{v}$ is an extreme point of P.

Defn: We say that a minimization problem is **unbounded** if for each real B, there is a feasible point of cost strictly less than B.

Theorem: In LPS, suppose that $\vec{b} \in T$. Then either LPS instance is unbounded or there is a vertex $\vec{v}$ of F satisfying $\langle \vec{c}, \vec{v} \rangle = \boxed{\vec{c}^{\,T} \vec{v} \le \vec{c}^{\,T} \vec{b}} = \langle \vec{c}, \vec{b} \rangle$

(either there is no optimal solution or there is an optimal vertex.)

Corollary. If in LPS, $F \neq \phi$ and $\vec{c}^T \vec{x} \geq B$ for all $\vec{x} \in F$, then there is an optimal vertex. (Provided that the cost function is bounded below, LP is a finite problem!)

Proof. $\vec{p} \in F$. Find $\vec{v}_1$ s.t. $\boxed{\langle \vec{c}, \vec{v}_1 \rangle} \leq \langle \vec{c}, \vec{p} \rangle$.

If $\langle \vec{c}, \vec{v}_1 \rangle$ is minimum, we are done.

If not, $\exists$ $\vec{p}_2 \in F$ s.t. $\langle \vec{c}, \vec{p}_2 \rangle < \langle \vec{c}, \vec{v}_1 \rangle$

$$\cdots < \langle \vec{c}, \vec{v}_2 \rangle < \langle \vec{c}, \vec{v}_1 \rangle$$